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Rössler attractor-based numerical solution of the fractal-fractional operator: A fixed point approach

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Abstract

In this article, we introduce an extended (α, c)-interpolative metric space and present an associated fixed-point theorem. With our main theorem, we present the existence of a solution for a fractal-fractional differential equation with a power-law kernel. In the end, we analyze the dynamics of the chaotic Rössler system.

Keywords: An extended (α, c)-interpolative metric space; Fractal-Fractional operator; Rössler attractor.
2010 MSC: 26A33, 35S10, 47H10, 54E35, 54E99.

1. Introduction

The role of fixed points in studying the stability and convergence of dynamical systems is central. These systems become more complex when being studied using fractional calculus: the traditional calculus is generalized to include non-integer order derivatives. Since fractional-order systems include memory and hereditary properties, such models tend to provide a more realistic description of natural and physical processes. Fixed points behavior changes with this extension, which affects stability conditions and extending the dynamical phenomena spectrum. The nonlocal nature of fractional derivatives provides a distinct context to analyze how trajectories approach, repel, or oscillate around fixed points, exploring instances where dependencies and lagged response are involved and complex [1]-[6].

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Chaotic attractors, known for their sensitivity to initial conditions and intricate geometrical structures, often coexist with fixed points in dynamical systems. This co-existence becomes even stronger in the fractional order systems, as the fractional nature of the trajectories is so perceptive of exploring the phase space. A continuum between order and chaos is shown to exist through the interplay between fixed points and chaotic attractors, where fixed points act as pivots in chaotic motion. Fractional derivatives introduce an additional level of complexity, generating paths where trajectories flow into chaotic attractors or into fixed points near which the trajectories are stabilized. This intricate relationship reveals the nonlinear dynamics and reveals how fractional order system performs as a bridge between deterministic structures and randomly behaving systems, opening the path for new modeling approaches to the physical, biological, and engineering systems [7]-[10].

The fractional order Lorenz system is a fractional extension of the classical Lorenz system, which is an example that connects the fixed points, fractional calculus and chaotic attractors. There exists a set of three coupled nonlinear differential equations, the classical Lorenz system, which are chaotic in some parameter conditions. When the system is extended to fractional-order derivative, the chaos behavior is richer, which reveals novel dynamics between fixed points and chaos [11]-[13].

In both the classical and fractional versions, fixed points are determined by solving for equilibrium solutions where the time derivatives vanish. These are the fixed-point steady states of the system. In the fractional version, however, the stability of these points also depends on the fraction order α, where $0 < \alpha < 1$. But the fractional derivatives can introduce memory effects that can induce trajectories to linger near fixed points for long periods of time, or explore more interesting topologies around them, but diverge, depending on the value of α.

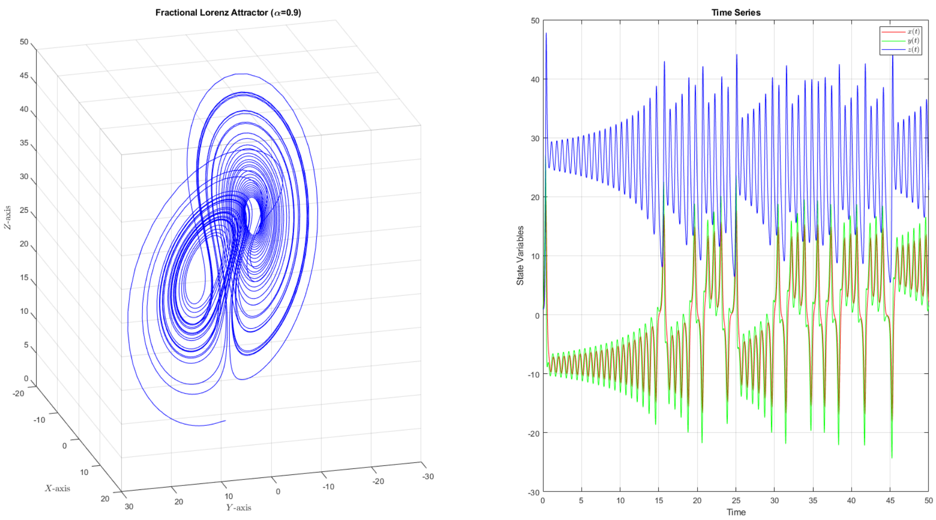

Figure 1: The behavior of the fractional-order Lorenz system with order α=0.9

Fixed points in the Lorenz system are the equilibrium points where the derivatives of all state variables vanish. For the Lorenz system, these are calculated by setting:

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0.$$

The equations for the system become:

$$\sigma(y - x) = 0 \quad \Rightarrow \quad y = x,$$

$$x(\rho - z) - y = 0 \quad \Rightarrow \quad x(\rho - z) - x = 0 \quad \Rightarrow \quad z = \rho - 1,$$

$$xy - \beta z = 0 \quad \Rightarrow \quad x^2 = \beta z.$$

From the above plots, we can notice that:

- **Trivial Fixed Point**: $(0, 0, 0)$
  - This corresponds to a state where the system is at rest.
- **Non-Trivial Fixed Points**:

$$(\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho-1).$$

  - These exist only if $\rho > 1$ (*i.e.*, when the parameter ρ exceeds the critical value).

In the fractional-order system, the fixed points remain the same as in the classical system because they are defined by the equilibrium conditions, where the time derivatives vanish. However, the stability and trajectories around these fixed points are influenced by fractional order α, leading to richer dynamics.

In the plotted attractor, the trajectories hover near the vicinity of the non-trivial fixed points:

$$(\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho-1),$$

but never settle due to the chaotic nature of the system. The attractor's shape shows how the system evolves around these fixed points in the phase space.

2. A Fixed Point Result

The concept of triangle inequality has evolved over time to accommodate broader mathematical frameworks. Initially, the **classical triangle inequality** was introduced in metric spaces, where for any three points $\varkappa, y, z$, the inequality $d(\varkappa, \mathfrak{z}) \leq d(\varkappa, y) + d(y, \mathfrak{z})$ holds. To extend this framework, the ***b*-metric space** [14],[15] was developed, relaxing the standard inequality by introducing a constant $s \geq 1$, leading to $d(\varkappa, \mathfrak{z}) \leq s.[d(\varkappa, y) + d(y, \mathfrak{z})]$. Further generalizations led to the **extended *b*-metric space** [16], which replaced the constant s with a variable function $\theta(\varkappa, \mathfrak{z}) \geq 1$, yielding $d(\varkappa, \mathfrak{z}) \leq \theta(\varkappa, \mathfrak{z}).[d(\varkappa, y) + d(y, \mathfrak{z})]$. This kind of extension extends the application of fixed point theorems to more general settings and increases the flexibility of analyzing complex systems.

In very recent times, the classical triangle inequality is extended to more general structures in generalized metric spaces, see, for example, [17]-[22]. Triangle Inequality in a suprametric space[17] in which a multiplicative term given by the product of intermediate distances appears. It is as follows:

$$d(\varkappa, y) \leq d(\varkappa, \mathfrak{z}) + d(\mathfrak{z}, y) + \rho d(\varkappa, \mathfrak{z}) d(\mathfrak{z}, y), \quad \text{where } \rho > 0 \text{ is a constant.}$$

Very recently, we have another triangle inequality altered, in an (α, c)**-interpolative metric space** [23] with another term involving powers of intermediate distances. Specifically, for any $\varkappa, y, \mathfrak{z} \in X$, the inequality is expressed as:

$$d(\varkappa, y) \leq d(\varkappa, \mathfrak{z}) + d(\mathfrak{z}, y) + c[(d(\varkappa, \mathfrak{z}))^{\alpha}(d(\mathfrak{z}, y))^{1-\alpha}], \text{ where } \alpha \in (0,1) \text{ and } c \geq 0.$$

Motivated and inspired by the above generalized triangle inequalities and distance spaces, we introduce below:

Definition 2.1. *Let X be a non-empty set and $\theta : X \times X \to [1, \infty)$. A function $d : X \times X \to [0, \infty)$ is called an extended (α, c)-interpolative metric space if for all $\varkappa, y, \mathfrak{z} \in X$ it satisfies*

1. $d(\varkappa, y) = 0$ iff $\varkappa = y$;
2. $d(\varkappa, y) = d(y, \varkappa)$;

3. there exists an $\alpha \in (0,1)$ and $c \geqslant 0$ such that

$$d(\varkappa, y) \leqslant \theta(\varkappa, y)[d(\varkappa, \mathfrak{z}) + d(\mathfrak{z}, y)] + c\left[(d(\varkappa, \mathfrak{z}))^{\alpha}(d(\mathfrak{z}, y))^{1-\alpha}\right].$$

Let (X, d) be an extended (α, c)-interpolative metric space. A sequence $\{\varkappa_n\}$ in X is said to be:

- "**Cauchy**" if the distance $d(\varkappa_n, \varkappa_m)$ between two terms of the sequence approaches zero as $n, m \to \infty$.
- "**convergent**" if there exists an element $\varkappa \in X$ such that the distance $d(\varkappa_n, \varkappa)$ approaches zero as $n \to \infty$, denoted by $\lim_{n\to\infty} \varkappa_n = \varkappa$.

Furthermore, the extended (α, c)-interpolative metric space (X, d) is said to be "**complete**" if every Cauchy sequence in the space converges to a point within X. These concepts highlight the fundamental characteristics of sequences and their relationship to the completeness of the space.

