Letters in Nonlinear Analysis and its Applications 3 (2025) No. 3, 139–150
Available online at www.lettersinnonlinearanalysis.com
Research Article

Letters in Nonlinear Analysis and its Applications
Peer Review Scientific Journal
ISSN: 2958-874x

Remark on some non-uniformly nonlinear elliptic equations

Ahmed Youssfi

University Sidi Mohamed Ben Abdellah, Laboratory of Applied Sciences and Innovative Technologies, National School of Applied Sciences, My Abdellah Avenue, Road Imouzer, P.O. Box 72, Fès-Principale 30 000 Fez, Morocco

Abstract

We consider the Dirichlet problem for a class of nonlinear elliptic equations with degenerate coercivity whose model is

$$\operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\right)=\operatorname{div}F,$$

with $0<\theta<1$ and $|F|\in L^s(\Omega)$. When θ is sufficiently close to 1, we prove that the solutions are not in Sobolev spaces.

Keywords: Nonlinear elliptic equations, Degenerate coercivity, Regularity, A priori estimates, Rearrangement.
2010 MSC: 46E30; 46E35; 35J60; 35J70

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, with $N\geq 2$, and p a real such that $1<p<N$. We consider the following problem

$$\begin{aligned}-\operatorname{div} a(x,u,\nabla u)&=-\operatorname{div}F \quad \text{in } \Omega,\\ u&=0 \quad \text{on } \partial\Omega,\end{aligned} \tag{1}$$

Email address: `ahmed.youssfi@usmba.ac.ma` (Ahmed Youssfi)

Received: January 07, 2025, Accepted: April 28, 2025 Online: May 3, 2025

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function that satisfied the following assumptions for almost every $x \in \Omega$, for every $s \in \mathbb{R}$ and for every ξ, ξ' in $\mathbb{R}^N$ with $\xi \neq \xi'$:

$$a(x, s, \xi) \cdot \xi \geq h^{p-1}(|s|)|\xi|^p \tag{2}$$

where for $t \in \mathbb{R}$, $h(t) = \frac{1}{(1+t)\theta}$ with $0 \leq \theta < 1$,

$$|a(x, s, \xi)| \leq \beta(a_0(x) + |s|^{p-1} + |\xi|^{p-1}) \tag{3}$$

where $\beta > 0$, a_0 is a non negative function in $L^{p'}(\Omega)$ with $p' = \frac{p}{p-1}$, and

$$(a(x, s, \xi) - a(x, s, \xi')) \cdot (\xi - \xi') > 0. \tag{4}$$

Regularity results for problems like (1) have been established by many authors when the function h is constant. The linear case with $p = 2$ has been studied by G. Stampacchia in [14, 15]. The nonlinear case with $1 < p < N$ has been investigated in [7, 8, 13]. For a solution u of (1), it is shown that if $|F|$ belongs to $L^s(\Omega)$ with $s < \frac{N}{p-1}$ then u belongs to $L^{(s(p-1))^*}(\Omega)$, while it is in $L^\infty(\Omega)$ if $s > \frac{N}{p-1}$. The limit case yields u in the Orlicz space $L_\phi(\Omega)$ generated by the N-function $\phi(t) = \exp(|t|^{\frac{N}{N-1}}) - 1$.

When h is not necessarily a constant function ($0 \leq \theta \leq 1$), the problem (1) was studied in [5] and in [19] where an L^∞ result was obtained for solutions of its parabolic counter-part. It has been established in [5] that if $|F|$ belongs to $L^s(\Omega)$, the problem (1) admits a solution u such as :

1. $s > \frac{N}{p-1} \quad \Rightarrow \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$,
2. $s = \frac{N}{p-1} \quad \Rightarrow \quad u \in W_0^{1,p(1-\theta)}(\Omega) \cap L_{\phi_{N,\theta}}(\Omega)$,
3. $\frac{Np'}{N-\theta(N-p)} \leq s < \frac{N}{p-1} \quad \Rightarrow \quad u \in W_0^{1,p}(\Omega) \cap L^r(\Omega)$,
4. $\max\left(p', \frac{Np'}{p((1-\theta)N+\theta)}\right) \leq s < \frac{Np'}{N-\theta(N-p)} \quad \Rightarrow \quad u \in W_0^{1,q}(\Omega) \cap L^r(\Omega)$,

where $L_{\phi_{N,\theta}}(\Omega)$ is the Orlicz space generated by the N-function

$$\phi_{N,\theta}(t) = \exp\left(t^{\frac{N(1-\theta)}{N-1}}\right) - 1, \quad p' = \frac{p}{p-1}, \quad r = \frac{(1-\theta)Ns(p-1)}{N-s(p-1)} \text{ and } q = \frac{(1-\theta)Ns(p-1)}{N-\theta s(p-1)}.$$

The following figure summarizes these different regularity results in light of the different areas to which the pairs (θ, s) belong.

