Letters in Nonlinear Analysis and its Applications **3** (2025) No. 2, 119-125
`https://doi.org/10.66147/lnaa.20253253`
Available online at www.lettersinnonlinearanalysis.com
**Research Article**

Letters in Nonlinear Analysis and its Applications
*Peer Review Scientific Journal*
**ISSN: 2958-874x**

# Coupled fixed points via simulation functions

Josefa Caballero[a,*], Kishin Sadarangani[a], Rayco Toledo[a]

[a] *Department of Mathematics, University of Las Palmas de Gran Canaria, Campus de Tafira Baja, 35017, Las Palmas, Spain*

**Abstract**

In the present paper, we show a result in complete metric spaces about the existence and uniqueness of coupled fixed points by using simulation functions. Moreover, we illustrate our result by presenting a theorem about the existence and uniqueness of solution to a general system of nonlinear functional-integral equations.

*Keywords:* Coupled fixed point, Simulation function, Functional-integral equation
*2010 MSC:* 47H10,39B05

## 1. Introduction and preliminaries

It is well known that the Banach contraction mapping principle and its generalizations constitute a very important tool in the theory of existence of solutions to functional, differential, and integral equations. Particularly, one of these generalizations uses the so-called simulation functions.

In the sequel, we present this class of functions and the above-mentioned fixed point theorem. This material appears in [1, 2].

**Definition 1.1.** A function $\xi : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ is said to be a simulation function if it satisfies the following conditions:

---

[*] Corresponding author

*Email addresses:* `josefa.caballero@ulpgc.es` (Josefa Caballero), `kishin.sadarangani@ulpgc.es` (Kishin Sadarangani), `rayco.toledo@ulpgc.es` (Rayco Toledo)

*Received January 09,2025, Accepted: March 26, 2025, Online: March 29, 2025*

- (i) $\xi(0,0)=0$.
- (ii) $\xi(t,s)<s-t$, for any $t,\, s>0$.
- (iii) Let $(t_n)$ and $(s_n)$ be sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n=\lim_{n\to\infty} s_n>0$, then $\limsup_{n\to\infty}\xi(t_n,s_n)<0$.

In what follows, by $\mathcal{J}$ we will denote the class of simulation functions. Examples of functions belonging to $\mathcal{J}$ are the following ones:

1. $\xi_1(t,s)=\phi_1(s)-\phi_2(t)$, for any $t,\, s\in[0,\infty)$, where $\phi_1,\, \phi_2:[0,\infty)\to[0,\infty)$ are continuous functions such that
   - (a) $\phi_1(t)=\phi_2(t)=0$ if and only if $t=0$,
   - (b) $\phi_1(t)<t\le\phi_2(t)$ for any $t>0$.
2. $\xi_2(t,s)=\varphi(s)-t$, for any $t,\, s\in[0,\infty)$, where $\varphi:[0,\infty)\to[0,\infty)$ is a continuous and increasing function and it satisfies that $\varphi(t)<t$ if $t>0$.

By $\mathcal{G}$ we denote the class of functions $\varphi$ satisfying the above mentioned conditions. It is clear that if $\varphi\in\mathcal{G}$ then $\varphi(0)=0$. Examples of functions in $\mathcal{G}$ are $\varphi(t)=\ln(1+t)$; $\varphi(t)=\arctan t$; $\varphi(t)=\dfrac{t}{t+1}$; $\varphi(t)=kx$ with $k\in(0,1)$.

Next, we present the result about fixed point by using simulation functions that it appears in Theorem 2.8 of [1].

**Theorem 1.2.** *Let $(X,d)$ be a complete metric space and $T:X\to X$ a mapping such that there exists $\xi\in\mathcal{J}$ satisfying*

$$\xi\big(d(T_x,T_y),d(x,y)\big)\ge 0,$$

*for any $x,y\in X$ with $x\ne y$. Then $T$ has a unique fixed point $x^*\in X$. Moreover, for any $x_0\in X$ the Picard sequence $(x_n)$, where $x_n=Tx_{n-1}$ for any $n\in\mathbb{N}$ converges to the fixed point $x^*\in X$.*

In this paper, we present a result about coupled fixed points. The concept of coupled fixed point was introduced by Guo and Lakshmikantham in [3] for the study of coupled quasi-solutions of an initial value problem for ordinary differential equations. Some papers on coupled fixed points have appeared in the literature (see [3, 4, 5, 6, 7, 8, 9, 10], among others). Moreover, as an application of our result, we study the existence and uniqueness of solutions to a coupled system of functional equations.

