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Rus-Hicks-Rhoades maps on quasi-metric spaces revisited

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Abstract

Motivated by recent researches conducted by Sehie Park, we here present various adjustments and clarifications concerning the quasi-metric extension of well-known fixed point theorems due to Rus, and Hicks and Rhoades, respectively. In particular, some pertinent examples are given.

Keywords: Quasi-metric space; Smyth complete; Rus-Kirk-Rhoades map; fixed point.
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1. Introduction and preliminaries

The study of the fixed point theory in quasi-metric spaces has received a renewed impetus in the last two years due, in large part, to the contributions by Park [6, 7, 8]. In particular, he explored the quasi-metric extension and possible improvement of relevant fixed point theorems due to Rus [11] and Hicks and Rhoades [5]. The purpose of this note is to update, adjust and clarify some aspects of such extension, including some pertinent examples.

At this point, and in order to help the reader, we recall (our notation and terminology are standard, see, e.g., [9]) that by a quasi-metric on a set X we mean a function d from $X \times X$ to $[0, \infty)$ that fulfills the following two conditions for all $x, y, z \in X$: (qm1) $d(x, y) = d(y, x) = 0$ if and only if $x = y$; (qm2) $d(x, y) \leq d(x, z) + d(z, y)$.

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In this case, the pair (X, d) is called a quasi-metric space.

If the quasi-metric d verifies the next condition stronger than (qm1): $d(x, y) = 0$ if and only if $x = y$, we say that d is a T_1 quasi-metric on X and that (X, d) is a T_1 quasi-metric space.

If d is a quasi-metric on a set X, the function $d^s : X \times X \to [0, \infty)$ given by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ for all $x, y \in X$, is a metric on X.

It is well known that each quasi-metric d on a set X induces a T_0 topology τ_d on X that has as a base the family of open balls $\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}$ where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and all $\varepsilon > 0$ (in particular, τ_d is T_1 if and only if d is a T_1 quasi-metric).

We say that a sequence $(x_n)_{n\in\mathbb{N}}$ in a quasi-metric space (X, d) is τ_d-convergent if it converges in the topological space (X, τ_d). Hence, $(x_n)_{n\in\mathbb{N}}$ is τ_d-convergent to $x \in X$ if and only if $d(x, x_n) \to 0$.

We will say that a self-mapping T of a quasi-metric space (X, d) is τ_d-continuous (resp. τ_{d^s}-continuous) if it continuous from (X, τ_d) into itself (resp. from (X,τ_{d^s}) into itself).

On the other hand, and for our goals here, we will consider the following notions of quasi-metric completeness.

A quasi-metric space (X, d) is called:

(i) Smyth complete if every left K-Cauchy sequence is τ_{d^s}-convergent, where a sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be left K-Cauchy if for each $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ whenever $n_\varepsilon \leq n \leq m$.

(ii) bicomplete if the metric space (X, d^s) is complete.

Clearly, every Smyth complete quasi-metric space is bicomplete. The converse does not hold in general, as the well-known Sorgenfrey quasi-metric line shows.

In [7], and according to the terminology proposed in [1, 4], right K-Cauchy sequences are called right Cauchy sequences, Smyth complete quasi-metric spaces are called right-complete quasi-metric spaces and bicomplete quasi-metric spaces are called complete quasi-metric spaces.

We emphasize that, as already was point out in [9, p. 71], completeness does not imply T-orbital completeness in Park's sense, contrary to what was stated in [7, p. 118]. However, it is clear that right-completeness (i.e., Smyth completeness) does imply T-orbital completeness.

The monographs [2, 3] provide suitable sources for a deep study of quasi-metric spaces and other related structures.

2. The metric case

In [11], Rus proved the following well-known result.

Theorem 2.1. *Let (X, d) be a complete metric space. If T is a continuous self-mapping of X such that there exists a constant $\alpha \in (0, 1)$ satisfying the inequality*

$$d(Tx, T^2x) \leq \alpha d(x, Tx),$$

for all $x \in X$, then, T has a fixed point.

Later, Hicks and Rhoades [5] proved the following.

Theorem 2.2. *Let (X, d) be a complete metric space. If T is a nonexpansive self-mapping of X such that there exists a constant $\alpha \in (0, 1)$ satisfying the inequality*

$$d(Tx, T^2x) \leq \alpha d(x, Tx),$$

for all $x \in X$, then, T has a fixed point.

Note that, in fact, Theorem 2.2 is a consequence of Theorem 2.1.

In an attempt of improving and unifying Theorems 2.1 and 2.2, Park stated in [6, Theorem H(γ_1)] the following.

Theorem 2.3. *Let (X, d) be a complete metric space. If T is a self-mapping of X such that there exists a constant $\alpha \in (0, 1)$ satisfying the inequality*

$$d(Tx, T^2x) \leq \alpha d(x, Tx),$$

for all $x \in X$, then, T has a fixed point.

Although, as formulated, Theorem 2.3 is not true (see Example 2.4 below), it remains true assuming that T is orbitally continuous in the terms described and developed by Park (see, e.g., Theorem 3.2 of [7]) and its proof).