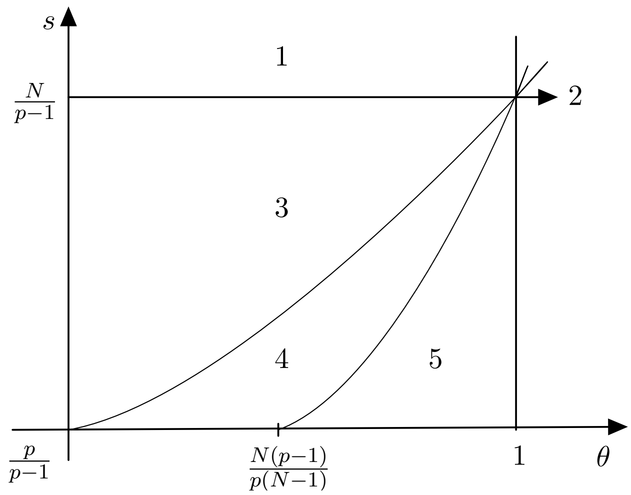

Figure 1: Different areas of regularity of solutions.

Notice that when θ is close to 1, precisely $\frac{N(p-1)}{p(N-1)} < \theta < 1$, we can decrease the summability of $|F|$, to cover the pairs (θ, s) belonging to the zone 5 which completes the study of the regularity of solutions of the problem (1) started in [5].

For $k > 0$, let $T_k : \mathbb{R} \to \mathbb{R}$ be the truncation function at levels $\pm k$ defined by

$$T_k(s) = \max(-k, \min(k, s)).$$

The main result is stated as follows.

Theorem 1.1. *Suppose that $\frac{N(p-1)}{p(N-1)} < \theta < 1$ and (2), (3) and (4) hold true. Let $|F| \in L^s(\Omega)$ with $p' < s < \frac{Np'}{p((1-\theta)N+\theta)}$. Then, there exists a measurable function u such that:*

$$u \in \mathcal{M}^r(\Omega) \text{ and } |\nabla u| \in \mathcal{M}^q(\Omega)$$

with

$$r = \frac{(1-\theta)Ns(p-1)}{N - s(p-1)} \text{ and } q = \frac{(1-\theta)Ns(p-1)}{N - \theta s(p-1)},$$

solution of (1) in the sense that

$$\begin{cases} T_k(u) \in W_0^{1,p}(\Omega) \\ \displaystyle\int_\Omega a(x, u, \nabla u) \cdot \nabla T_k(u - v)\, dx \le \int_\Omega F \cdot \nabla T_k(u - v)\, dx \end{cases} \tag{5}$$

for every $k > 0$ and for every v in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

We point out that the results in Theorem 1.1 cover these in [1, Theorem] and vice-versa. The result we provide here for a datum in divergence form is exactly the one obtained in [1, Theorem 1.9] for a source term in suitable Lebesgue spaces. Indeed, going back to [1, Theorem 1.9], and pick a function f in $L^m(\Omega)$. By duality arguments f can be written as $f = -\mathrm{div}(F)$, where $|F| \in L^s(\Omega)$ with $s = \frac{Nm}{N-m}$. Thus, by [1, Theorem 1.9] we obtain the results in Theorem 1.1. Reciprocally, let us put ourselves in the conditions of Theorem 1.1 with $|F| \in L^s(\Omega)$. Choosing a function $f \in L^m(\Omega)$ with $m = \frac{Ns}{N+s}$. The exponent m is such that $1 < m < \frac{N}{(p-1)(N(1-\alpha)+\alpha)+1}$. So that one can clearly recover the regularity results in [1, Theorem 1.9].