## 2. Main result

Suppose that $(X,d)$ is a complete metric space and $G:X\times X\to X$ a mapping.

**Definition 2.1.** An element $(x_0,y_0)\in X\times X$ is said to be a coupled fixed point of the mapping $G$ if $G(x_0,y_0)=x_0$ and $G(y_0,x_0)=y_0$.

Now, we are ready to present our main result.

**Theorem 2.2.** *Let $(X,d)$ be a complete metric space and $G:X\times X\to X$ be a mapping, such that there exists $\xi\in\mathcal{J}$ satisfying the following condition*

$$\xi\Big(d\big(G(x,y),G(u,v)\big),\, \max\big(d(x,u),d(y,v)\big)\Big)\ge 0,$$

*for any $(x,y),(u,v)\in X\times X$, with $(x,y)\ne(u,v)$. Then $G$ has a unique coupled fixed point.*

*Proof.* Consider the metric space $\left(X\times X,\tilde{d}\right)$ where

$$\tilde{d}\big((x,y),(u,v)\big)=\max\big(d(x,u),d(y,v)\big),$$

for any $(x,y),(u,v)\in X\times X$.

It is well known that $(X\times X,\tilde{d})$ is also a complete metric space.

Next, we define the following mapping $\tilde{G}:X\times X\to X\times X$ by

$$\tilde{G}(x,y)=\big(G(x,y),G(y,x)\big).$$

In what follows, we check that $\tilde{G}$ satisfies assumptions of Theorem 1.2.

In fact, for any $x,y,u,v\in X$, we have that

$$\begin{aligned}\tilde{d}\big(\tilde{G}(x,y),\tilde{G}(u,v)\big)&=\tilde{d}\Big(\big(G(x,y),G(y,x)\big),\big(G(u,v),G(v,u)\big)\Big)=\\&=\max\Big(d\big(G(x,y),G(u,v)\big),d\big(G(y,x),G(v,u)\big)\Big).\end{aligned}$$

In this point, we can consider two cases:

(a) $\tilde{d}\big(\tilde{G}(x,y),\tilde{G}(u,v)\big)=d\big(G(x,y),G(u,v)\big)$,

(b) $\tilde{d}\big(\tilde{G}(x,y),\tilde{G}(u,v)\big)=d\big(G(y,x),G(v,u)\big)$.

**Case (a).** Taking into account our assumption, we infer

$$\begin{aligned}&\xi\Big(\tilde{d}\big(\tilde{G}(x,y),\tilde{G}(u,v)\big),\tilde{d}\big((x,y),(u,v)\big)\Big)=\\&=\xi\Big(d\big(G(x,y),G(u,v)\big),\max\big(d(x,u),d(y,v)\big)\Big)\geq 0.\end{aligned}$$

This proves that the condition appearing in Theorem 1.2 is satisfied for the complete metric space $(X\times X,\tilde{d})$.

**Case (b).** By using similar argument to case (a), we get

$$\begin{aligned}&\xi\Big(\tilde{d}\big(\tilde{G}(x,y),\tilde{G}(u,v)\big),\tilde{d}\big((x,y),(u,v)\big)\Big)=\\&=\xi\Big(d\big(G(y,x),G(v,u)\big),\max\big(d(x,u),d(y,v)\big)\Big)=\\&=\xi\Big(d\big(G(y,x),G(v,u)\big),\max\big(d(y,v),d(x,u)\big)\Big)\geq 0.\end{aligned}$$

In this case, we have also proved that the condition of Theorem 1.2 holds.

Now, by Theorem 1.2, the mapping $\tilde{G}$ has a unique fixed point. This is, there exists a unique pair $(u_0,v_0)\in X\times X$ such that

$$\tilde{G}(u_0,v_0)=(u_0,v_0).$$

Taking into account the definition of $\tilde{G}$, we deduce that

$$\tilde{G}(u_0,v_0)=\big(G(u_0,v_0),G(v_0,u_0)\big)=(u_0,v_0),$$

and from this

$$G(u_0,v_0)=u_0\quad\text{and}\quad G(v_0,u_0)=v_0.$$

This is, $(u_0,v_0)\in X\times X$ is a coupled fixed point of the mapping $G$.