Example 2.2. *Let $X = [0, 1)$. Define $d : X \times X \to [0, \infty)$ and $\theta : X \times X \to [1, \infty)$ as follows:*

$$d(\varkappa, y) = |\varkappa - y|^4; \forall \varkappa, y \in X,$$

and $\theta(\varkappa, y) = 1 + |\varkappa - y|, \forall \varkappa, y \in X$. Then d is an extended $(1/3, 14)$-interpolative metric space.

Example 2.3. *Let $X = \mathbb{R}$, define $d : X \times X \to [0, \infty)$ and $\theta : X \times X \to [1, \infty)$ as follows:*

$$d(\varkappa, y) = \begin{cases} |\varkappa - y|; \text{ if } |\varkappa - y| \leq 1, \\ |\varkappa - y|^2; \text{ if } |\varkappa - y| > 1. \end{cases} \quad \text{and } \theta(\varkappa, y) = 1 + |\varkappa - y|, \forall \varkappa, y \in X.$$

Then d is extended $(1/2, 1)$-interpolative metric space.

Example 2.4. *Let X be the space of all sequences of numbers. Let $\varkappa = \{\varkappa_i\}_{i\geq 1}$, and $y = \{y_i\}_{i\geq 1}$ be elements of X. Define $d : X \times X \to [0, \infty)$ and $\theta : X \times X \to [1, \infty)$ as follows:*

$$d(\varkappa, y) = \sum_{i=1}^{\infty} \frac{|\varkappa_i - y_i|^2}{1 + |\varkappa_i - y_i|_1^2}; \quad \text{and} \quad \theta(\varkappa, y) = 1 + \sup_{i\geqslant 1} \frac{|\varkappa_i - y_i|^2}{1 + |\varkappa_i - y_i|^2}$$

such that $|\varkappa_i - y_i|$ in bounded, so $\frac{|\varkappa_i - y_i|^2}{1+|\varkappa_i - y_i{}^2} \leqslant 1$ ensuring convergence of the sum.
It is immediate that $d(\varkappa, y) \geq 0$, $d(\varkappa, y) = 0 \Leftrightarrow \varkappa = y$ and that $d(\varkappa, y) = d(y, \varkappa)$.
For the generalized triangle inequality, we have

$$\begin{aligned} |\varkappa_i - y_i| &= |\varkappa_i - \mathfrak{z}_i + \mathfrak{z}_i - y_i| \\ &\leq |\varkappa_i - \mathfrak{z}_i| + |\mathfrak{z}_i - y_i|. \end{aligned}$$

We can write it as

$$|\varkappa_i - y_i|^2 \leqslant |\varkappa_i - \mathfrak{z}_i|^2 + |\mathfrak{z}_i - y_i|^2 + 2|\varkappa_i - \mathfrak{z}_i||\mathfrak{z}_i - y_i|.$$

Thus,

$$\begin{aligned} &\frac{|\varkappa_i - y_i|^2}{1 + |\varkappa_i - y_i|^2} \\ &\leqslant \frac{|\varkappa_i - \mathfrak{z}_i|^2}{1 + |\varkappa_i - \mathfrak{z}_i|^2} + \frac{|y_i - \mathfrak{z}_i|^2}{1 + |y_i - \mathfrak{z}_i|^2} + \frac{2|\varkappa_i - \mathfrak{z}_i||\mathfrak{z}_i - y_i|}{\left(1 + |\varkappa_i - \mathfrak{z}_i|^2\right)\left(1 + |\mathfrak{z}_i - y_i|^2\right)} \end{aligned}$$

$$\leqslant \left(1+\sup_{i\geqslant 1}\frac{|\varkappa_i-y_i|^2}{1+|\varkappa_i-y_i|^2}\right)\left(\frac{|\varkappa_i-\mathfrak{z}_i|^2}{1+|\varkappa_i-\mathfrak{z}_i|^2}+\frac{|\mathfrak{z}_i-y_i|^2}{1+|\ \mathfrak{z}_i-y_i)^2}\right)+\frac{2|\varkappa_i-\mathfrak{z}_i||\mathfrak{z}_i-y_i|}{\left(1+|\varkappa_i-\mathfrak{z}_i|^2\right)\left(1+|\mathfrak{z}_i-y_i|^2\right)}$$

$$\leqslant \left(1+\sup_{i\geqslant 1}\frac{|\varkappa_i-y_i|^2}{1+|\varkappa_i-y_i|^2}\right)\left(\frac{|\varkappa_i-\mathfrak{z}_i|^2}{1+|\varkappa_i-\mathfrak{z}_i|^2}+\frac{|\mathfrak{z}_i-y_i|^2}{1+|\mathfrak{z}_i-y_i|^2}\right)+\frac{2|\varkappa_i-\mathfrak{z}_i||\mathfrak{z}_i-y_i|}{\sqrt{\left(1+|\varkappa_i-\mathfrak{z}_i|^2\right)(1+|\ \mathfrak{z}_i-y_i|^2)}}$$

$$=\left(1+\sup_{i\geqslant 1}\frac{|\varkappa_i-y_i|^2}{1+|\varkappa_i-y_i|^2}\right)\left(\frac{|\varkappa_i-\mathfrak{z}_i|^2}{1+|\varkappa_i-\mathfrak{z}_i|^2}+\frac{|\mathfrak{z}_i-y_i|^2}{1+|\mathfrak{z}_i-y_i|^2}\right)+2\left(\frac{|\varkappa_i-\mathfrak{z}_i|^2}{1+|\varkappa_i-\mathfrak{z}_i|^2}\right)^{\frac{1}{2}}\left(\frac{|\mathfrak{z}_i-y_i|^2}{1+|\mathfrak{z}_i-y_i|^2}\right)^{\frac{1}{2}}$$

$$\leqslant \left(1+\sup_{i\geqslant 1}\frac{|\varkappa_i-y_i|^2}{1+|\varkappa_i-y_i|^2}\right)\left(\frac{|\varkappa_i-\mathfrak{z}_i|^2}{1+|\varkappa_i-\mathfrak{z}_i|^2}+\frac{|\mathfrak{z}_i-y_i|^2}{1+|\mathfrak{z}_i-y_i|^2}\right)+c\left(\frac{|\varkappa_i-\mathfrak{z}_i|^2}{1+|\varkappa_i-\mathfrak{z}_i|^2}\right)^{\alpha}\left(\frac{|\mathfrak{z}_i-y_i|^2}{1+|\mathfrak{z}_i-y_i|^2}\right)^{1-\alpha}$$

for $c>2$ and $\alpha=\frac{1}{2}$.
Finally we can conclude that

$$\sum_{i=1}^{\infty}\frac{|\varkappa_i-y_i|^2}{1+|\varkappa_i-y_i|^2}\leq\left(1+\sup_{i\geqslant 1}\frac{|\varkappa_i-y_i|^2}{1+|\varkappa_i-y_i|^2}\right)\left(\sum_{i=1}^{\infty}\frac{|\varkappa_i-\mathfrak{z}_i|^2}{1+|\varkappa_i-\mathfrak{z}_i|^2}+\sum_{i=1}^{\infty}\frac{|\mathfrak{z}_i-y_i|^2}{1+|\mathfrak{z}_i-y_i|^2}\right)$$
$$+c\left(\sum_{i=1}^{\infty}\frac{|\varkappa_i-\mathfrak{z}_i|^2}{1+|\varkappa_i-\mathfrak{z}_i|^2}\right)^{\alpha}\left(\sum_{i=1}^{\infty}\frac{|\mathfrak{z}_i-y_i|^2}{1+|\mathfrak{z}_i-y_i|^2}\right)^{1-\alpha},$$

i.e., $d(\varkappa,y)\leqslant\theta(\varkappa,y)(d(\varkappa,\mathfrak{z})+d(\mathfrak{z},y))+c(d(\varkappa,\mathfrak{z}))^{\alpha}(d(\mathfrak{z},y))^{1-\alpha}$.
Thus, (X,d) is an extended (α,c)-interpolative metric space for $c>2$ and $\alpha=\frac{1}{2}\in(0,1)$.

Remark 2.5. *Note that, in general, an extended (α,c)-interpolation metric is not a continuous functional.*

Lemma 2.1. Let (X,d) be an extended (α,c)-interpolation metric. If d is continuous, then every convergent sequence has a unique limit.

Theorem 2.6. Let (X,d) be a complete extended (α,c)-interpolative metric space such that d is a continuous function. Let $T:X\to X$ satisfying

$$d(T\varkappa,Ty)\leq kd(\varkappa,y),\quad\text{for all } \varkappa,y\in X, \tag{1}$$

where $k\in[0,1)$ be such that for each $\varkappa_0\in X,\lim_{n,m\to\infty}\theta(\varkappa_n,\varkappa_m)<\frac{1}{k}$, here $\varkappa_n=T^n\varkappa_0, n=1,2,\dots$ Then T has precisely one fixed point ε.

Proof: To start the proof, let us take an arbitrary point $\varkappa\in X$ and use it to create a sequence. We assume that $\varkappa_0=x$ is the first element of the chosen sequence. The following is the definition of the structure of the created sequence $\{\varkappa_n\}$: $\varkappa_{n+1}=T\varkappa_n$, $\forall\ n\in\mathbb{N}$. Before going any further with the proof, we inspect and remove the trivial case: It $\varkappa_{n_0}=\varkappa_{n_0+1}$ for any $n_0\in\mathbb{N}$ then $\varkappa_{n_0}=\varkappa_{n_0+1}=T\varkappa_{n_0}$. In other words, $\varkappa_{n_0}$ constitutes the requisite fixed point of the specified mapping T, so concluding the proof.