Remark 1.2. *We emphasize that $\frac{N(p-1)}{p(N-1)} < \theta$ implies that the range of s is nonempty.*

We underline that the gradient of the function u which appears in (5) is defined in [3, lemma 2.1] as the unique measurable function $v : \Omega \to \mathbb{R}^N$ satisfying

$$\nabla T_k(u) = v\chi_{\{|u|<k\}}, \text{ for almost every } x \in \Omega, \ \forall k > 0,$$

where χ_E is the characteristic function of a measurable set E of Ω. Moreover, if $u \in W_0^{1,1}(\Omega)$ then v coincides with usual distributional gradient of u. Notice that the use of $T_k(u - v)$ as test function yields a meaning to each term in (5), although ∇u does not belong to $(L^{p'}(\Omega))^N$. In fact, both integrals of (5) are only on the set $|u - v| \le k$, and on this set $|u| \le k + \|v\|_{L^\infty(\Omega)} = M$. Therefore, since assumption (2) implies that $a(x, s, 0) = 0$, we have

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla T_k(u - v)\, dx = \int_\Omega a(x, T_M(u), \nabla T_M(u)) \cdot \nabla T_k(u - v)\, dx,$$

which finite by the growth condition (3) and

$$\int_\Omega F \cdot \nabla T_k(u - v)\, dx = \int_\Omega F \cdot \nabla T_k(T_M(u) - v)\, dx$$

which is finite thanks to Hölder's inequality since $\nabla T_k(T_M(u)-v)$ belongs to $(L^p(\Omega))^N$.

It is worth recalling that when h is not necessarily a constant function, there is a difficulty in dealing with problem (1). Note that since no bounds are assumed on the function h, the operator $-\operatorname{div} a(x,u,\nabla u)$ acting from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ may degenerates when its second argument u has large values and hence it is not coercive. As a consequence, the classical theory of existence of solutions for (1) can not be applied. To overcome this problem, we consider approximate equations in which we introduce a truncation.

2. Preliminary results

Let Ω be a bounded open set of $\mathbb{R}^N$. If u is a measurable function in Ω, we denote by $\mu_u(t)$ its distribution function, that is

$$\mu_u(t) = |\{x\in\Omega : |u(x)|>t\}|, \qquad t\geq 0,$$

where $|E|$ denotes the Lebesgue measure of a measurable subset E of $\mathbb{R}^N$.
The decreasing rearrangement u^* of u is defined by

$$u^*(s) = \inf\{t\geq 0 : \mu_u(t)\leq s\} \qquad \text{for } \ s\in[0,|\Omega|].$$

We refer to [4, 16, 17, 18] for a detailed exposition of basic facts on rearrangements.

For $0<q<+\infty$, the Marcinkiewicz space $\mathcal{M}^q(\Omega)$ consists of all measurable functions $u:\Omega\to\mathbb{R}$ such that for all $t>0$

$$\|u_n\|_{\mathcal{M}^q(\Omega)} := t^q\mu_u(t)\leq c,$$

for some constant $c>0$. We observe that this condition is equivalent to say that

$$\tau^{\frac{1}{q}}u^*(\tau)\leq c$$

for all $\tau\in]0,|\Omega|[$ and for some constant $c>0$. Recall that the quantity $\|.\|_{\mathcal{M}^q(\Omega)}$ does not define a norm on $\mathcal{M}^q(\Omega)$ since the triangle inequality is not satisfied (see [11]). We also recall the following connection between Marcinkiewicz and Lebesgue spaces (see, for instance, [11])

$$L^q(\Omega)\subset\mathcal{M}^q(\Omega)\subset L^r(\Omega)$$

for $0<r<q$. Indeed for the last inclusion, if $|\{x\in\Omega : |u(x)|>t\}|\leq Ct^{-q}$ for all $t\geq 1$, then we have

$$\begin{aligned}\int_\Omega |u|^r dx &= \int_0^\infty |\{x\in\Omega : |u(x)|^r>t\}|dt\\ &\leq |\Omega| + C\int_1^\infty t^{-\frac{q}{r}}dt\\ &= |\Omega| + \frac{rC}{q-r}.\end{aligned}$$

Let us point out that in the literature, Marcinkiewicz spaces are also known as weak-Lebesgue spaces. The following lemma (see [1]) provides a sufficient condition for a measurable function to be in a Marcinkiewicz space.

Lemma 2.1. *Let u be a measurable function belonging to $\mathcal{M}^r(\Omega)$ for some $r>0$, such that, for every $k>0$, $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$, $p>1$. Suppose that*

$$\int_\Omega |\nabla T_k(u)|^p dx\leq ck^\lambda, \qquad \forall k>k_0,$$

for some non-negative λ, c and k_0. Then $|\nabla u|$ belongs to $\mathcal{M}^{\frac{rp}{r+\lambda}}(\Omega)$.

