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Study of a similarity boundary layer equation by using the shooting method

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Abstract

In this paper we are interested in the study of the existence or nonexistence and uniqueness or nonuniqueness of the solutions of the boundary value problem involving a third order ordinary nonlinear autonomous differential equation satisfying a boundary conditions. Its solutions are similarity solutions to a problem of boundary-layer theory.

Keywords: Third order nonlinear differential equation, Boundary-Layer, Mixed convection, Boundary value problem, Shooting method.
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1. Introduction

Nonlinear autonomous differential equations are mathematical models in which the rate of change of a variable depends nonlinearly on its current value and the equation lacks explicit dependence on time. These equations are significant in modeling various natural phenomena, from population dynamics to chemical reactions and electrical circuits. Unlike linear equations, their solutions often exhibit complex behaviors such as stability, periodicity, or chaotic dynamics. Understanding nonlinear autonomous differential equations is crucial in physics, engineering, and biology for predicting and analyzing systems that evolve over time without external influences. For recent results on various types of nonlinear differential equations, see the following [1, 2, 3, 8, 9, 10, 11, 11, 21].

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In this paper, we consider Let $\beta \in \mathbb{R}$. We consider the following boundary value problem:

$$\begin{cases} f'''(t) + f(t)f''(t) + \beta f'(t)\left[f'(t) - 1\right] = 0; \\ f(0) = a, \\ f'(0) = b, \\ f'(t) \to \lambda \text{ as } t \to +\infty, \end{cases} \qquad (\mathcal{P}_{\beta;a,b,\lambda})$$

where $\beta \in \mathbb{R}$, $a, \lambda \in \mathbb{R}$, $b, t \in [0, +\infty)$. This problem is associated with

$$f'''(t) + f(t)f''(t) + \beta f'(t)\left[f'(t) - 1\right] = 0. \qquad (1)$$

In [4], the authors give only a partial results exactly in the case $\beta > 1$, of the problem $(\mathcal{P}_{\beta;a,b,\lambda})$ which arises in the boundary-layer theory in some contexts of fluids mechanics, in particular the mixed convection phenomena created by a heated plate and embedded in a porous medium saturated with a fluid. How is modeling of boundary-layer of a system of partial differential equations which reduces to a system involving a third order differential equation (see [18, 19, 22]). We obtain the following equation with a boundary conditions which appears from others equations and others boundary conditions that are obtained during the modeling of a phenomenon of mixed convection.

$$\frac{\partial^3\Psi}{\partial y^3} + \frac{\partial\Psi}{\partial x}\frac{\partial^2\Psi}{\partial y^2} - \frac{\partial\Psi}{\partial y}\left(\frac{\partial^2\Psi}{\partial x\partial y} - mx^{m-1}\right) = 0, \qquad (2)$$

with the boundary conditions

$$\frac{\partial\Psi}{\partial x}(x,0) = -\omega x^{\frac{m-1}{2}}, \quad \frac{\partial\Psi}{\partial y}(x,0) = x^m \quad \text{and} \quad \frac{\partial\Psi}{\partial y}(x,+\infty) = 0. \qquad (3)$$

We are looking for similarity solutions of Eq. (2), with the boundary conditions (3), which reduces to a system involving a third order differential equation by introducing the dimensionless similarity variables (see [6, 14]):

$$\Psi(x,y) = \sqrt{2}x^{\frac{m+1}{2}}F(t) \quad \text{with} \quad t(x,y) = \frac{1}{\sqrt{2}}x^{\frac{m-1}{2}}y, \quad \text{and} \quad \Theta(x,y) = x^m\theta(t).$$

Therefore, we obtain the following third order autonomous nonlinear differential equation

$$F'''(t) + (m+1)F(t)F''(t) - 2mF'(t)\left[F'(t) - 1\right] = 0, \qquad (4)$$

with the boundary conditions $F(0) = a_0 \in \mathbb{R}$, $F'(0) = b \geq 0$ and $F'(+\infty) = \lambda \in \{0, 1\}$.
Then, Ψ is a solution of Eq. (2) if and only if F is a solution of Eq. (4). The function Ψ is called similarity solution of (2) and the variable t is called the similarity variable.