For all $n\in\mathbb{N}$, we will therefore assume that $\varkappa_n\neq\varkappa_{n+1}$ throughout the proof. Finally, we notice that $d(\varkappa_n,\varkappa_{n+1})>0$ for all $n\in\mathbb{N}$. Because of the presumption (1) of the Theorem, we figured out that $d(\varkappa_n,\varkappa_{n+1})\leq kd(\varkappa_{n-1},\varkappa_n)$, for all $n\in\mathbb{N}$.
Taking into account the above inequality, by repetition, we get that

$$d(\varkappa_n,\varkappa_{n+1})\leq k^n d(\varkappa_0,\varkappa_1),\quad\forall\ n\in\mathbb{N}. \tag{2}$$

By the generalized triangle inequality and (2), for $m>n$, we get:

$d(\varkappa_n,\varkappa_m)$

$$
\begin{aligned}
&\leqslant \theta(\varkappa_n,\varkappa_m)\left[d(\varkappa_n,\varkappa_{n+1})+d(\varkappa_{n+1},\varkappa_m)\right]+c\left[(d(\varkappa_n,\varkappa_{n+1}))^{\alpha}(d(\varkappa_{n+1},\varkappa_m))^{1-\alpha}\right]\\
&\leqslant \theta(\varkappa_n,\varkappa_m)d(\varkappa_n,\varkappa_{n+1})+\theta(\varkappa_n,\varkappa_m)d(\varkappa_{n+1},\varkappa_m)+c\left[(d(\varkappa_n,\varkappa_{n+1}))^{\alpha}(d(\varkappa_{n+1},\varkappa_m))^{1-\alpha}\right]\\
&\leqslant \theta(\varkappa_n,\varkappa_m)d(\varkappa_n,\varkappa_{n+1})+\theta(\varkappa_n,\varkappa_m)\Bigg[\theta(\varkappa_{n+1},\varkappa_m)\left(d(\varkappa_{n+1},\varkappa_{n+2})+d(\varkappa_{n+2},\varkappa_m)\right)\\
&\quad+c\left((d(\varkappa_{n+1},\varkappa_{n+2}))^{\alpha}(d(\varkappa_{n+2},\varkappa_m))^{1-\alpha}\right)\Bigg]+c\left[(d(\varkappa_n,\varkappa_{n+1}))^{\alpha}(d(\varkappa_{n+1},\varkappa_m))^{1-\alpha}\right]\\
&=\theta(\varkappa_n,\varkappa_m)d(\varkappa_n,\varkappa_{n+1})+\theta(\varkappa_n,\varkappa_m)\theta(\varkappa_{n+1},\varkappa_m)d(\varkappa_{n+1},\varkappa_{n+2})\\
&\quad+\theta(\varkappa_n,\varkappa_m)\theta(\varkappa_{n+1},\varkappa_m)d(\varkappa_{n+2},\varkappa_m)\\
&\quad+\theta(\varkappa_n,\varkappa_m)c\left[(d(\varkappa_{n+1},\varkappa_{n+2}))^{\alpha}(d(\varkappa_{n+2},\varkappa_m))^{1-\alpha}\right]+c\left[(d(\varkappa_n,\varkappa_{n+1}))^{\alpha}(d(\varkappa_{n+1},\varkappa_m))^{1-\alpha}\right]\\
&\leqslant \theta(\varkappa_n,\varkappa_m)d(\varkappa_n,\varkappa_{n+1})+\theta(\varkappa_n,\varkappa_m)\theta(\varkappa_{n+1},\varkappa_m)d(\varkappa_{n+1},\varkappa_{n+2})\\
&\quad+\theta(\varkappa_n,\varkappa_m)\theta(\varkappa_{n+1},\varkappa_m)\Bigg[\theta(\varkappa_{n+2},\varkappa_m)\left(d(\varkappa_{n+2},\varkappa_{n+3})+d(\varkappa_{n+3},\varkappa_m)\right)\\
&\quad+c\left((d(\varkappa_{n+2},\varkappa_{n+3}))^{\alpha}(d(\varkappa_{n+3},\varkappa_m))^{1-\alpha}\right)\Bigg]+\theta(\varkappa_n,\varkappa_m)c\left[(d(\varkappa_{n+1},\varkappa_{n+2}))^{\alpha}(d(\varkappa_{n+2},\varkappa_m))^{1-\alpha}\right]\\
&\quad+c\left[(d(\varkappa_n,\varkappa_{n+1}))^{\alpha}(d(\varkappa_{n+1},\varkappa_m))^{1-\alpha}\right]\\
&\leqslant \theta(\varkappa_n,\varkappa_m)k^n d(\varkappa_0,\varkappa_1)+\theta(\varkappa_n,\varkappa_m)\theta(\varkappa_{n+1},\varkappa_m)k^{n+1}d(\varkappa_0,\varkappa_1)\\
&\quad+\theta(\varkappa_n,\varkappa_m)\theta(\varkappa_{n+1},\varkappa_m)\theta(\varkappa_{n+2},\varkappa_m)k^{n+2}d(\varkappa_0,\varkappa_1)+\theta(\varkappa_n,\varkappa_m)\theta(\varkappa_{n+1},\varkappa_m)\theta(\varkappa_{n+2},\varkappa_m)d(\varkappa_{n+3},\varkappa_m)\\
&\quad+\theta(\varkappa_n,\varkappa_m)\theta(\varkappa_{n+1},\varkappa_m)c\left[(d(\varkappa_{n+2},\varkappa_{n+3}))^{\alpha}(d(\varkappa_{n+3},\varkappa_m))^{1-\alpha}\right]\\
&\quad+\theta(\varkappa_n,\varkappa_m)c\left[(d(\varkappa_{n+1},\varkappa_{n+2}))^{\alpha}(d(\varkappa_{n+2},\varkappa_m))^{1-\alpha}\right]+c\left[(d(\varkappa_n,\varkappa_{n+1}))^{\alpha}(d(\varkappa_{n+1},\varkappa_m))^{1-\alpha}\right].
\end{aligned}
$$

By recurrent the same method, we will obtain:

$$
\begin{aligned}
d(\varkappa_n,\varkappa_m)&\leqslant d(\varkappa_0,\varkappa_1)\sum_{n=1}^{\infty}k^n\prod_{i=1}^{n}\theta(\varkappa_i,\varkappa_m)+c\sum_{n=1}^{\infty}(d(\varkappa_n,\varkappa_{n+1}))^{\alpha}(d(\varkappa_{n+1},\varkappa_m))^{1-\alpha}\prod_{i=1}^{n-1}\theta(\varkappa_i,\varkappa_m) \qquad (3)\\
&\leqslant d(\varkappa_0,\varkappa_1)\sum_{n=1}^{\infty}k^n\prod_{i=1}^{n}\theta(\varkappa_i,\varkappa_m)+c(d(\varkappa_0,\varkappa_1))^{\alpha}\sum_{n=1}^{\infty}k^{n\alpha}(d(\varkappa_{n+1},\varkappa_m))^{1-\alpha}\prod_{i=1}^{n-1}\theta(\varkappa_i,\varkappa_m).
\end{aligned}
$$

Before continuing proceeding further, we verify that

$$(d(\varkappa_{n+1},\varkappa_m))^{1-\alpha}<1.$$

For this purpose, we are going to use elementary induction to establish it.
By generalized triangle inequality

$$d(\varkappa_n,\varkappa_{n+2})\leqslant\theta(\varkappa_n,\varkappa_{n+2})\left[d(\varkappa_n,\varkappa_{n+1})+d(\varkappa_{n+1},\varkappa_{n+2})\right]+c\left[(d(\varkappa_n,\varkappa_{n+1}))^{\alpha}(d(\varkappa_{n+1},\varkappa_{n+2}))^{1-\alpha}\right]. \qquad (4)$$

By taking limits as $n\to\infty$ in the above inequality and taking the inequality (2) into account, we conclude that

$$\lim_{n\to\infty}d(\varkappa_n,\varkappa_{n+2})=0. \qquad (5)$$

In addition, we have $\lim_{n\to\infty}d(\varkappa_n,\varkappa_{n+3})=0$, by using (5) & following the same pattern as we did to get (4).
We now assume that our statement's common case is true, which means we have:

$$\lim_{n\to\infty}d(\varkappa_n,\varkappa_{n+r})=0,\quad\text{for some}\quad r\in\mathbb{N}. \qquad (6)$$

As a consequence, due to the generalized triangle inequality, we have

$$d(\varkappa_n, \varkappa_{n+r+1}) \leqslant \theta(\varkappa_n, \varkappa_{n+r+1})\left[d(\varkappa_n, \varkappa_{n+r}) + d(\varkappa_{n+r}, \varkappa_{n+r+1})\right] + c\left[(d(\varkappa_n, \varkappa_{n+r}))^{\alpha}(d(\varkappa_{n+r}, \varkappa_{n+r+1}))^{1-\alpha}\right].$$