3. A priori estimates

The proof is based on approximation introducing truncations. Let $n \in \mathbb{N}$, we define the operator A_n by

$$A_n(u) := -\mathrm{div}\, a(\cdot, T_n(u), \nabla u).$$

From (2), we have

$$\int_\Omega a(x, T_n(u), \nabla u) \cdot \nabla u \, dx \geq h(n) \int_\Omega |\nabla u|^p \, dx,$$

so that A_n is coercive and satisfies the classical Leray-Lions conditions. It follows from [12], that A_n is surjective from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$. Since $s > p'$, the term $\mathrm{div}F$ is an element of $W^{-1,p'}(\Omega)$. Therefore, there exists at least one solution u_n in $W_0^{1,p}(\Omega)$ of

$$\begin{aligned} -\,\mathrm{div}\ a(x, T_n(u_n), \nabla u_n) &= -\mathrm{div} F \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{6}$$

in the sense that

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla v \, dx = \int_\Omega F \cdot \nabla v \, dx \tag{7}$$

for every v in $W_0^{1,p}(\Omega)$.

Theorem 3.1. *Let $|F| \in L^s(\Omega)$ with $p' < s < \frac{Np'}{p((1-\theta)N+\theta)}$ and let u_n be a solution of (6). Then*

$$u_n \in \mathcal{M}^r(\Omega) \ \ with \ \ r = \frac{(1-\theta)Ns(p-1)}{N - s(p-1)}, \tag{8}$$

and

$$\|u_n\|_{\mathcal{M}^r(\Omega)} \leq \left(\frac{1-\theta}{NC_N^{\frac{1}{N}}} \|F\|_{(L^s(\Omega))^N}^{\frac{p'}{p}} \left(\frac{N(s(p-1)-1)}{N - s(p-1)} \right)^{1-\frac{p'}{ps}} + |\Omega|^{\frac{1-\theta}{r}} \right)^{\frac{r}{1-\theta}},$$

where C_N is the measure of the unit ball in $\mathbb{R}^N$. Furthermore we have

$$|\nabla u_n| \in \mathcal{M}^q(\Omega) \ \ with \ \ q = \frac{(1-\theta)Ns(p-1)}{N - \theta s(p-1)}, \tag{9}$$

and

$$\begin{aligned} \|\nabla u_n\|_{\mathcal{M}^q(\Omega)} \ &\leq \ \|F\|_{(L^s(\Omega))^N}^{p'} (|\Omega|^{\frac{1}{r}} + c)^{\theta p} \left(\frac{r(s-p')}{r(s-p') - \theta ps} \right)^{\frac{s-p'}{s}} \\ &\quad \times \left(|\Omega|^{\frac{r(s-p')-\theta ps}{rs}} + c^{\frac{r(s-p')-\theta ps}{rs}} \right). \end{aligned}$$

Proof. For $\epsilon > 0$ and $t > 0$, we use in the formulation of solution (7) the test function $v = T_\epsilon(u_n - T_t(u_n))$, which belongs to $W_0^{1,p}(\Omega)$, obtaining

$$\int_{\{t<|u_n|\leq t+\epsilon\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx = \int_{\{t<|u_n|\leq t+\epsilon\}} F \cdot \nabla u_n \, dx,$$

where $\{t < |u_n| \leq t + \epsilon\}$ denotes the set $\{x \in \Omega : t < |u_n(x)| \leq t + \epsilon\}$.
Assumption (2) yields

$$h^{p-1}(t+\epsilon) \int_{\{t<|u_n|\leq t+\epsilon\}} |\nabla u_n|^p dx \leq \int_{\{t<|u_n|\leq t+\epsilon\}} F \cdot \nabla u_n dx.$$

Since F belongs at least to $(L^{p'}(\Omega))^N$, Hölder's inequality gives

$$h^{p-1}(t+\epsilon)\int_{\{t<|u_n|\le t+\epsilon\}}|\nabla u_n|^p dx \le \left(\int_{\{t<|u_n|\le t+\epsilon\}}|F|^{p'}dx\right)^{\frac{1}{p'}}\left(\int_{\{t<|u_n|\le t+\epsilon\}}|\nabla u_n|^p dx\right)^{\frac{1}{p}}.$$