It is easily seen that the uniqueness of the coupled fixed point is $(u_0,v_0)$ is obtained of the uniqueness of the fixed point $(u_0,v_0)$ for $\tilde{G}$.

This completes the proof.

□

## 3. Applications

In this section, we illustrate our result studying the existence and uniqueness of solutions to the following coupled system of functional-integral equations

$$\begin{cases} x(t) = f\left(t, x(t), y(t), \int_0^t g(s, x(s), y(s))\, ds\right) \\ y(t) = f\left(t, y(t), x(t), \int_0^t g(s, y(s), x(s))\, ds\right), \end{cases} \tag{1}$$

for $t \in [0,1]$, in $C[0,1] \times C[0,1]$.

In the following theorem, we present a sufficient condition for the existence and uniqueness of solutions to Problem (1).

**Theorem 3.1.** *Suppose the following assumptions:*

(i) $f : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous functions.*

(ii) *There exists* $\varphi \in \mathcal{G}$ *such that*

$$\big|f(t,x,y,z) - f(t,u,v,w)\big| \le \varphi\big(max(|x-u|, |y-v|, |z-w|)\big),$$

*for any* $t \in [0,1]$ *and* $x, y, z, u, v, w \in \mathbb{R}$.
*(Here,* $\mathcal{G}$ *is the class of functions appearing in section 1).*

(iii) *The following inequality*

$$\big|g(t,x,y) - g(t,u,v)\big| \le \max\big(|x-u|, |y-v|\big),$$

*for any* $t \in [0,1]$ *and* $x, y, u, v \in \mathbb{R}$*, holds.*
*Then Problem (1) has a unique solution* $(x,y) \in C[0,1] \times C[0,1]$.

*Proof.* Consider the operator $F$ defined on $C[0,1] \times C[0,1]$ by

$$F(x,y)(t) = f\left(t, x(t), y(t), \int_0^t g\big(s, x(s), y(s)\big) ds\right),$$

for any $t \in [0,1]$.

From i, it follows that $F$ applies $C[0,1] \times C[0,1]$ into $C, [0,1]$.

Now, we check that $F$ satisfies the condition appearing in Theorem 2.2.

In fact, by using our assumptions, we have, for any $(x,y), (u,v) \in C[0,1] \times C[0,1]$ with $(x,y) \ne (u,v)$, that

$$\begin{aligned} d\big(F(x,y), F(u,v)\big) &= \underset{0\le t\le 1}{sup} \Big|F(x,y)(t) - F(u,v)(t)\Big| = \\ &= \underset{0\le t\le 1}{sup} \Bigg| f\left(t, x(t), y(t), \int_0^t g\left(s, x(s), y(s)\right) ds\right) \\ &\quad - f\left(t, u(t), v(t), \int_0^t g\left(s, u(s), v(s)\right) ds\right) \Bigg| \\ &\le \underset{0\le t\le 1}{sup}\ \varphi\Big(max\big(|x(t)-u(t)|, |y(t)-v(t)|\big), \Big| \int_0^t g\left(s, x(s), y(s)\right) - g\left(s, u(s), v(s)\right)\, ds\Big|\Big) \\ &\le \underset{0\le t\le 1}{sup}\ \varphi\Big(max\big(d(x,u), d(y,v), \int_0^t \big|g(s,x(s),y(s)) - g(s,u(s),v(s))\big| ds\Big) \\ &\le \underset{0\le t\le 1}{sup}\ \varphi\Big(max\big(d(x,u), d(y,v), \int_0^t \max\big(|x(s)-u(s)|, |y(s)-v(s)|\big) ds\Big) \\ &\le \underset{0\le t\le 1}{sup}\ \varphi\Big(max\big(d(x,u), d(y,v), \max\big(d(x,u), d(y,v)\big)\Big) \\ &\le \underset{0\le t\le 1}{sup}\ \varphi\Big(max\big(d(x,u), d(y,v)\big)\Big), \end{aligned}$$

where we have used the fact that $\varphi$ is increasing.