Remark 1.1. *When $m > -1$, Eq. (4) is equivalent to the following equation*

$$f'''(t) + f(t)f''(t) + \beta f'(t)\left[f'(t) - 1\right] = 0, \qquad (5)$$

with $f(t) = \sqrt{m+1}F\left(\frac{t}{\sqrt{m+1}}\right)$ and β is a constant depending on m.

1. *For $m = -1$, Eq. (4) reduces to $F'''(t) + 2F'(t)\left[F'(t) - 1\right] = 0$. This equation has a first integral.*
2. *For $m = 0$ (resp. $\beta = 0$), Eq. (4) (resp. Eq. (5)) reduces to the Blasius equation (a lot of papers have dealt with it).*

2. Description of our method

As we mentioned in the introduction, we interested in studying solutions of the Eq. (1) and associated with this equation the following boundary value problem:

$$\begin{cases} f'''(t) + f(t)f''(t) + \beta f'(t)\,[f'(t) - 1] = 0, \\ f(0) = a = \sqrt{m+1}a_0, \\ f'(0) = b, \\ f'(t) \to \lambda \text{ as } t \to +\infty, \end{cases} \qquad (\mathcal{P}_{\beta;a,b,\lambda})$$

where $t \in [0, +\infty)$, $a, \lambda \in \mathbb{R}$, $b \geq 0$ and the parameter $\beta > 1$ is a temperature power-law profile and $b = 1 + \varepsilon$ is the mixed convection parameter. Note that, if $\lambda \notin \{0, 1\}$, then the problem $(\mathcal{P}_{\beta;a,b,\lambda})$ do not have any solutions (see [15]).

Let $(\mathcal{Q}_{\beta;a,b,c})$ be the following initial value problem, we denote by f_c the solution of this problem on the right maximal interval of its existence $[0, T_c)$:

$$\begin{cases} f'''(t) + f(t)f''(t) + \beta f'(t)\,[f'(t) - 1] = 0, \\ f(0) = a, \\ f'(0) = b, \\ f''(0) = c. \end{cases} \qquad (\mathcal{Q}_{\beta;a,b,c})$$

The approach used to study the boundary value problem $(\mathcal{P}_{\beta;a,b,\lambda})$ is a shooting method that consists in finding values of a real parameter $c = f''_c(0)$ for which the solution of Eq. (1) satisfying the initial conditions of $(\mathcal{Q}_{\beta;a,b,c})$ exists up to infinity and such that $f'_c(t) \to \lambda \in \{0, 1\}$ as $t \to +\infty$.
This approach has used in the study of different cases, the case where $a \geq 0$, $b \geq 0$, $0 < \beta < 1$ was treated in [4], and for $a \in \mathbb{R}$, $b \leq 0$, $0 < \beta < 1$ see [5], and the results obtained generalize the ones of [20], for $a \in \mathbb{R}$, $b \leq 0$, $\beta \geq 1$ see [12]. In [16], have established some results about the existence and uniqueness of convex and concave solution of $(\mathcal{P}_{\beta;a,b,1})$ where $-2 < \beta < 0$ and $b > 0$. These results can be recovered from [17], where the general equation $f''' + ff'' + \mathbf{g}(f') = 0$ is studied. For the case $a \in \mathbb{R}$, $b \geq 0$, $\beta < 0$, see [13]. In [23, 24], some theoretical results can be found about the problem $(\mathcal{P}_{\beta;0,b,1})$ with $-2 < \beta < 0$, and $b < 0$, the authors use the method based on the fixed point theorem allows them to prove the existence of a convex solution for the case $a = 0$.

3. On Blasius Equation

We recall some results about subsolutions and supersolutions of the Blasius equation (see [17]). Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function.

Definition 3.1. *We say that f is a subsolution (resp. a supersolution) of the Blasius equation if f is of class C^3 and if $f''' + ff'' \leq 0$ on I (resp. $f''' + ff'' \geq 0$ on I).*

Definition 3.2. *Let $\epsilon > 0$, We say that f is a ϵ-subsolution (resp. a ϵ-supersolution) of the Blasius equation if f is of class C^3 and if $f''' + ff'' \leq -\epsilon$ on I (resp. $f''' + ff'' \geq \epsilon$ on I).*

Proposition 3.3. *Let $t_0 \in \mathbb{R}$. There does not exist nonpositive concave subsolution of the Blasius equation on the interval $[t_0, +\infty)$.*

