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Coupled fixed points via simulation functions

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Abstract

In the present paper, we show a result in complete metric spaces about the existence and uniqueness of coupled fixed points by using simulation functions. Moreover, we illustrate our result by presenting a theorem about the existence and uniqueness of solution to a general system of nonlinear functional-integral equations.

Keywords: Coupled fixed point, Simulation function, Functional-integral equation
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1. Introduction and preliminaries

It is well known that the Banach contraction mapping principle and its generalizations constitute a very important tool in the theory of existence of solutions to functional, differential, and integral equations. Particularly, one of these generalizations uses the so-called simulation functions.
In the sequel, we present this class of functions and the above-mentioned fixed point theorem. This material appears in [1, 2].

Definition 1.1. A function $\xi : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ is said to be a simulation function if it satisfies the following conditions:

(i) $\xi(0, 0) = 0$.

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(ii) $\xi(t,s) < s - t$, for any t, $s > 0$.

(iii) Let (t_n) and (s_n) be sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \xi(t_n, s_n) < 0$.

In what follows, by $\mathcal{J}$ we will denote the class of simulation functions. Examples of functions belonging to $\mathcal{J}$ are the following ones:

(1) $\xi_1(t,s) = \phi_1(s) - \phi_2(t)$, for any t, $s \in [0,\infty)$, where ϕ_1, $\phi_2 : [0,\infty) \to [0,\infty)$ are continuous functions such that
  (a) $\phi_1(t) = \phi_2(t) = 0$ if and only if $t = 0$,
  (b) $\phi_1(t) < t \le \phi_2(t)$ for any $t > 0$.

(2) $\xi_2(t,s) = \varphi(s) - t$, for any t, $s \in [0,\infty)$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and increasing function and it satisfies that $\varphi(t) < t$ if $t > 0$.

By $\mathcal{G}$ we denote the class of functions φ satisfying the above mentioned conditions. It is clear that if $\varphi \in \mathcal{G}$ then $\varphi(0) = 0$. Examples of functions in $\mathcal{G}$ are $\varphi(t) = \ln(1+t)$; $\varphi(t) = \arctan t$; $\varphi(t) = \dfrac{t}{t+1}$; $\varphi(t) = kx$ with $k \in (0,1)$.

Next, we present the result about fixed point by using simulation functions that it appears in Theorem 2.8 of [1].

Theorem 1.2. *Let (X,d) be a complete metric space and $T : X \to X$ a mapping such that there exists $\xi \in \mathcal{J}$ satisfying*

$$\xi\big(d(T_x, T_y), d(x,y)\big) \ge 0,$$

for any $x, y \in X$ with $x \ne y$. Then T has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$ the Picard sequence (x_n), where $x_n = Tx_{n-1}$ for any $n \in \mathbb{N}$ converges to the fixed point $x^* \in X$.

In this paper, we present a result about coupled fixed points. The concept of coupled fixed point was introduced by Guo and Lakshmikantham in [3] for the study of coupled quasi-solutions of an initial value problem for ordinary differential equations. Some papers on coupled fixed points have appeared in the literature (see [3, 4, 5, 6, 7, 8, 9, 10], among others). Moreover, as an application of our result, we study the existence and uniqueness of solutions to a coupled system of functional equations.

2. Main result

Suppose that (X,d) is a complete metric space and $G : X \times X \to X$ a mapping.

Definition 2.1. An element $(x_0, y_0) \in X \times X$ is said to be a coupled fixed point of the mapping G if $G(x_0, y_0) = x_0$ and $G(y_0, x_0) = y_0$.

Now, we are ready to present our main result.

Theorem 2.2. *Let (X,d) be a complete metric space and $G : X \times X \to X$ be a mapping, such that there exists $\xi \in \mathcal{J}$ satisfying the following condition*

$$\xi\Big(d\big(G(x,y), G(u,v)\big), \max\big(d(x,u), d(y,v)\big)\Big) \ge 0,$$

for any $(x,y), (u,v) \in X \times X$, with $(x,y) \ne (u,v)$. Then G has a unique coupled fixed point.

Proof. Consider the metric space $\left(X \times X, \tilde{d}\right)$ where

$$\tilde{d}\big((x,y),(u,v)\big) = \max\big(d(x,u), d(y,v)\big),$$

for any $(x,y), (u,v) \in X \times X$.

It is well known that $(X \times X, \tilde{d})$ is also a complete metric space.