From the last inequality, we deduce that

$$\varphi\Big(\max\big(d(x,u),d(y,v)\big)\Big)-d\big(F(x,y),F(u,v)\big)\geq 0. \tag{2}$$

Now, taking into account that if we put $\xi(t,s)=\varphi(s)-t$, then, as we saw in section 1, $\xi\in\mathcal{J}$ and (2)

$$\xi\Big(d\big(F(x,y),F(u,v)\big),\max\big(d(x,u),d(y,v)\big)\Big)\geq 0.$$

This proves that assumption of Theorem 2.2 is satisfied.

Therefore, $F$ has a unique coupled fixed point $(x_0,y_0)$. This means that $(x_0,y_0)\in C[0,1]\times C[0,1]$, $F(x_0,y_0)=x_0$ and $F(y_0,x_0)=y_0$, or, equivalently, for any $t\in[0,1]$

$$x_0(t)=f\Big(t,x_0(t),y_0(t),\int_0^t g\big(s,x_0(s),y_0(s)\big)\,ds\Big)$$
$$y_0(t)=f\Big(t,y_0(t),x_0(t),\int_0^t g\big(s,y_0(s),x_0(s)\big)\,ds\Big).$$

This proves that $(x_0,y_0)\in C[0,1]\times C[0,1]$ is a unique solution to Problem (1). $\square$

Next, we present a particular numerical example.

**Remark 3.2.** In our example, we will use the function $f:\mathbb{R}_+\to\mathbb{R}$ given by $f(x)=\ln(1+x)$. This function, as it is concave and $f(0)=0$, a well known result says us that

$$\big|\ln(1+x)-\ln(1+y)\big|\leq\ln\big(1+|x-y|\big),$$

for any $x,y\in\mathbb{R}_+$.

We are ready to present our numerical example.

**Example 3.3.** Consider the following coupled system of integral equations

$$\begin{cases} x(t) &= t^2+\dfrac{1}{3}\Big(\ln\Big(1+\dfrac{|x(t)|}{2}\Big)+\ln\big(1+|y(t)|\big)\Big)+\\ &\quad +\ln\Big(1+\int_0^t\lambda\big(s^2+|x(s)|+|y(s)|\big)ds\Big)\\ y(t) &= t^2+\dfrac{1}{3}\Big(\ln\Big(1+\dfrac{|y(t)|}{2}\Big)+\ln\big(1+|x(t)|\big)\Big)+\\ &\quad +\ln\Big(1+\int_0^t\lambda\big(s^2+|x(s)|+|y(s)|\big)ds\Big), \end{cases} \tag{3}$$

for any $t\in[0,1]$ with $\lambda>0$. Notice that Problem (3) is a particular of Problem (1) with

$$f(t,x,y,z)=t^2+\frac{1}{3}\left(\ln\left(1+\frac{|x|}{2}\right)+\ln(1+|y|)+\ln(1+|z|)\right),$$

and

$$g(t,x,y)=\lambda\,\left(t^2+|x|+|y|\right).$$

It is clear that $f$ and $g$ are continuous functions on $[0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ and on $[0,1]\times\mathbb{R}\times\mathbb{R}$, respectively. Therefore, assumption i of Theorem 3.1 is satisfied. Moreover, for any $t\in[0,1]$ and $x,y,z,u,v,w\in\mathbb{R}$, we have