Taking into consideration (2) and (6), and applying $n \to \infty$, we get:

$$\lim_{n\to\infty} d(\varkappa_n, \varkappa_{n+r+1}) = 0.$$

By using this observation, we come to an end that,
$d(\varkappa_{n+1}, \varkappa_m) < 1$ and hence $(d(\varkappa_{n+1}, \varkappa_m))^{1-\alpha} < 1$.
The inequality's right-hand side (3) might be assessed in light of this fact in the following way:

$$\leqslant d(\varkappa_0, \varkappa_1) \sum_{n=1}^{\infty} k^n \prod_{i=1}^{n} \theta(\varkappa_i, \varkappa_m) + c(d(\varkappa_0, \varkappa_1))^{\alpha} \sum_{n=1}^{\infty} k^{n\alpha} \prod_{i=1}^{n-1} \theta(\varkappa_i, \varkappa_m)$$

Since, $\lim_{n,m\to\infty} \theta(\varkappa_{n+1}, \varkappa_m) k < 1$, so that the series $\sum_{n=1}^{\infty} k^n \prod_{i=1}^{n} \theta(\varkappa_i, \varkappa_m)$ converges by ratio test for each $m \in \mathbb{N}$, let

$$S = \sum_{n=1}^{\infty} k^n \prod_{i=1}^{n} \theta(\varkappa_i, \varkappa_m); \quad S_n = \sum_{j=1}^{n} k^j \prod_{i=1}^{j} \theta(\varkappa_i, \varkappa_m)$$

$$\text{similarly, } \sum_{n=1}^{\infty} k^{n\alpha} \prod_{i=1}^{n-1} \theta(\varkappa_i, \varkappa_m) \text{ also converges.}$$

In summary, $\{\varkappa_n\}$ is a Cauchy iterative sequence.
With reference to (X, d) is a complete an extended (α, c)-interpolation metric space, the sequence $\{\varkappa_n\}$ converges to $\varkappa^* \in X$. We claim that $\varkappa^*$ is the fixed point of T. Contrarily, suppose that $\varkappa^*$ is not the fixed point of T. Thus $d(\varkappa^*, T\varkappa^*) > 0$. Note that $d(\varkappa_{n+1}, T\varkappa^*) = d(T\varkappa_n, T\varkappa^*) \leqslant kd(\varkappa_n, \varkappa^*)$

Applying "$\limsup$" on each side, we get $d(\varkappa^*, T\varkappa^*) \leqslant kd(\varkappa^*, \varkappa^*) = 0$, which is a contradiction as we assumed that $d(\varkappa^*, T\varkappa^*) > 0$. Thus, $\varkappa^*$ is the fixed point of T. Moreover, (1) makes it simple to determine if the fixed point is unique.

3. Existence of Unique Solution to the Fractal-Fractional Operator

Definition 3.1. [24][25] *Let $g(\gamma)$ be a differentiable function on the interval (a, b), defined under fractional order α and fractal order β. The **Fractal-Fractional Derivative** in the Riemann–Liouville sense with a power law kernel is given by:*

$$_{\mathbf{P}}^{\mathbb{FF}-\mathbb{RL}}\mathcal{D}_{\gamma}^{\alpha,\beta} g(\gamma) = \frac{1}{\Gamma(m-\alpha)} \frac{d^{\beta}}{d\gamma^{\beta}} \int_a^{\gamma} g(\varkappa)(\gamma - \varkappa)^{m-\alpha-1}\, d\varkappa, \tag{7}$$

where:

- The **Gamma function** $\Gamma(\mathfrak{z})$ is defined as:

$$\Gamma(\mathfrak{z}) = \int_0^{\infty} t^{z-1} e^{-t}\, dt, \quad z > 0.$$

- The fractional order α satisfies $m - 1 < \alpha \leq m$, ensuring fractional differentiation.
- The fractal order β lies in the range $0 < m - 1 < \beta \leq m$, accounting for fractal characteristics.
- The operator $\frac{d^{\beta}}{d\gamma^{\beta}}$ applies the fractal differentiation rule:

$$\frac{d^{\beta} g(\gamma)}{d\gamma^{\beta}} = \lim_{\gamma\to\varkappa} \frac{g(\gamma) - g(\varkappa)}{\gamma^{\beta} - \varkappa^{\beta}}. \tag{8}$$

This derivative uniquely incorporates both *memory effects* and *scaling dynamics* characteristic of fractals, making it an essential tool for modeling nonlocal and complex systems. It captures phenomena where the interplay between fractional memory and fractal geometry is fundamental.

Definition 3.2. [24][25] *Let $g(\gamma)$ be a function differentiable on (a,b) with fractional order α and fractal order β. The **fractal-fractional derivative** in the Caputo sense with a power law kernel is defined as:*

$$ {}^{\mathbb{FF}-\mathbf{P}}_{\mathbf{P}}\mathcal{D}^{\alpha,\beta}_{\gamma} g(\gamma) = \frac{1}{\Gamma(m-\alpha)} \int_a^{\gamma} \frac{d^{\beta} g(\varkappa)}{d\varkappa^{\beta}} (\gamma-\varkappa)^{m-\alpha-1}\, d\varkappa, \tag{9} $$

where:

- $m-1 < \alpha \leq m$ is the fractional order,
- $0 < m-1 < \beta \leq m$ is the fractal order,
- The fractal differentiation operator is:

$$ \frac{d^{\beta} g(\varkappa)}{d\varkappa^{\beta}} = \lim_{\gamma \to \varkappa} \frac{g(\gamma)-g(\varkappa)}{\gamma^{\beta}-\varkappa^{\beta}}. $$

This definition extends the classical Caputo derivative by integrating fractal scaling, allowing for the modeling of systems with both memory and fractal properties.

Definition 3.3. [24][25] *The **fractal-fractional integral** of a function $g(\gamma)$ with fractional order α and fractal order β, using a power law kernel, is defined as:*

$$ {}^{\mathbb{FF}-\mathbf{P}}_{0}\mathcal{I}^{\alpha}_{\gamma} g(\gamma) = \frac{\beta}{\Gamma(\alpha)} \int_a^{\gamma} \varkappa^{\alpha-1} u(\varkappa)(\gamma-\varkappa)^{\alpha-1}\, d\varkappa, \tag{10} $$

Definition 3.4. [24][25] *The numerical solution for a nonlinear fractional-order ordinary differential equation in the Caputo sense is expressed as:*

$$ \mathbb{C}^{\alpha}_{0}\mathfrak{z}(\gamma) = f(\gamma, \mathfrak{z}(\gamma)), \quad \mathfrak{z}(0) = \mathfrak{z}_0, \tag{11} $$

where $\mathbb{C}^{\alpha}_{0}$ denotes the Caputo fractional derivative.
The corresponding numerical algorithm for solving (11) can be found in [26] for circuit based:

$$ \mathfrak{z}_{n+1} = \mathfrak{z}_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \left[\begin{array}{l} \frac{h^{\alpha} f(\gamma_j, \mathfrak{z}(\gamma_j))}{\alpha(\alpha+1)} \{(n+1-j)^{\alpha}(n+2-r+\alpha) - (n-j)^{\alpha}(n+2-r+2\alpha)\} \\ -\frac{h^{\alpha} f(\gamma_{j-1}, \mathfrak{z}(\gamma_{j-1}))}{\alpha(\alpha+1)} \left\{(n+1-j)^{\alpha+1} - (n-j)^{\alpha}(n+2-r+\alpha)\right\} \end{array} \right], \tag{12} $$

where:

- h is the step size,
- $\Gamma(\alpha)$ is the Gamma function,
- $\gamma_j = jh$ represents discrete time steps.

For fractal-fractional derivative associated results, one can refer [26], [27] and [28]. The scheme precisely approximates fractional order system dynamics, which incorporate memory effects present in fractional calculus. The system of ordinary differential equations in (13) models the dynamics of an integer-order chaotic system with three state variables: $\varkappa_1$, y_1, and $\mathfrak{z}_1$. The equations are defined as:

$$\begin{cases} \frac{d\varkappa_1}{d\gamma} & = -y_1 - \mathfrak{z}_1, \\ \frac{dy_1}{d\gamma} & = \varkappa_1 + ay_1, \\ \frac{d\mathfrak{z}_1}{d\gamma} & = b + \mathfrak{z}_1(\varkappa_1 - c). \end{cases} \tag{13}$$

The terms a, b, and c are non-negative parameters that directly control the system's chaotic dynamics. This is a nonlinear coupled system, containing linear and nonlinear interactions described by these equations. The initial conditions for this system are specified as:

$$\varkappa_1(0) = \varkappa_1^*, \quad y_1(0) = y_1^*, \quad \mathfrak{z}_1(0) = \mathfrak{z}_1^*, \tag{14}$$

here, the initial values of the state variables are $\varkappa_1^*$, y_1^*, and $\mathfrak{z}_1^*$.
A Fractal-Fractional differential operator with a power law kernel is applied to generalize the system. This transforms the integer-order chaotic system into a Fractal-Fractional order chaotic system, as defined below:

$$\begin{cases} {}^{\mathbb{FFP}}_{0}\mathcal{D}^{\alpha,\beta}_{\gamma} & = -y_1 - \mathfrak{z}_1, \\ {}^{\mathbb{FFP}}_{0}\mathcal{D}^{\alpha,\beta}_{\gamma} & = \varkappa_1 + ay_1, \\ {}^{\mathbb{FFP}}_{0}\mathcal{D}^{\alpha,\beta}_{\gamma} & = b + \mathfrak{z}_1(\varkappa_1 - c). \end{cases} \tag{15}$$

In this formulation:

- ${}^{\mathbb{FFP}}\mathcal{D}^{\alpha,\beta}_{0}$ denotes the Fractal-Fractional differential operator with a power-law kernel having fractional order α and fractal order β,
- $0 < \alpha, \beta \leq 1$, representing the orders of the fractional and fractal components.