Proof. See [17], Proposition 2.11. □

Proposition 3.4. *Let $t_0 \in \mathbb{R}$. There does not exist nonpositive convex supersolution of the Blasius equation on the interval $[t_0, +\infty)$.*

Proof. [17], Proposition 2.5. □

Proposition 3.5. *Let $t_0 \in \mathbb{R}$. There does not exist ϵ-subsolution of the Blasius equation on the interval $[t_0, +\infty)$.*

Proof. See [17], Proposition 2.18. □

4. Preliminary Results

Proposition 4.1 ([12])**.** *Let us suppose that f be a solution of Eq. (1) on the maximal interval $I = (T_-, T_+)$, for all $t \in I$, we have*

1. *Let $H_0 = f'' + f(f' - \beta)$. Then $H_0' = (1 - \beta)f'^2$;*
2. *Let $H_1 = f'' + f(f' - 1)$. Then $H_1' = (1 - \beta)f'(f' - 1)$;*
3. *Let $H_2 = 3f''^2 + \beta f'^2(2f' - 3)$. Then $H_2' = -6ff''^2$;*
4. *Let $H_3 = 2ff'' - f'^2 + (2f' - \beta)f^2$. Then $H_3' = 2(2 - \beta)ff'^2$.*

Proposition 4.2. *Let f be a solution of the Eq. (1) on some maximal interval $I = (T_-, T_+)$ and $\beta > 1$.*

1. *If F is any anti-derivative of f on I, then $(f''e^F)' = -\beta f'(f' - 1)e^F$.*
2. *Assume that $T_+ = +\infty$ and that $f'(t) \to \lambda \in \mathbb{R}$ as $t \to +\infty$. If, moreover, f is of constant sign at infinity, then $f''(t) \to 0$ as $t \to +\infty$.*
3. *If $T_+ = +\infty$ and if $f'(t) \to \lambda \in \mathbb{R}$ as $t \to +\infty$, then $\lambda = 0$ or $\lambda = 1$.*
4. *If $T_+ < +\infty$, then f'' and f' are unbounded near T_+.*
5. *If there exists a point $t_0 \in I$ satisfying $f''(t_0) = 0$ and $f'(t_0) = \mu$, where $\mu = 0$ or 1 then for all $t \in I$, we have $f(t) = \mu(t - t_0) + f(t_0)$.*
6. *If $f'(t) \to 0$ as $t \to +\infty$, then $f(t)$ does not tend to $-\infty$ or $+\infty$ as $t \to +\infty$.*

Proof. The first statement follows immediately from Eq.(1). For the proof of statements (2)-(5), see [17], and statement (6), see [4]. □

Proposition 4.3. *Let $\beta > 1$ and f be a solution of (1) on some right maximal interval $[\tau, T_+)$. If there exists $t_0 \in [\tau, T_+)$ such that $f_c'(t_0) = 0$ and $f_c''(t_0) < 0$, then for all $t > t_0$, $f_c''(t) < 0$.*

Proof. Let f_c be a solution of (1) on some right maximal interval of existence $[0, T_+)$. Let there exists $t_0 \in [\tau, T_+)$, such that $f_c'(t_0) = 0$, $f_c''(t_0) < 0$. We suppose that there exists $t_1 > t_0$, such that $f_c''(t_1) = 0$, it follows that $f_c' < 0$ on (t_0, t_1), then from Eq.(1), we have

$$f_c'''(t_1) = -\beta f_c'(t_1)(f_c'(t_1) - 1) < 0,$$

which yields a contradiction. □

5. The boundary value problem $(P_{\beta;a,b,\lambda})$ with $b \geq 0$

In the following, we will study the boundary value problem $(P_{\beta;a,b,\lambda})$ for $\beta > 1$, $a \in \mathbb{R}$ and $b \geq 0$. For more details, see [17], there is a results about this problem with the function g no vanishes between b and λ . The case where $0 < \beta \leq 1$, with $b \geq 0$, was an almost complete study in [4]. Moreover, in our case, we are interested here in concave, convex, concave-convex and convex-concave solutions of this problem. We consider the case $b \geq 0$, we will distinguish between the cases $0 \leq b < 1$ and $b \geq 1$.