$$
\begin{aligned}
\big|f(t,x,y,z)-f(t,u,v,w)\big| &= \frac{1}{3}\Bigg|\ln\left(1+\frac{|x|}{2}\right)+\ln\left(1+|y|\right)+\ln\left(1+|z|\right)-\\
&\quad -\ln\left(1+\frac{|u|}{2}\right)-\ln\left(1+|v|\right)-\ln\left(1+|w|\right)\Bigg|\\
&\le \frac{1}{3}\left[\left|\ln\left(1+\frac{|x|}{2}\right)-\ln\left(1+\frac{|u|}{2}\right)\right|+\right.\\
&\quad \left.+\Big|\ln\left(1+|y|\right)-\ln\left(1+|v|\right)\Big|+\Big|\ln\left(1+|z|\right)-\ln\left(1+|w|\right)\Big|\right]\\
&\le \frac{1}{3}\left[\ln\left(1+\left|\frac{|x|}{2}-\frac{|u|}{2}\right|\right)+\ln\Big(1+\big||y|-|v|\big|\Big)+\right.\\
&\quad \left.+\ln\Big(1+\big||z|-|w|\big|\Big)\right]\\
&\le \frac{1}{3}\left[\ln\left(1+\frac{|x-u|}{2}\right)+\ln\left(1+|y-v|\right)+\ln\left(1+|z-u|\right)\right]\\
&\le \frac{1}{3}\Big[\ln\left(1+|x-u|\right)+\ln\left(1+|y-v|\right)+\ln\left(1+|z-u|\right)\Big]\\
&\le \frac{1}{3}\Big[3\ln\Big(1+\max\left(|x-u|,|y-v|,|z-u|\right)\Big)\Big]\\
&\le \ln\Big(1+\max\left(|x-u|,|y-v|,|z-u|\right)\Big),
\end{aligned}
\tag{4}
$$

where we have used Remark 3.2, the inequality $\big||x|-|y|\big|\le|x-y|$, for any $x,y\in\mathbb{R}$ and the increasing character of the function $f(x)=\ln(1+x)$ for $x\ge 0$. From 4, we infer that assumption ii of Theorem 3.1 is satisfied with the function $\varphi(t)=\ln(1+t)$. It is easily seen that $\varphi\in\mathcal{G}$.

On the other hand, for any $t\in[0,1]$ and $x,y,u,v\in\mathbb{R}$, we have

$$
\begin{aligned}
\big|g(t,x,y)-g(t,u,v)\big| &= \lambda\,\big||x|+|y|-|u|-|v|\big| \le\\
&\le \lambda\Big(\big||x|-|u|\big|+\big||y|-|v|\big|\Big)\le\\
&\le \lambda\Big(\big|x-u\big|+\big|y-v\big|\Big)\le\\
&\le 2\lambda\max\,\big(\big|x-u\big|,\big|y-v\big|\big).
\end{aligned}
$$

Therefore, if $2\lambda\le 1$, this is, if $0\le\lambda\le\dfrac{1}{2}$, then assumption iii of Theorem 3.1 is satisfied.

Finally, by Theorem 3.1, we deduce that if $0\le\lambda\le\dfrac{1}{2}$, Problem (3) has a unique solution $(x_0,y_0)\in C[0,1]\times C[0,1]$.

## Acknowledgements

The authors, J.C. and K. S., are partially supported by the project PID2023-148028NB-I00.

## References

[1] Khojasteh, F., Shukla, S., Radenovic, S.,*A New Approach to the Study of Fixed Point Theory for Simulation Functions*, Filomat **29** (2015) 1189-1194.

[2] Karapinar, E.,*Fixed Points Results via Simulation Functions*, Filomat **30:8** (2016) 2343-2350.

[3] Guo, D., Lakshmikantham, V.,*Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal. Th. Methods Appl. **11** (1987) 623-632.

[4] Chen, Y.Z.,*Existence theorems of coupled fixed points*, J. Math. Anal. Appl. **154** (1991) 142-150.

[5] Gnana Bhaskar, T., Lakshmikantham, V., *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Analysis **65** (2006) 1379-1393.

[6] Samet, B.,*Coupled fixed pint theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces*, Nonlinear Anal. **72** (2010) 4508-4517.

[7] Luong, N.V., Thuan, N.X.,*Coupled fixed points in partially ordered metric spaces and applications*, Nonlinear Analysis **74** (2011) 983-992.

[8] Urs, C.,*Coupled fixed points theorems and applications to periodic boundary value problems*, Miskolc Math. Notes vol. **14** (2013) 323-333.

[9] Harjani, J., Rocha, J., Sadarangani, K.,$\alpha-$*Coupled fixed points and their application in dynamic programming*, Abstract and Applied Analysis, vol. **2014**, Article ID 593645, 4 pag.

[10] Afshari, H., Kalantari, S., Karapinar, E., *Solution of fractional differential equations via coupled fixed point*, Elec. J. Diff. Eq., vol. **2015** (2015) No. 286 (1-12).