It generalizes the dynamics by introducing memory and the fractal scaling effects into the dynamics, improving the complexity of the description of complex systems.
To establish the existence and uniqueness of the fractal-fractional model, the system described in (15) can be reformulated using the Fractal-Fractional Riemann–Liouville derivative as follows:

$$\begin{cases} {}^{\mathbb{RL}}_{0}\mathcal{D}^{\beta}_{\gamma}(\varkappa_1(\gamma)) = \beta\gamma^{\beta-1}\psi_1(\gamma, \varkappa_1), \\ {}^{\mathbb{RL}}_{0}\mathcal{D}^{\beta}_{\gamma}(y_1(\gamma)) = \beta\gamma^{\beta-1}\psi_2(\gamma, y_1), \\ {}^{\mathbb{RL}}_{0}\mathcal{D}^{\beta}_{\gamma}(\mathfrak{z}_1(\gamma)) = \beta\gamma^{\beta-1}\psi_3(\gamma, \mathfrak{z}_1). \end{cases} \tag{16}$$

where ${}^{\mathbb{RL}}_{0}\mathcal{D}^{\beta}_{\gamma}$ denotes the Riemann–Liouville fractal-fractional operator.
The functions ψ_1, ψ_2, and ψ_3 are expressed as:

$$\begin{cases} \psi_1(\gamma, \varkappa_1) & = -y_1 - \mathfrak{z}_1, \\ \psi_2(\gamma, y_1) & = \varkappa_1 + ay_1, \\ \psi_3(\gamma, \mathfrak{z}_1) & = b + \mathfrak{z}_1(\varkappa_1 - c). \end{cases} \tag{17}$$

We can rewrite the system (16) as:

$$\begin{cases} {}^{\mathbb{RL}}_{0}\mathcal{D}^{\omega}_{\gamma}\mathscr{G}(\gamma) = \beta\gamma^{\beta-1}\Phi(\gamma, \mathscr{G}(\gamma)) \\ \mathscr{G}(0) = \mathscr{G}^*. \end{cases} \tag{18}$$

where

$$\mathscr{G}(\gamma) = \begin{cases} \varkappa_1(\gamma) \\ y_1(\gamma), \\ \mathfrak{z}_1(\gamma) \end{cases} \quad \mathscr{G}(0) = \begin{cases} \varkappa_1(0) \\ y_1(0) \\ \mathfrak{z}_1(0) \end{cases}, \quad \Phi(\gamma, \mathscr{G}(0)) = \begin{cases} \Psi_1(\gamma, \varkappa_1) \\ \Psi_1(\gamma, y_1) \\ \Psi_1(\gamma, \mathfrak{z}_1). \end{cases}$$

Now, by replacing ${}_{0}^{\mathbb{RL}}\mathcal{D}_{\gamma}^{\alpha}$ with ${}_{0}^{\mathbb{C}}\mathcal{D}_{\gamma}^{\alpha}$ and applying the fractional integral, we get:

$$\mathscr{G}(\gamma) = \mathscr{G}(0) + \frac{\beta}{\Gamma(\alpha)} \int_a^{\gamma} \kappa^{\beta-1}(\gamma-\kappa)^{\alpha-1}\Phi(\kappa, \mathscr{G}(\kappa))d\kappa. \tag{19}$$

Let $P(\psi)$ be an extended (α, c) interpolation metric space of real-valued continuous functions on the interval $\psi = [0, T]$. The extended (α, c) interpolation metric on $P(\psi)$ can be defined as the supremum extended (α, C) interpolation metric:

$$d(f, g) = \sup_{\gamma \in \psi} |f(\gamma) - g(\gamma)|^2 = \|f(\gamma) - g(\gamma)\|$$

where $f, g \in P_2(\psi)$ are functions in the space, $\theta(f, g) = 1 + \sup_{\gamma\in\psi} |f(\gamma) - g(\gamma)|^2,\ c > 2$ and $\alpha = \frac{1}{2}$.
Let $X = P(\psi) \times P(\psi) \times P(\psi)$ with the norm

$$\|f(\gamma) - g(\gamma)\| = \sup_{\gamma\in\psi} \max\left\{|f_1(\gamma) - g_1(\gamma)|^2, |f_2(\gamma) - g_2(\gamma)|^2, |f_3(\gamma) - g_3(\gamma)|^2\right\}.$$

To turn the problem into a fixed-point problem, we need to express it in terms of an operator $F : X \to X$ such that a solution to the problem (19) corresponds to a fixed point of F.
Make the problem (15) a fixed point problem now. Define an operator $F : X \to X$ as

$$F(\mathscr{G}(\gamma)) = \mathscr{G}(0) + \frac{\beta}{\Gamma(\alpha)} \int_a^{\gamma} \kappa^{\beta-1}(\gamma-\kappa)^{\alpha-1}\Phi(\kappa, \mathscr{G}(\kappa))d\kappa. \tag{20}$$

Theorem 3.5. Under the following assumption the problem (15) has unique solution

1. Let $\Phi : \psi \times X \longrightarrow \mathbb{R}$ is a continuous function.
2. There is a constant $v_\phi > 0$ such that for every $\mathscr{G}, \overline{\mathscr{G}} \in X$, we have

$$|\Phi(\gamma, \mathscr{G}) - \Phi(\gamma, \overline{\mathscr{G}})| \leqslant v_\phi |\mathscr{G} - \overline{\mathscr{G}}|.$$

3.

$$E < 1, \text{ where } E = \frac{\gamma^{\alpha+\beta-1}\Gamma(\beta+1)\left[\omega\left(\nu_\phi^2 + 2\omega_\psi\right) + \omega_\phi^2\right]}{\Gamma(\alpha+\beta)}.$$

Proof: Let we define $\max_{\gamma\to\psi} |\Phi(\gamma, 0)| = \omega_\phi < \infty$.
We show that $F(R_\omega) \subset R_\omega$ where $R_\omega = \{\mathscr{G} \in X \diagup \|\mathscr{G}\| \leq \omega\}$.
For $\mathscr{G} \in R_\omega$, we have,

$$\begin{aligned}
\|F(\mathscr{G})\| \leq & \frac{\beta^2}{(\Gamma(\alpha))^2} \max_{\gamma\in\psi} \left| \int_0^{\gamma} \kappa^{\beta-1}(\gamma-\kappa)^{\alpha-1}(\Phi(\gamma, \mathscr{G}(\gamma)) - \Phi(\gamma, 0) + \Phi(\gamma, 0))d\kappa \right|^2 \\
\leqslant & \frac{\beta^2}{(\Gamma(\alpha)^2)} \max_{\gamma\in\gamma} \left\{ \left| \int_0^{\gamma} \kappa^{\beta-1}(\gamma-\kappa)^{\alpha-1}(\Phi(\gamma, \mathscr{G}(\gamma)) - \bar{\Phi}(\gamma, 0))d\kappa \right|^2 + \left| \int_0^{T} \kappa^{\beta-1}(\gamma-\kappa)^{\alpha-1}\bar{\Phi}(\gamma, 0)d\kappa \right|^2 \right. \\
& \left. + 2\int_0^{\gamma} \kappa^{\beta-1}(\gamma-\kappa)^{\alpha-1}(\Phi(\gamma, \mathscr{G}(\gamma)) - \Phi(\gamma, 0))d\kappa \int_0^{\gamma} \kappa^{\beta-1}(\gamma-\kappa)^{\alpha-1}\Phi(\gamma, 0)d\kappa \right\} \\
\leqslant & \frac{\beta^2}{(\Gamma(\alpha))^2} \max_{\gamma\in\psi} \left\{ \left| \int_0^{\gamma} \kappa^{\beta-1}(\gamma-\kappa)^{\alpha-1}(\Phi(\gamma, \mathscr{G}(\gamma)) - \Phi(\gamma, 0))d\kappa \right|^2 + \left| \int_0^{\gamma} \kappa^{\beta-1}(\gamma-\kappa)^{\alpha-1}\Phi(\gamma, 0)d\kappa \right|^2 \right. \\
& + 2\left(\int_0^{\gamma} \kappa^{\beta-1}(\gamma-\kappa)^{\alpha-1}(\Phi(\gamma, \mathscr{G}(\gamma)) - \Phi(\gamma, 0))^2 d\kappa \right) \left(\int_0^{\gamma} \kappa^{\beta-1}(\gamma-\kappa)^{\alpha-1}(\Phi(\gamma, 0))d\kappa \right) \\
= & \frac{\beta^2}{(\Gamma(\alpha))^2} \left\{ \left(\gamma^{\alpha+\beta-1}\boldsymbol{\beta}(\alpha, \beta) \right)^2 v_\phi^2 \|\mathscr{G}\| + \left(\gamma^{\alpha+\beta-1}\boldsymbol{\beta}(\alpha, \beta) \right)^2 \omega_\phi^2 \right.
\end{aligned}$$

$$
\begin{aligned}
&\quad +2\left(\gamma^{\alpha+\beta-1}\boldsymbol{\beta}(\alpha,\beta)\right)^2\|\mathscr{G}\|\omega_\phi\Big\} \\
&= \frac{\gamma^{\alpha+\beta-1}\Gamma(\beta+1)}{\Gamma(\alpha+\beta)}\left(v_\phi^2\omega + w_\phi^2 + 2\omega\omega_\phi\right) \\
&= \frac{\gamma^{\alpha+\beta-1}\Gamma(\beta+1)\left[\omega\left(v_\phi^2+2\omega_\phi\right)+\omega_\phi^2\right]}{\Gamma(\alpha+\beta)} \\
&< 1.
\end{aligned}
$$