Example 2.4. Let $X = [0, 1]$ and let d be the usual metric on X. Define $T : X \to X$ as $T0 = 1$ and $Tx = x/2$ otherwise. Of course, (X, d) is complete and the self-mapping T of X has no fixed points. However, we have

$$d(T0, T^20) = d(1, \frac{1}{2}) = \frac{1}{2} = \frac{1}{2}d(0, 1) = \frac{1}{2}d(0, T0),$$

and

$$d(Tx, T^2x) = d(\frac{x}{2}, \frac{x}{4}) = \frac{x}{4} = \frac{1}{2}d(x, \frac{x}{2}) = \frac{1}{2}d(x, Tx),$$

for all $x \in (0, 1]$. So, all conditions of Theorem 2.3 are satisfied for $\alpha = 1/2$.

3. The quasi-metric case

In [6, 7], Park discussed, among other types of contractions, the so-called RHR-maps in the framework of quasi-metric spaces.

A self-mapping T of a quasi-metric space (X, d) is said to be a Rus-Hicks-Rohades map, or simply a RHR-map, if there is a constant $\alpha \in (0, 1)$ such that $d(Tx, T^2x) \leq \alpha d(x, Tx)$ for all $x \in X$.

In [7, Theorem 6.3] it was stated the following quasi-metric generalization of Theorem 2.3.

Theorem 3.1. *Let (X, d) be a bicomplete quasi-metric space. Then, every HRH-map on (X, d) has a fixed point.*

As formulated, Theorem 3.1 is not true (compare Theorem 2.3). In fact, the next example shows that it does not hold even if (X, d) is Smyth complete and T is a τ_d-continuous RHR-map.

Example 3.2. Let $X = \mathbb{N} \cup \{\infty\}$ and let d be the quasi-metric on X given by $d(x, x) = 0$ for all $x \in X$, $d(n, \infty) = 0$ for all $n \in \mathbb{N}$, $d(\infty, n) = 1/n$ for all $n \in \mathbb{N}$, and $d(n, m) = 1/m$ for all $n, m \in \mathbb{N}$ with $n \neq m$.

Then, (X, d) is a Smyth complete quasi-metric space (see, e.g., [9, p. 72]).

Now, let T be the self mapping of X defined as $T\infty = 1$, and $Tn = 2n$ for all $n \in \mathbb{N}$.

It is clear that T is τ_d-continuous. Moreover, we have that $d(Tx, T^2x) = d(x, Tx)/2$ for all $x \in X$ [10, p. 104]. So, T is also an RHR-map on (X, d).

Finally, note that T is not τ_{d^s}-continuous because $d^s(\infty, n) = 1/n$ but $d^s(T\infty, Tn) = 1$ for all $n \in \mathbb{N}$.

However, Park [7, Theorem 3.2] obtained a positive result that we adapt to our context as follows.

Proposition 3.3. (*Park*) *Let (X, d) be a Smyth complete quasi-metric space. Then, every τ_{d^s}-continuous RHR-map on (X, d) has a fixed point.*

Note that Example 3.2 shows that Proposition 3.3 cannot be extended to τ_d-continuous RHR-maps. Nevertheless, we can obtain the following.

Proposition 3.4. *Let T be a τ_d-continuous RHR-map on a Smyth complete quasi-metric space (X, d). Then, there exists $\xi \in X$ such that $d(T\xi, \xi) = 0$.*

Proof. (We sketch the proof) By hypothesis, there exists a constant $\alpha \in (0,1)$ such that $d(Tx, T^2x) \leq \alpha d(x, Tx)$ for all $x \in X$.

Fix $x_0 \in X$. Then, $d(T^n x_0, T^{n+1}x_0) \leq \alpha^n d(x_0, Tx_0)$ for all $n \in \mathbb{N}$, and, hence, $(T^n x_0)_{n\in\mathbb{N}}$ is a left K-Cauchy sequence in the Smyth complete quasi-metric space (X, d). Hence, there is $\xi \in X$ such that $d(\xi, T^n x_0) \to 0$ and $d(T^n x, \xi) \to 0$ as $n \to \infty$. From the τ_d-continuity of T it follows that $d(T\xi, T^{n+1}x_0) \to 0$. Since

$$d(T\xi, \xi) \leq d(T\xi, T^{n+1}x_0) + d(T^{n+1}x_0, \xi),$$

for all $n \in \mathbb{N}$, we deduce that $d(T\xi, \xi) = 0$, which concludes the proof. □

Corollary 3.5. *Every τ_d-continuous RHR-map on a Smyth complete T_1 quasi-metric space (X, d) has a fixed point.*

Remark 3.6. Note that Proposition 3.4 applies to Example 3.2. In fact, we have $d(T\infty, \infty) = 0$, but T has no fixed points.

We conclude the paper with an example showing that Propositions 3.3 and 3.4 cannot be generalized to bicomplete quasi-metric spaces.

Example 3.7. Let $X = (0, 1)$ and let d be the restriction on X of the classical Sorgenfrey quasi-metric. Thus, $d(x, y) = y - x$ if $x \leq y$, and $d(x, y) = 1$ if $x > y$. Note that d^s is the discrete metric on X, so (X, d^s) is a complete metric space, i.e., (X, d) is bicomplete.

Now, let T be the self-mapping of X defined as $Tx = (1+x)/2$ for all $x \in X$. Obviously, T is τ_d-continuous and τ_{d^s}-continuous, and it has no fixed points in X. Moreover, T is a RHR-map on (X, d). Indeed, for each $x \in X$ we get

$$d(Tx, T^2x) = d(\frac{1+x}{2}, \frac{3+x}{4}) = \frac{3+x}{4} - \frac{1+x}{2} = \frac{1-x}{4} = \frac{1}{2}(\frac{1+x}{2} - x) = \frac{1}{2}d(x, Tx).$$

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