Remark 5.1. *It is clear for $0 \leq b < 1$, the boundary value problem $(P_{\beta;a,b,1})$ does not have a concave solution on $[0, +\infty)$, and the boundary value problem $(P_{\beta;a,b,0})$ does not have a convex solution on $[0, +\infty)$. For $b \geq 1$, the boundary value problem $(P_{\beta;a,b,1})$ does not have a convex solution on $[0, +\infty)$.*

Proposition 5.2. *Let $\beta > 1$. There does not exist $c \in \mathbb{R}$, such that f'_c is a solution on its right maximal interval of existence $[0, T_+)$ and $f'_c(t) \to +\infty)$ as $t \to T_c$, with $T_c < +\infty$.*

Proof. Let $\beta > 1$ and f_c be a solution of the problem $(P_{\beta;a,b,+\infty})$, i.e. $f'_c(T_c) = +\infty$. There exists $t_0 \in [0, T_c)$ such that $f'_c(t) > 1$ for all $t \in [t_0, T_c)$, we have

$$f'''_c(t) + f_c(t)f''_c(t) = -\beta f'_c(t)(f'_c(t) - 1) < -\beta f'_c(t_0)(f'_c(t_0) - 1) = -\epsilon.$$

Then, f_c is a ϵ-subsolution of the Blasius equation on $[t_0, T_c)$. Therefore from Proposition 3.5, we have $T_c < +\infty$. Furthermore the function H_1 is decreasing for $t > t_0$. Hence for all $t \in [t_0, T_c)$, $H_1(t) \leq H_1(t_0)$, then we have

$$f_c(t)(f'_c(t) - 1) < f''_c(t) + f_c(t)(f'_c(t) - 1) \leq H_1(t_0) < f''_c(t_0) + f_c(t_0)f'_c(t_0),$$

which is a contradiction with the fact that $f'_c(T_c) = +\infty$. □

5.1. The case $a \leq 0$

Remark 5.3. *This case has been studied in [17], it shows that with either $\beta \geq 1$ and $a < 0$, or $\beta > 1$ and $a \leq 0$, the boundary value problem $(P_{\beta;a,b,\lambda})$ has no concave solution if $b > 1$, and no convex solution if $b \leq 1$.*

5.1.1. ***The case*** $0 \leq b < 1$

We consider the following sets C_0, C_1, C_2 and C_3 defined by:

$$C_0 = \{c < 0;\ f'_c > 0 \text{ on } [0, T_c)\},$$

$$C_1 = \{c < 0;\ \exists s_c \in [0, T_c) \text{ s.t } f'_c > 0 \text{ on } [0, s_c) \text{ and } f'_c(s_c) = 0\},$$

$$C_2 = \{c \geq 0;\ \ f''_c \geq 0 \text{ on } [0, T_c)\},$$

$$C_3 = \{c \geq 0;\ \exists t_c \in [0, T_c),\ \exists \epsilon_c > 0 \text{ s.t } f'_c < 1 \text{ on } [0, t_c),$$
$$f'_c > 1 \text{ on } (t_c, t_c + \epsilon_c) \text{ and } f''_c > 0 \text{ on } (0, t_c + \epsilon_c)\}.$$

Remark 5.4. *This is obvious that C_0, C_1, C_2 and C_3 are disjoint sets and that there union is the whole line of real numbers.*

Proposition 5.5. *Let $\beta \geq 2$ and $c \in C_0$, then there exists $c \geq c^*$, such that $C_0 = [c^*, 0[$.*

Proof. Let $c \in C_0$ and $\beta \geq 2$ and f_c be a solution of the problem $(P_{\beta;a,b,\lambda})$. Either there exists $t_0 \in [0, T_c[$ such that $f_c(t_0) = 0$ or $f_c < 0$ and $f'_c \to 0$ as $t \to +\infty$. Thus the function H_3 is increasing on $[0, t_0)$ or on $[0, +\infty)$, and from Proposition 4.2, item 2, we have $2ac - b^2 + (2b - \beta)a^2 \leq 0$, which implies that $c \geq \frac{b^2+(\beta-2b)a^2}{2a}$. □

Remark 5.6. *Thanks to the previous Proposition and from Proposition 4.3, thus $C_1 \neq \emptyset$, with $f_c(s_c) < 0$, the point s_c be as in definition of C_1 i.e $f_c < 0$ on $[0, s_c)$ and $f_c'(s_c) = 0$. Hence f_c is not defined on the whole interval $[0, +\infty)$*