For every $\mathscr{G},\overline{\mathscr{G}} \in X$, we have $\max_{T\in\Psi}$

$$
\begin{aligned}
&|F(\mathscr{G})-F(\overline{\mathscr{G}})|^2 \\
&= \left(\frac{\beta}{\sqrt{\alpha}}\right)^2 \max_{\gamma\in\psi}\left|\int_0^\gamma \kappa^{\beta-1}(\gamma-\kappa)^{\alpha-1}\Phi(\kappa,\mathscr{G}(\kappa))d\kappa - \int_0^\gamma \kappa^{\beta-1}(\gamma-\kappa)^{\alpha-1}\Phi(\kappa,\overline{\mathscr{G}}(\kappa))d\kappa\right|^2 \\
&\leqslant \left(\frac{\beta}{\sqrt{\alpha}}\right)^2 \max_{\gamma\in\psi}\left\{\left|\int_0^\gamma \kappa^{\beta-1}(\gamma-\kappa)^{\alpha-1}\Phi(\kappa,\mathscr{G}(\kappa))d\kappa\right|^2 + \left|\int_0^\gamma \kappa^{\beta-1}(\gamma-\kappa)^{\alpha-1}\Phi(\kappa,\overline{\mathscr{G}}(\kappa))d\kappa\right|^2\right. \\
&\quad \left. -\,2\left|\int_0^\gamma \kappa^{\beta-1}(\gamma-\kappa)^{\alpha-1}\Phi(\kappa,\mathscr{G}(\kappa))d\kappa\right|\left|\int_0^\gamma \kappa^{\beta-1}(\gamma-\kappa)^{\alpha-1}\Phi(\kappa,\overline{\mathscr{G}}(\kappa))d\kappa\right|\right. \\
&\leqslant \left(\frac{\beta}{\sqrt{\alpha}}\right)^2\left(\gamma^{\alpha+\beta-1}\boldsymbol{\beta}(\alpha,\beta)\right)^2 \max_{\gamma\in\psi}|\Phi(\kappa,\mathscr{G}(\kappa))-\Phi(\kappa,\overline{\mathscr{G}}(\kappa))|^2 \\
&\leqslant \left(\frac{\beta}{\sqrt{\alpha}}\right)^2\left(\gamma^{\alpha+\beta-1}\boldsymbol{\beta}(\alpha,\beta)\right)^2 v_\phi^2 \max_{\gamma\in\psi}|\mathscr{G}-\overline{\mathscr{G}}|^2 \\
&\leq \left(\gamma^{\alpha+\beta-1}\sqrt{\beta+1}\right)^2 v_\phi^2\|\mathscr{G}-\overline{\mathscr{G}}\|^2 \\
&= \eta\|\mathscr{G}-\overline{\mathscr{G}}\|^2,
\end{aligned}
$$

where $\eta = \left(\gamma^{\alpha+\beta-1}\sqrt{\beta+1}\right)^2 v_\phi^2 < 1$.
Thus, all the conditions of the Theorem.2.6 satisfied and hence the problem (15) will have unique solution.

3.1. Rössler Attractor-based Numerical Solution

We examine the Rössler chaotic system that has been introduced in [29], [30].

$$
\begin{cases}
\dfrac{d\varkappa_1}{d\kappa} = -y_1 - \mathfrak{z}_1, \\
\dfrac{dy_1}{d\kappa} = \varkappa_1 + ay_1, \\
\dfrac{d\mathfrak{z}_1}{d\kappa} = b + \mathfrak{z}_1(\varkappa_1 - c).
\end{cases} \tag{21}
$$

The following will be obtained by using fractal-fractional derivatives:

$$
\begin{cases}
\mathscr{D}_\kappa^{\omega,\mu}\varkappa_1(\kappa) = -y_1 - \mathfrak{z}_1, \\
\mathscr{D}_\kappa^{\omega,\mu}y_1(\kappa) = \varkappa_1 + ay_1, \\
\mathscr{D}_\kappa^{\omega,\mu}\mathfrak{z}_1(\kappa) = b + \mathfrak{z}_1(\varkappa_1 - c).
\end{cases} \tag{22}
$$

Using the connection that exists between fractal and traditional derivatives, we obtain

$$
\frac{df(\kappa)}{d\kappa^\omega} = \lim_{\kappa_1\to\kappa}\frac{f(\kappa_1)-f(\kappa)}{\kappa_1^\omega-\kappa^\omega},\ \omega > 0
$$

$$
\begin{aligned}
&= \lim_{\kappa_1 \to \kappa} \frac{f(\kappa_1) - f(\kappa)}{\kappa_1 - \kappa} \frac{\kappa_1 - \kappa}{\kappa_1^{\omega} - \kappa^{\omega}} \\
&= \lim_{\kappa_1 \to \kappa} \frac{\frac{f(\kappa_1) - f(\kappa)}{\kappa_1 - \kappa}}{\frac{\kappa_1^{\omega} - \kappa^{\omega}}{\kappa_1 - \kappa}} \\
&= \frac{f'(\kappa)}{\lim_{\kappa_1 \to \kappa} \frac{\kappa_1^{\omega} - \kappa^{\omega}}{\kappa_1 - \kappa}} \\
&= \frac{f'(\kappa)}{\lim_{\kappa_1 \to \kappa} \omega \kappa_1^{\omega - 1}} \\
&= \frac{1}{\omega \kappa^{\omega - 1}} f'(\kappa)
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{D}_{\kappa}^{\omega,\mu} \varkappa_1(\kappa) &= \frac{1}{\Gamma(1-\omega)} \frac{d}{d\kappa^{\mu}} \int_0^{\kappa} (\kappa - \rho)^{-\omega} \varkappa_1(\rho) d\rho \\
&= \frac{1}{\Gamma(1-\omega)} \frac{1}{\mu \kappa^{\mu - 1}} \frac{d}{d\kappa} \int_0^{\kappa} (\kappa - \rho)^{-\omega} \varkappa_1(\rho) d\rho \\
&= \frac{1}{\mu \kappa^{\mu - 1}} \, {}_0^{\mathbb{RL}}\mathscr{D}_{\kappa}^{\omega} \varkappa_1(\kappa).
\end{aligned}
$$

So,

$$
\mathscr{D}_{\kappa}^{\omega,\mu} y_1(\kappa) = \frac{1}{\mu \kappa^{\mu - 1}} \, {}_0^{\mathbb{RL}}\mathscr{D}_{\kappa}^{\omega} y_1(\kappa).
$$

$$
\mathscr{D}_{\kappa}^{\omega,\mu} \mathfrak{z}_1(\kappa) = \frac{1}{\mu \kappa^{\mu - 1}} \, {}_0^{\mathbb{RL}}\mathscr{D}_{\kappa}^{\omega} \mathfrak{z}_1(\kappa).
$$

Therefore, we obtain:

$$
\begin{cases}
{}_0^{\mathbb{RL}}\mathscr{D}_{\kappa}^{\omega} \varkappa_1(\kappa) = \mu \kappa^{\mu - 1}[-y_1 - \mathfrak{z}_1], \\
{}_0^{\mathbb{RL}}\mathscr{D}_{\kappa}^{\omega} y_1(\kappa) = \mu \kappa^{\mu - 1}[\varkappa_1 + a y_1], \\
{}_0^{\mathbb{RL}}\mathscr{D}_{\kappa}^{\omega} \mathfrak{z}_1(\kappa) = \mu \kappa^{\mu - 1}[b + \mathfrak{z}_1(\varkappa_1 - c)].
\end{cases} \tag{23}
$$

For simplification,

$$
\begin{cases}
\mathscr{A}(\kappa, \varkappa_1, y_1, \mathfrak{z}_1) = \mu \kappa^{\mu - 1}[-y_1 - \mathfrak{z}_1], \\
\mathscr{B}(\kappa, \varkappa_1, y_1, \mathfrak{z}_1) = \mu \kappa^{\mu - 1}[\varkappa_1 + a y_1], \\
\mathscr{C}(\kappa, \varkappa_1, y_1, \mathfrak{z}_1) = \mu \kappa^{\mu - 1}[b + \mathfrak{z}_1(\varkappa_1 - c)].
\end{cases} \tag{24}
$$

Then we get,

$$
\begin{cases}
{}_0^{\mathbb{RL}}\mathscr{D}_{\kappa}^{\omega} \varkappa_1(\kappa) = \mathscr{A}(\kappa, \varkappa_1, y_1, \mathfrak{z}_1), \\
{}_0^{\mathbb{RL}}\mathscr{D}_{\kappa}^{\omega} y_1(\kappa) = \mathscr{B}(\kappa, \varkappa_1, y_1, \mathfrak{z}_1), \\
{}_0^{\mathbb{RL}}\mathscr{D}_{\kappa}^{\omega} \mathfrak{z}_1(\kappa) = \mathscr{C}(\kappa, \varkappa_1, y_1, \mathfrak{z}_1).
\end{cases} \tag{25}
$$