Since $s>p'$, we apply again Hölder's inequality to get

$$h^{p}(t+\epsilon)\int_{\{t<|u_n|\le t+\epsilon\}}|\nabla u_n|^p dx \le \left(\int_{\{t<|u_n|\le t+\epsilon\}}|F|^{s}dx\right)^{\frac{p'}{s}}|\{t<|u_n|\le t+\epsilon\}|^{1-\frac{p'}{s}}.$$

Dividing both sides of the previous inequality by ϵ and then letting ϵ tends to 0^+, being h continuous, it follows that for almost every $t>0$

$$h^p(t)\left(-\frac{d}{dt}\int_{\{|u_n|>t\}}|\nabla u_n|^p dx\right)\le\left(-\frac{d}{dt}\int_{\{|u_n|>t\}}|F|^s dx\right)^{\frac{p'}{s}}(-\mu_n'(t))^{1-\frac{p'}{s}}, \tag{10}$$

where $\{|u_n|>t\}$ denotes the set $\{x\in\Omega : |u_n(x)|>t\}$ and $\mu_n(t)$ stands for the distribution function of u_n, that is $\mu_n(t)=|\{x\in\Omega : |u_n(x)|>t\}|$.
On the other hand, from Fleming-Rishel coarea formula (see [10]) and isoperimetric inequality (see [9]), we have for almost every $t>0$

$$NC_N^{\frac{1}{N}}(\mu_n(t))^{\frac{N-1}{N}}\le -\frac{d}{dt}\int_{\{|u_n|>t\}}|\nabla u_n|dx, \tag{11}$$

where C_N is the measure of the unit ball in $\mathbb{R}^N$. Using the Hölder inequality we obtain that for almost every $t>0$

$$-\frac{d}{dt}\int_{\{|u_n|>t\}}|\nabla u_n|dx\le\left(-\frac{d}{dt}\int_{\{|u_n|>t\}}|\nabla u_n|^p dx\right)^{\frac{1}{p}}(-\mu_n'(t))^{1-\frac{1}{p}}. \tag{12}$$

Then, combining (10), (11) and (12) we obtain that for almost every $t>0$

$$h(t)\le\frac{1}{NC_N^{\frac{1}{N}}}\left(-\frac{d}{dt}\int_{\{|u_n|>t\}}|F|^s dx\right)^{\frac{p'}{ps}}\frac{(-\mu_n'(t))^{1-\frac{p'}{ps}}}{(\mu_n(t))^{\frac{N-1}{N}}},$$

Integrating between 0 and $\tau>0$ both sides of this inequality and then using Hölder's inequality (since $ps>p'$) with exponents $\frac{ps}{p'}$ and $\frac{ps}{ps-p'}$, we obtain

$$\begin{aligned}&H(\tau)\\&\le\frac{1}{NC_N^{\frac{1}{N}}}\left(\int_0^\tau\left(-\frac{d}{dt}\int_{\{|u_n|>t\}}|F|^s dx\right)dt\right)^{\frac{p'}{ps}}\left(\int_0^\tau\frac{-\mu_n'(t)}{\mu_n(t)^{\frac{N-1}{N}\frac{ps}{ps-p'}}}dt\right)^{1-\frac{p'}{ps}},\end{aligned}$$

where $H(s)=\int_0^s h(t)dt$. Taking into account that $|F|$ belongs to $L^s(\Omega)$, and making a change of variables in the last integral, we get

$$H(\tau)\le\frac{1}{NC_N^{\frac{1}{N}}}\|F\|^{\frac{p'}{p}}_{(L^s(\Omega))^N}\left(\int_{\mu_n(\tau)}^{|\Omega|}\frac{d\sigma}{\sigma^{\frac{N-1}{N}\frac{ps}{ps-p'}}}\right)^{1-\frac{p'}{ps}}.$$

A straightforward calculation of the integral in the right-hand side and the fact that $\tau^{1-\theta} \leq (1-\theta)H(\tau)+1$, for $\tau > 0$, allow us to have

$$\tau^{1-\theta} \leq \frac{1-\theta}{NC_N^{\frac{1}{N}}} \|F\|_{(L^s(\Omega))^N}^{\frac{p'}{p}} \left(\frac{N(s(p-1)-1)}{N-s(p-1)}\right)^{1-\frac{p'}{ps}} \mu_n(\tau)^{-\frac{N-s(p-1)}{Ns(p-1)}} + 1,$$

which gives

$$\begin{aligned} &\tau^{\frac{(1-\theta)Ns(p-1)}{N-s(p-1)}} \mu_n(\tau) \\ &\leq \left(\frac{1-\theta}{NC_N^{\frac{1}{N}}} \|F\|_{(L^s(\Omega))^N}^{\frac{p'}{p}} \left(\frac{N(s(p-1)-1)}{N-s(p-1)}\right)^{1-\frac{p'}{ps}} + |\Omega|^{\frac{N-s(p-1)}{Ns(p-1)}}\right)^{\frac{Ns(p-1)}{N-s(p-1)}}. \end{aligned}$$