Next, we define the following mapping $\tilde{G} : X \times X \to X \times X$ by

$$\tilde{G}(x,y) = \big(G(x,y), G(y,x)\big).$$

In what follows, we check that $\tilde{G}$ satisfies assumptions of Theorem 1.2.

In fact, for any $x, y, u, v \in X$, we have that

$$\begin{aligned}\tilde{d}\big(\tilde{G}(x,y), \tilde{G}(u,v)\big) &= \tilde{d}\Big(\big(G(x,y), G(y,x)\big), \big(G(u,v), G(v,u)\big)\Big) = \\ &= \max\Big(d\big(G(x,y), G(u,v)\big), d\big(G(y,x), G(v,u)\big)\Big).\end{aligned}$$

In this point, we can consider two cases:

(a) $\tilde{d}\big(\tilde{G}(x,y), \tilde{G}(u,v)\big) = d\big(G(x,y), G(u,v)\big)$,

(b) $\tilde{d}\big(\tilde{G}(x,y), \tilde{G}(u,v)\big) = d\big(G(y,x), G(v,u)\big)$.

Case (a). Taking into account our assumption, we infer

$$\begin{aligned}&\xi\Big(\tilde{d}\big(\tilde{G}(x,y), \tilde{G}(u,v)\big), \tilde{d}\big((x,y),(u,v)\big)\Big) = \\ =&\xi\Big(d\big(G(x,y), G(u,v)\big), \max\big(d(x,u), d(y,v)\big)\Big) \geq 0.\end{aligned}$$

This proves that the condition appearing in Theorem 1.2 is satisfied for the complete metric space $(X \times X, \tilde{d})$.

Case (b). By using similar argument to case (a), we get

$$\begin{aligned}&\xi\Big(\tilde{d}\big(\tilde{G}(x,y), \tilde{G}(u,v)\big), \tilde{d}\big((x,y),(u,v)\big)\Big) = \\ =&\xi\Big(d\big(G(y,x), G(v,u)\big), \max\big(d(x,u), d(y,v)\big)\Big) = \\ =&\xi\Big(d\big(G(y,x), G(v,u)\big), \max\big(d(y,v), d(x,u)\big)\Big) \geq 0.\end{aligned}$$

In this case, we have also proved that the condition of Theorem 1.2 holds.

Now, by Theorem 1.2, the mapping $\tilde{G}$ has a unique fixed point. This is, there exists a unique pair $(u_0, v_0) \in X \times X$ such that

$$\tilde{G}(u_0, v_0) = (u_0, v_0).$$

Taking into account the definition of $\tilde{G}$, we deduce that

$$\tilde{G}(u_0, v_0) = \big(G(u_0, v_0), G(v_0, u_0)\big) = (u_0, v_0),$$

and from this

$$G(u_0, v_0) = u_0 \quad \text{and} \quad G(v_0, u_0) = v_0.$$

This is, $(u_0, v_0) \in X \times X$ is a coupled fixed point of the mapping G.

It is easily seen that the uniqueness of the coupled fixed point is (u_0, v_0) is obtained of the uniqueness of the fixed point (u_0, v_0) for $\tilde{G}$.

This completes the proof.

□

3. Applications

In this section, we illustrate our result studying the existence and uniqueness of solutions to the following coupled system of functional-integral equations

$$\begin{cases} x(t) = f\left(t, x(t), y(t), \int_0^t g(s, x(s), y(s))\, ds\right) \\ y(t) = f\left(t, y(t), x(t), \int_0^t g(s, y(s), x(s))\, ds\right), \end{cases} \tag{1}$$

for $t \in [0,1]$, in $C[0,1] \times C[0,1]$.

In the following theorem, we present a sufficient condition for the existence and uniqueness of solutions to Problem (1).

Theorem 3.1. *Suppose the following assumptions:*

(i) $f : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous functions.*

(ii) *There exists* $\varphi \in \mathcal{G}$ *such that*

$$\left|f(t,x,y,z) - f(t,u,v,w)\right| \leq \varphi\big(max(|x-u|, |y-v|, |z-w|)\big),$$

for any $t \in [0,1]$ *and* $x, y, z, u, v, w \in \mathbb{R}$.
(Here, $\mathcal{G}$ *is the class of functions appearing in section 1).*

(iii) *The following inequality*

$$\left|g(t,x,y) - g(t,u,v)\right| \leq \max\big(|x-u|, |y-v|\big),$$

for any $t \in [0,1]$ *and* $x, y, u, v \in \mathbb{R}$*, holds.*
Then Problem (1) has a unique solution $(x,y) \in C[0,1] \times C[0,1]$.