When the Riemann-Liouville integral is applied to both equation sides, the following results are obtained:

$$\begin{cases} \varkappa_1(\kappa) - \varkappa_1(0) = \dfrac{1}{\Gamma(\omega)} \displaystyle\int_0^{\kappa} (\kappa-\rho)^{\omega-1} \mathscr{A}(\rho, \varkappa_1, y_1, \mathfrak{z}_1) d\rho, \\ y_1(\kappa) - y_1(0) = \dfrac{1}{\Gamma(\omega)} \displaystyle\int_0^{\kappa} (\kappa-\rho)^{\omega-1} \mathscr{B}(\rho, \varkappa_1, y_1, \mathfrak{z}_1) d\rho, \\ \mathfrak{z}_1(\kappa) - \mathfrak{z}_1(0) = \dfrac{1}{\Gamma(\omega)} \displaystyle\int_0^{\kappa} (\kappa-\rho)^{\omega-1} \mathscr{C}(\rho, \varkappa_1, y_1, \mathfrak{z}_1) d\rho. \end{cases} \tag{26}$$

Then we discritize at κ_{n+1}:

$$\begin{cases} \varkappa_1(\kappa_{n+1}) - \varkappa_1(0) = \dfrac{1}{\Gamma(\omega)} \displaystyle\int_0^{\kappa_{n+1}} (\kappa_{n+1}-\rho)^{\omega-1} \mathscr{A}(\rho, \varkappa_1, y_1, \mathfrak{z}_1) d\rho, \\ y_1(\kappa_{n+1}) - y_1(0) = \dfrac{1}{\Gamma(\omega)} \displaystyle\int_0^{\kappa_{n+1}} (\kappa_{n+1}-\rho)^{\omega-1} \mathscr{B}(\rho, \varkappa_1, y_1, \mathfrak{z}_1) d\rho, \\ \mathfrak{z}_1(\kappa_{n+1}) - \mathfrak{z}_1(0) = \dfrac{1}{\Gamma(\omega)} \displaystyle\int_0^{\kappa_{n+1}} (\kappa_{n+1}-\rho)^{\omega-1} \mathscr{C}(\rho, \varkappa_1, y_1, \mathfrak{z}_1) d\rho. \end{cases} \tag{27}$$

Thus, we get,

$$\begin{cases} \varkappa_1(\kappa_{n+1}) - \varkappa_1(0) = \dfrac{1}{\Gamma(\omega)} \displaystyle\sum_{r=0}^{n} \int_{\kappa_r}^{\kappa_{r+1}} (\kappa_{r+1}-\rho)^{\omega-1} \mathscr{A}(\rho, \varkappa_1, y_1, \mathfrak{z}_1) d\rho, \\ y_1(\kappa_{n+1}) - y_1(0) = \dfrac{1}{\Gamma(\omega)} \displaystyle\sum_{r=0}^{n} \int_{\kappa_r}^{\kappa_{r+1}} (\kappa_{r+1}-\rho)^{\omega-1} \mathscr{B}(\rho, \varkappa_1, y_1, \mathfrak{z}_1) d\rho, \\ \mathfrak{z}_1(\kappa_{n+1}) - \mathfrak{z}_1(0) = \dfrac{1}{\Gamma(\omega)} \displaystyle\sum_{r=0}^{n} \int_{\kappa_r}^{\kappa_{r+1}} (\kappa_{r+1}-\rho)^{\omega-1} \mathscr{C}(\rho, \varkappa_1, y_1, \mathfrak{z}_1) d\rho. \end{cases} \tag{28}$$

After applying two-step Lagrange interpolation, we now obtain:

$$P_r(\rho) \approx \frac{f(\kappa_r, y_{1_r})}{m}(\rho - \kappa_{r-1}) - \frac{f(\kappa_{r-1}, y_{1_{r-1}})}{m}(\rho - \kappa_r)$$

$$\begin{cases} \varkappa_1(\kappa_{n+1}) - \varkappa_1(0) = \dfrac{1}{\Gamma(\omega)} \displaystyle\sum_{r=0}^{n} \int_{\kappa_r}^{\kappa_{r+1}} (\kappa_{r+1} - \rho)^{\omega-1} \\ \qquad \left[\dfrac{\mathscr{A}(\kappa_r, \varkappa_{1_r}, y_{1_r}, \mathfrak{z}_{1_r})}{m} (\rho - \kappa_{r-1}) - \dfrac{\mathscr{A}(\kappa_{r-1}, \varkappa_{1_{r-1}}, y_{1_{r-1}}, \mathfrak{z}_{1_{r-1}})}{m} (\rho - \kappa_r) \right] d\rho, \\ y_1(\kappa_{n+1}) - y_1(0) = \dfrac{1}{\Gamma(\omega)} \displaystyle\sum_{r=0}^{n} \int_{\kappa_r}^{\kappa_{r+1}} (\kappa_{r+1} - \rho)^{\omega-1} \\ \qquad \left[\dfrac{\mathscr{B}(\kappa_r, \varkappa_{1_r}, y_{1_r}, \mathfrak{z}_{1_r})}{m} (\rho - \kappa_{r-1}) - \dfrac{\mathscr{B}(\kappa_{r-1}, \varkappa_{1_{r-1}}, y_{1_{r-1}}, \mathfrak{z}_{1_{r-1}})}{m} (\rho - \kappa_r) \right] d\rho, \\ \mathfrak{z}_1(\kappa_{n+1}) - \mathfrak{z}_1(0) = \dfrac{1}{\Gamma(\omega)} \displaystyle\sum_{r=0}^{n} \int_{\kappa_r}^{\kappa_{r+1}} (\kappa_{r+1} - \rho)^{\omega-1} \\ \qquad \left[\dfrac{\mathscr{C}(\kappa_r, \varkappa_{1_r}, y_{1_r}, \mathfrak{z}_{1_r})}{m} (\rho - \kappa_{r-1}) - \dfrac{\mathscr{C}(\kappa_{r-1}, \varkappa_{1_{r-1}}, y_{1_{r-1}}, \mathfrak{z}_{1_{r-1}})}{m} (\rho - \kappa_r) \right] d\rho. \end{cases} \tag{29}$$

Therefore,

$$\begin{aligned} &\varkappa_1(\kappa_{n+1}) - \varkappa_1(0) \\ &= \frac{1}{\Gamma(\omega)} \Bigg[\sum_{r=0}^{n} \frac{\mathscr{A}(\kappa_r, \varkappa_{1_r}, y_{1_r}, \mathfrak{z}_{1_r})}{m} \int_{\kappa_r}^{\kappa_{r+1}} (\kappa_{n+1} - \rho)^{\omega-1} (\rho - \kappa_{r-1}) d\rho \\ &\quad - \sum_{r=0}^{n} \frac{\mathscr{A}(\kappa_{r-1}, \varkappa_{1_{r-1}}, y_{1_{r-1}}, \mathfrak{z}_{1_{r-1}})}{m} \int_{\kappa_r}^{\kappa_{r+1}} (\kappa_{n+1} - \rho)^{\omega-1} (\rho - \kappa_r) d\rho \Bigg] \\ &= \frac{1}{\Gamma(\omega)} \Bigg[\sum_{r=0}^{n} \frac{\mathscr{A}(\kappa_r, \varkappa_{1_r}, y_{1_r}, \mathfrak{z}_{1_r})}{m} \Bigg[\frac{(-\kappa_r + \kappa_{n+1})^{\omega} (-(1+\omega)\kappa_{r-1} + \omega\kappa_r + \kappa_{n+1})}{\omega(\omega+1)} \\ &\qquad - \frac{(-\kappa_{r+1} + \kappa_{n+1})^{\omega} (-(1+\omega)\kappa_{r-1} + \omega\kappa_{r+1} + \kappa_{n+1})}{\omega(\omega+1)} \Bigg] \\ &\quad - \sum_{r=0}^{n} \frac{\mathscr{A}(\kappa_{r-1}, \varkappa_{1_{r-1}}, y_{1_{r-1}}, \mathfrak{z}_{1_{r-1}})}{m} \Bigg[\frac{(-\kappa_r + \kappa_{n+1})^{\omega+1}}{\omega(\omega+1)} - \frac{(-\kappa_{r+1} + \kappa_{n+1})^{\omega} \times (-(1+\omega)\kappa_{r-1} + \omega\kappa_{r+1} + \kappa_{n+1})}{\omega(\omega+1)} \Bigg] \Bigg]. \end{aligned}$$