Lemma 5.7. *Let $c \in C_0$ and $c \leq a(\beta - b)$, if there exists t_0 such that $f_c(t_0) = 0$ then $f_c''(t_0) \leq 0$.*

Proof. Let $c \in C_0$ and $c \leq a(\beta - b)$. Assume for contradiction that there exists $t_0 \in [0, T_c)$ such that $f_c(t_0) = 0$ and $f_c''(t_0) > 0$. Thus the function H_0 is decreasing on $[0, t_0)$, hence $c + a(b - \beta) > 0$, this is a contradiction. □

5.1.2. ***The case*** $b \geq 1$

In this case it is easy to see that $\mathbb{R}$ can be partitioned into the following sets C_0', C_1', C_2', C_3' and C_4' defined by:

$$C_0' = \{c > 0 : f_c'' > 0 \text{ on } [0, T_c)\},$$

$$C_1' = \{c > 0 : f_c'' > 0 \text{ on } [0, s_c) \text{ and } f_c'' < 0 \text{ on } (s_c, s_c + \varepsilon_c)\},$$

$$C_2' = \{c \leq 0 : f_c'' \leq 0 \text{ on } [0, T_c)\},$$

$$C_3' = \{c \leq 0 : \exists t_c \in [0, T_c), \exists \varepsilon_c > 0 \text{ s.t } f_c' > 1 \text{ on } (0, t_c), \\ f_c' < 1 \text{ on } (t_c, t_c + \varepsilon_c) \text{ and } f_c'' < 0 \text{ on } (0, t_c + \varepsilon_c)\},$$

$$C_4' = \{c \leq 0 : \exists s_c \in [0, T_c), \exists \varepsilon_c > 0 \text{ s.t } f_c'' < 0 \text{ on } (0, s_c), \\ f_c'' > 0 \text{ on } (s_c, s_c + \varepsilon_c) \text{ and } f_c' > 1 \text{ on } (0, s_c + \varepsilon_c)\}.$$

Remark 5.8. *From the Proposition 5.2 and as we say at the beginning of this section, this is obvious that $C_0' = \emptyset$ and thus $C_1' =]0, +\infty[$. And from Proposition 4.2, item 1, we can deduce that $C_4' = \emptyset$. Thus that $C_2' \cup C_3' =]-\infty, 0]$.*

Lemma 5.9. *Let $c > 0$ and $b > \frac{3}{2}$, then the problem $(P_{\beta;a,b,0})$ has no nonpositive convex-concave solution.*

Proof. Let $c > 0$, $b > \frac{3}{2}$ and f_c be a nonpositive convex-concave solution of the problem $(P_{\beta;a,b,0})$, this implies that the function H_2 is increasing on $[0, +\infty)$, we have $H_2(0) \leq H_2(t)$. It follows that

$$\forall t \in [0, +\infty), \quad 3c^2 + \beta b^2(2b - 3) < H_2(t).$$

Thanks to Proposition 4.2, item 2 and 3, $H_2(+\infty) = 0$, a contradiction. □

Remark 5.10. *If $c \in C_1'$, $b \geq \frac{3}{2}$ and $f_c'(+\infty) = 0$, then the solution f_c changes the sign.*

5.2. The case $a > 0$

Proposition 5.11. *Let $1 < \beta \leq 2$, $b \geq 0$. There exists $c \geq c^*$ such that the problem $(P_{\beta;a,b,1})$ has infinitely many solutions.*

Proof. Let $1 < \beta \leq 2$ and $c \in \mathbb{R}$, we assume for contradiction that f' does not tend to 1, from the Proposition 5.2, it follows that, either f' tend to 0 or there exists $t_0 \in [0, T_c)$, such that $f_c'(t_0) = 0$. Hence in the first case and from the Proposition 4.2, item 4, the function H_3 is increasing on $[0, +\infty)$, we have for all $t > 0$, $H_3(0) < H_3(t)$, and from Proposition 4.2 item 6, we get $H_3(+\infty) = -\beta\ell^2$, where ℓ is the limit of f_c at infinity, it follows that $2ac - b^2 + (2b - \beta)a^2 \leq 0$, we obtain $c \leq \frac{b^2 + (\beta - 2b)a^2}{2a}$. For the second case the function H_3 is also increasing on $[0, t_0)$, thus $H_3(0) < H_3(t_0) < 0$, then we obtain the results as above. □