Proof. Consider the operator F defined on $C[0,1] \times C[0,1]$ by

$$F(x,y)(t) = f\left(t, x(t), y(t), \int_0^t g\big(s, x(s), y(s)\big) ds\right),$$

for any $t \in [0,1]$.

From i, it follows that F applies $C[0,1] \times C[0,1]$ into $C,[0,1]$.

Now, we check that F satisfies the condition appearing in Theorem 2.2.

In fact, by using our assumptions, we have, for any $(x,y),(u,v) \in C[0,1] \times C[0,1]$ with $(x,y) \neq (u,v)$, that

$$\begin{aligned}
& d\big(F(x,y), F(u,v)\big) = \sup_{0\leq t\leq 1}\Big|F(x,y)(t) - F(u,v)(t)\Big| = \\
& = \sup_{0\leq t\leq 1}\Bigg|f\left(t, x(t), y(t), \int_0^t g\left(s, x(s), y(s)\right) ds\right) \\
& - f\left(t, u(t), v(t), \int_0^t g\left(s, u(s), v(s)\right) ds\right)\Bigg| \\
& \leq \sup_{0\leq t\leq 1} \varphi\Big(max\big(|x(t)-u(t)|, |y(t)-v(t)|\big), \Big|\int_0^t g\left(s, x(s), y(s)\right) - g\left(s, u(s), v(s)\right) ds\Big|\Big) \\
& \leq \sup_{0\leq t\leq 1} \varphi\Big(max\big(d(x,u), d(y,v), \int_0^t \big|g(s, x(s), y(s)) - g(s, u(s), v(s))\big| ds\Big) \\
& \leq \sup_{0\leq t\leq 1} \varphi\Big(max\big(d(x,u), d(y,v), \int_0^t \max\big(|x(s)-u(s)|, |y(s)-v(s)|\big) ds\Big) \\
& \leq \sup_{0\leq t\leq 1} \varphi\Big(max\big(d(x,u), d(y,v), \max\big(d(x,u), d(y,v)\big)\Big) \\
& \leq \sup_{0\leq t\leq 1} \varphi\Big(max\big(d(x,u), d(y,v)\big)\Big),
\end{aligned}$$

where we have used the fact that φ is increasing.
From the last inequality, we deduce that

$$\varphi\Big(\max\big(d(x,u),d(y,v)\big)\Big)-d\big(F(x,y),F(u,v)\big)\geq 0. \tag{2}$$

Now, taking into account that if we put $\xi(t,s)=\varphi(s)-t$, then, as we saw in section 1, $\xi\in\mathcal{J}$ and (2)

$$\xi\Big(d\big(F(x,y),F(u,v)\big),\max\big(d(x,u),d(y,v)\big)\Big)\geq 0.$$

This proves that assumption of Theorem 2.2 is satisfied.

Therefore, F has a unique coupled fixed point (x_0,y_0). This means that $(x_0,y_0)\in C[0,1]\times C[0,1]$, $F(x_0,y_0)=x_0$ and $F(y_0,x_0)=y_0$, or, equivalently, for any $t\in[0,1]$

$$x_0(t)=f\Big(t,x_0(t),y_0(t),\int_0^t g\big(s,x_0(s),y_0(s)\big)\,ds\Big)$$
$$y_0(t)=f\Big(t,y_0(t),x_0(t),\int_0^t g\big(s,y_0(s),x_0(s)\big)\,ds\Big).$$

This proves that $(x_0,y_0)\in C[0,1]\times C[0,1]$ is a unique solution to Problem (1). □

Next, we present a particular numerical example.

Remark 3.2. In our example, we will use the function $f:\mathbb{R}_+\to\mathbb{R}$ given by $f(x)=\ln(1+x)$. This function, as it is concave and $f(0)=0$, a well known result says us that

$$\big|\ln(1+x)-\ln(1+y)\big|\leq\ln\big(1+|x-y|\big),$$

for any $x,y\in\mathbb{R}_+$.

We are ready to present our numerical example.