Similarly,

$$\begin{aligned} &y_1(\kappa_{n+1}) - y_1(0) \\ &= \frac{1}{\Gamma(\omega)} \Bigg[\sum_{r=0}^{n} \frac{\mathscr{B}(\kappa_r, \varkappa_{1_r}, y_{1_r}, \mathfrak{z}_{1_r})}{m} \int_{\kappa_r}^{\kappa_{r+1}} (\kappa_{n+1} - \rho)^{\omega-1} (\rho - \kappa_{r-1}) d\rho \\ &\quad - \sum_{r=0}^{n} \frac{\mathscr{B}(\kappa_{r-1}, \varkappa_{1_{r-1}}, y_{1_{r-1}}, \mathfrak{z}_{1_{r-1}})}{m} \int_{\kappa_r}^{\kappa_{r+1}} (\kappa_{n+1} - \rho)^{\omega-1} (\rho - \kappa_r) d\rho \Bigg] \\ &= \frac{1}{\Gamma(\omega)} \Bigg[\sum_{r=0}^{n} \frac{\mathscr{B}(\kappa_r, \varkappa_{1_r}, y_{1_r}, \mathfrak{z}_{1_r})}{m} \Bigg[\frac{(-\kappa_r + \kappa_{n+1})^{\omega} (-(1+\omega)\kappa_{r-1} + \omega\kappa_r + \kappa_{n+1})}{\omega(\omega+1)} \\ &\qquad - \frac{(-\kappa_{r+1} + \kappa_{n+1})^{\omega} (-(1+\omega)\kappa_{r-1} + \omega\kappa_{r+1} + \kappa_{n+1})}{\omega(\omega+1)} \Bigg] \end{aligned}$$

$$
-\sum_{r=0}^{n}\frac{\mathscr{B}(\kappa_{r-1},\varkappa_{1_{r-1}},y_{1_{r-1}},\mathfrak{z}_{1_{r-1}})}{m}\left[\frac{(-\kappa_r+\kappa_{n+1})^{\omega+1}}{\omega(\omega+1)}-\frac{\begin{array}{c}(-\kappa_{r+1}+\kappa_{n+1})^{\omega}\\ \times(-(1+\omega)\kappa_{r-1}+\omega\kappa_{r+1}+\kappa_{n+1})\end{array}}{\omega(\omega+1)}\right]\Bigg].
$$

$$
\begin{aligned}
&\mathfrak{z}_1(\kappa_{n+1})-\mathfrak{z}_1(0)\\
&=\frac{1}{\Gamma(\omega)}\Bigg[\sum_{r=0}^{n}\frac{\mathscr{C}(\kappa_r,\varkappa_{1_r},y_{1_r},\mathfrak{z}_{1_r})}{m}\int_{\kappa_r}^{\kappa_{r+1}}(\kappa_{n+1}-\rho)^{\omega-1}(\rho-\kappa_{r-1})d\rho\\
&\quad-\sum_{r=0}^{n}\frac{\mathscr{C}(\kappa_{r-1},\varkappa_{1_{r-1}},y_{1_{r-1}},\mathfrak{z}_{1_{r-1}})}{m}\int_{\kappa_r}^{\kappa_{r+1}}(\kappa_{n+1}-\rho)^{\omega-1}(\rho-\kappa_r)d\rho\Bigg]\\
&=\frac{1}{\Gamma(\omega)}\Bigg[\sum_{r=0}^{n}\frac{\mathscr{C}(\kappa_r,\varkappa_{1_r},y_{1_r},\mathfrak{z}_{1_r})}{m}\Bigg[\frac{(-\kappa_r+\kappa_{n+1})^{\omega}(-(1+\omega)\kappa_{r-1}+\omega\kappa_r+\kappa_{n+1})}{\omega(\omega+1)}\\
&\qquad-\frac{(-\kappa_{r+1}+\kappa_{n+1})^{\omega}(-(1+\omega)\kappa_{r-1}+\omega\kappa_{r+1}+\kappa_{n+1})}{\omega(\omega+1)}\Bigg]\\
&\quad-\sum_{r=0}^{n}\frac{\mathscr{C}(\kappa_{r-1},\varkappa_{1_{r-1}},y_{1_{r-1}},\mathfrak{z}_{1_{r-1}})}{m}\left[\frac{(-\kappa_r+\kappa_{n+1})^{\omega+1}}{\omega(\omega+1)}-\frac{\begin{array}{c}(-\kappa_{r+1}+\kappa_{n+1})^{\omega}\\ \times(-(1+\omega)\kappa_{r-1}+\omega\kappa_{r+1}+\kappa_{n+1})\end{array}}{\omega(\omega+1)}\right]\Bigg].
\end{aligned}
$$

3.2. Numerical Simulations

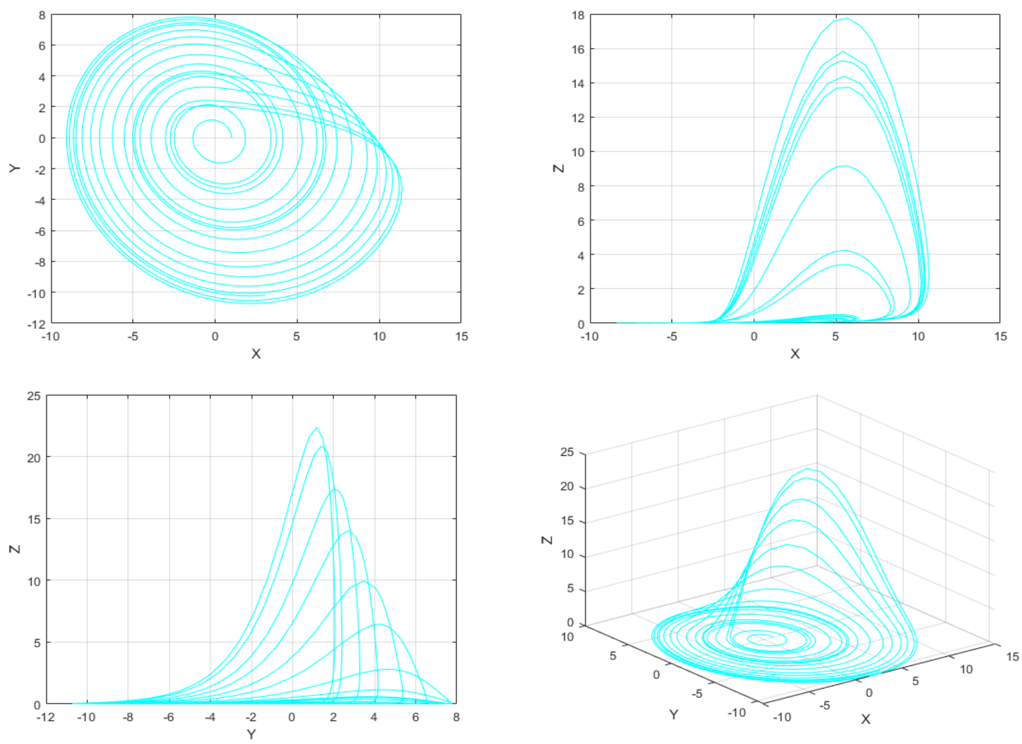

Figure 2: Dynamical behavior of the Rössler attractor for $a = 0.2$, $b = 0.2$ and $c = 5.7$; as obtained for $\omega = 0.95, \mu = 1$

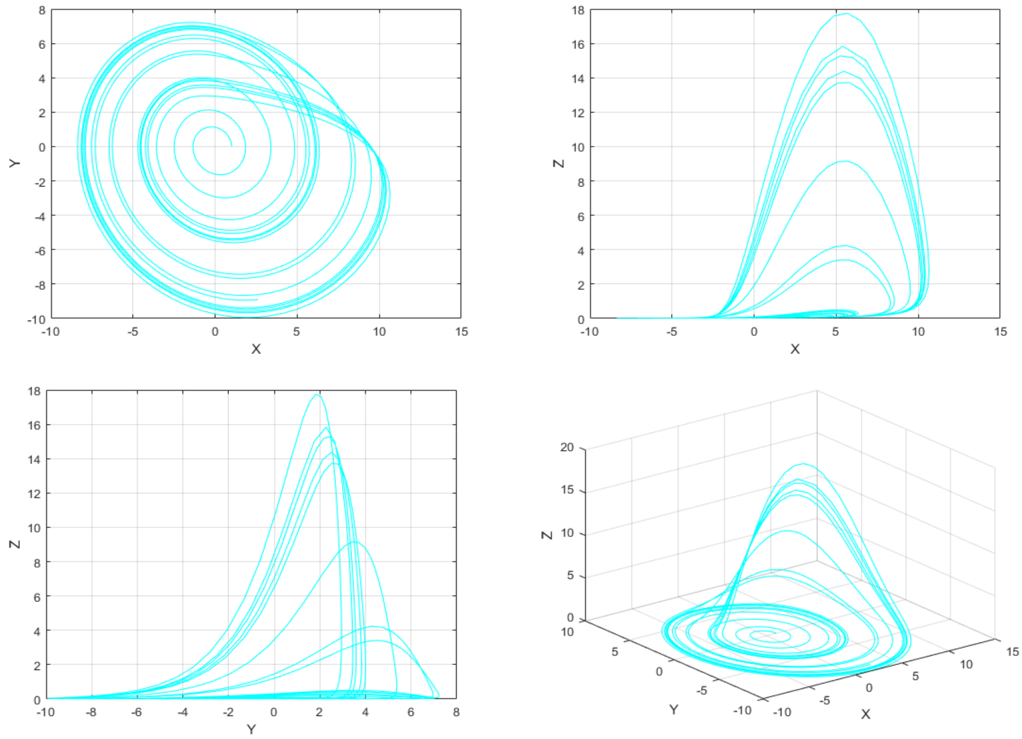

Figure 3: Dynamical behavior of the Rössler attractor for $a = 0.2$, $b = 0.2$ and $c = 5.7$; as obtained for $\omega = 0.85, \mu = 1$

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