5.2.1. ***The case*** $0 \leq b < 1$

Let us divide the sets C_0 into the following two subsets:

$$C_{0.1} = \{c \in C_0;\ f_c'(t) \to 0 \text{ as } \ t \to +\infty\},$$

$$C_{0.2} = \{c \in C_0;\ f_c'(t) \to 1 \text{ as } \ t \to +\infty\}.$$

Proposition 5.12. *Let $c \in C_{1.1} \cup C_2$. Then there exists $c_* < -b\sqrt{\frac{\beta(3-2b)}{3}}$, such that $c < c_*$.*

Proof. Let $c \in C_{1.1} \cup C_2$, then f_c and f_c' are positive on $[0, +\infty)$, if $c \in C_{1.1}$, thus the function H_2 is decreasing on $[0, +\infty)$, we have $H_2(0) > H_2(+\infty)$, we get $3c^2 + \beta b^2(2b-3) > 0$, we obtain the result. And for $c \in C_2$, the function H_2 is also increasing on $[0, t_0)$, thus $H_2(0) < H_2(t_0)$, where t_0 be the point such that $f_c'(t_0) = 0$, we obtain the results. □

Remark 5.13. *From the previous proposition, the problem $(P_{\beta;a,b,1})$ has infinitely many solutions which changes the concavity, and thus $[c_*, 0) \subset C_{0.2}$.*

Let us divide the sets C_3 into the following subsets:

$$C_{3.1} = \{c \in C_3;\ f_c'(t) \to 1 \text{ as } \ t \to +\infty\}$$

$$C_{3.2} = \{c \in C_3;\ f_c'(t) \to 0 \text{ as } \ t \to +\infty\}$$

$$C_{3.3} = \{c \in C_3;\ \exists s_c \in [0, T_c) \text{ s.t } f_c' > 0 \text{ on } [0, s_c) \text{ and } f_c'(s_c) = 0\}$$

Proposition 5.14. *Let $1 < \beta \leq 2$ and $c \in C_{0.1} \cup C_1 \cup C_{3.2} \cup C_{3.3}$, then there exists c^* such that $c < c^*$.*

Proof. Let $1 < \beta \leq 2$ and $c \in C_{0.1} \cup C_1 \cup C_{3.2} \cup C_{3.3}$. From the Proposition 4.2 item 2 and 3, if $c \in C_1 \cup C_{3.3}$, since f_c' and f_c are positive on $[0, s_c)$, hence the function H_3 is increasing on $[0, s_c)$, we have $2ac - b^2 + (2b-\beta)a^2 \leq 0$, which implies that $c < \frac{b^2+(\beta-2b)a^2}{2a}$. And from Proposition 4.2 item 4 and 6, if $c \in C_{0.1} \cup C_{3.2}$, we have H_3 is increasing on $[0, +\infty)$, we obtain the results. □

5.2.2. ***The case*** $b > 1$

Proposition 5.15. *Let $1 < \beta \leq 2$, then the problem $(P_{\beta;a,b,1})$ has infinitely many solutions which changes the convexity if $c > 0$.*

Proof. see [4] and from Remark 5.8 the solution of the problem $(P_{\beta;a,b,1})$ changes the convexity if $c \in C_1'$. □

Lemma 5.16. *Let $c \in C_1'$ and if there exists $t_0 > s_c$ such that $f_c'(t_0) = 0$, or $f_c'(t) \to 0$ as $t \to +\infty$, then $c > -c_*$.*

Proof. Let $c \in C_1'$ and t_0 such that $f_c'(t_0) = 0$. Thus the function H_2 is decreasing on $[0, t_0)$, hence $3c^2 + \beta b^2(2b-3) > 3f_c''^2(t_0) > 0$. And the case if we have $f_c'(+\infty) = 0$, from the proposition 4.2 item 2 and 4, we obtain the results. □

All this, from the previous Propositions we have the following theorem.

Theorem 5.17. *Let $\beta > 1$ and $b \geq 0$.*

1. *If $a < 0$ and $b \geq \frac{3}{2}$, the boundary value problem $(P_{\beta;a,b,0})$ has no nonpositive convex-concave solution on $[0, +\infty)$.*
2. *For all $a \in \mathbb{R}$, the boundary value problem $(P_{\beta;a,b,1})$ has infinitely many solutions on $[0, +\infty)$.*

Declarations

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