Example 3.3. Consider the following coupled system of integral equations

$$\begin{cases} x(t) &= t^2+\frac{1}{3}\Big(\ln\Big(1+\frac{|x(t)|}{2}\Big)+\ln\big(1+|y(t)|\big)\Big)+ \\ & \quad +\ln\Big(1+\int_0^t\lambda\big(s^2+|x(s)|+|y(s)|\big)ds\Big) \\ y(t) &= t^2+\frac{1}{3}\Big(\ln\Big(1+\frac{|y(t)|}{2}\Big)+\ln\big(1+|x(t)|\big)\Big)+ \\ & \quad +\ln\Big(1+\int_0^t\lambda\big(s^2+|x(s)|+|y(s)|\big)ds\Big), \end{cases} \tag{3}$$

for any $t\in[0,1]$ with $\lambda>0$. Notice that Problem (3) is a particular of Problem (1) with

$$f(t,x,y,z)=t^2+\frac{1}{3}\left(\ln\left(1+\frac{|x|}{2}\right)+\ln(1+|y|)+\ln(1+|z|)\right),$$

and

$$g(t,x,y)=\lambda\,\left(t^2+|x|+|y|\right).$$

It is clear that f and g are continuous functions on $[0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ and on $[0,1]\times\mathbb{R}\times\mathbb{R}$, respectively. Therefore, assumption i of Theorem 3.1 is satisfied. Moreover, for any $t\in[0,1]$ and $x,y,z,u,v,w\in\mathbb{R}$, we have

$$
\begin{aligned}
\big|f(t,x,y,z) - f(t,u,v,w)\big| &= \frac{1}{3}\Bigg| \ln\left(1+\frac{|x|}{2}\right) + \ln\left(1+|y|\right) + \ln\left(1+|z|\right) - \\
&- \ln\left(1+\frac{|u|}{2}\right) - \ln\left(1+|v|\right) - \ln\left(1+|w|\right)\Bigg| \\
&\le \frac{1}{3}\Bigg[\left|\ln\left(1+\frac{|x|}{2}\right) - \ln\left(1+\frac{|u|}{2}\right)\right| + \\
&+ \Big|\ln\left(1+|y|\right) - \ln\left(1+|v|\right)\Big| + \Big|\ln\left(1+|z|\right) - \ln\left(1+|w|\right)\Big|\Bigg] \\
&\le \frac{1}{3}\left[\ln\left(1+\left|\frac{|x|}{2}-\frac{|u|}{2}\right|\right) + \ln\left(1+\big||y|-|v|\big|\right) + \right. \\
&\left. + \ln\left(1+\big||z|-|w|\big|\right)\right] \\
&\le \frac{1}{3}\left[\ln\left(1+\frac{|x-u|}{2}\right) + \ln\left(1+|y-v|\right) + \ln\left(1+|z-u|\right)\right] \\
&\le \frac{1}{3}\Big[\ln\left(1+|x-u|\right) + \ln\left(1+|y-v|\right) + \ln\left(1+|z-u|\right)\Big] \\
&\le \frac{1}{3}\Big[3\ln\Big(1+\max\left(|x-u|,|y-v|,|z-u|\right)\Big)\Big] \\
&\le \ln\Big(1+\max\left(|x-u|,|y-v|,|z-u|\right)\Big),
\end{aligned} \tag{4}
$$

where we have used Remark 3.2, the inequality $\big||x|-|y|\big| \le |x-y|$, for any $x, y \in \mathbb{R}$ and the increasing character of the function $f(x) = \ln(1+x)$ for $x \ge 0$. From 4, we infer that assumption ii of Theorem 3.1 is satisfied with the function $\varphi(t) = \ln(1+t)$. It is easily seen that $\varphi \in \mathcal{G}$.

On the other hand, for any $t \in [0,1]$ and $x, y, u, v \in \mathbb{R}$, we have

$$
\begin{aligned}
\big|g(t,x,y) - g(t,u,v)\big| &= \lambda\,\big||x|+|y|-|u|-|v|\big| \le \\
&\le \lambda\Big(\big||x|-|u|\big| + \big||y|-|v|\big|\Big) \le \\
&\le \lambda\Big(\big|x-u\big| + \big|y-v\big|\Big) \le \\
&\le 2\lambda \max\left(\big|x-u\big|, \big|y-v\big|\right).
\end{aligned}
$$

Therefore, if $2\lambda \le 1$, this is, if $0 \le \lambda \le \frac{1}{2}$, then assumption iii of Theorem 3.1 is satisfied.

Finally, by Theorem 3.1, we deduce that if $0 \le \lambda \le \frac{1}{2}$, Problem (3) has a unique solution $(x_0, y_0) \in C[0,1] \times C[0,1]$.

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