This means that

$$u_n \in \mathcal{M}^r(\Omega) \text{ with } r = \frac{(1-\theta)Ns(p-1)}{N-s(p-1)}.$$

We now turn to the estimation of the gradients of solutions u_n. Going back to (10), dividing by $h^p(t)$, integrating both sides of the inequality between 0 and k, $(k \geq 1)$, and then using Hölder's inequality we obtain

$$\begin{aligned} \int_{\{|u_n|\leq k\}} |\nabla u_n|^p dx &\leq \left(\int_{\{|u_n|\leq k\}} |F|^s dx\right)^{\frac{p'}{s}} \left(\int_0^k \frac{-\mu_n'(t)}{h^{\frac{ps}{s-p'}}(t)} dt\right)^{1-\frac{p'}{s}} \\ &\leq \|F\|_{(L^s(\Omega))^N}^{p'} \left(\int_{\mu_n(k)}^{|\Omega|} (1+u_n^*(\sigma))^{\frac{\theta ps}{s-p'}} d\sigma\right)^{1-\frac{p'}{s}}. \end{aligned}$$

Since $u_n \in \mathcal{M}^r(\Omega)$, there exists a constant $c > 0$ such that $t^{\frac{1}{r}} u_n^*(t) \leq c$. Hence,

$$\begin{aligned} \int_{\{|u_n|\leq k\}} |\nabla u_n|^p dx &\leq \|F\|_{(L^s(\Omega))^N}^{p'} \left(\int_{\mu_n(k)}^{|\Omega|} (1+c\sigma^{-\frac{1}{r}})^{\frac{\theta ps}{s-p'}} d\sigma\right)^{1-\frac{p'}{s}} \\ &\leq \|F\|_{(L^s(\Omega))^N}^{p'} (|\Omega|^{\frac{1}{r}}+c)^{\theta p} \left(\int_{\mu_n(k)}^{|\Omega|} \sigma^{-\frac{\theta ps}{s-p'}} d\sigma\right)^{1-\frac{p'}{s}} \\ &= C_1 \left(|\Omega|^{\frac{r(s-p')-\theta ps}{rs}} - \mu_n^{\frac{r(s-p')-\theta ps}{rs}}(k)\right). \end{aligned}$$

where $C_1 = \|F\|_{(L^s(\Omega))^N}^{p'} (|\Omega|^{\frac{1}{r}}+c)^{\theta p} \left(\frac{r(s-p')}{r(s-p')-\theta ps}\right)^{\frac{s-p'}{s}}$. Again, since $u_n \in \mathcal{M}^r(\Omega)$ we have $\mu_n(k) \leq \dfrac{c}{k^r}$. Therefore, we obtain

$$\int_{\{|u_n|\leq k\}} |\nabla u_n|^p dx \leq C_1 \left(|\Omega|^{\frac{r(s-p')-\theta ps}{rs}} + c^{\frac{r(s-p')-\theta ps}{rs}} k^{\lambda}\right),$$

where $\lambda = \theta p - r\left(1 - \dfrac{p'}{s}\right)$. The assumption made on s ensures that $\lambda > 0$.
Then, for every $k \geq 1$ we obtain

$$\int_{\{|u_n|\leq k\}} |\nabla u_n|^p dx \leq C_2 k^{\lambda},$$

where

$$C_2 = C_1 \left(|\Omega|^{\frac{r(s-p')-\theta ps}{rs}} + c^{\frac{r(s-p')-\theta ps}{rs}}\right).$$

A straightforward calculation shows that

$$r+\lambda = p\frac{r}{q}.$$

An application of Lemma 2.1 implies that

$$|\nabla u_n| \in \mathcal{M}^q(\Omega).$$

□

Lemma 3.2. *Let u_n be a solution of (6). Then, there exists a measurable function u such that:*

$$u_n \to u \quad \textit{in measure and a.e. in } \Omega \tag{13}$$

$$T_k(u_n) \rightharpoonup T_k(u) \quad \textit{weakly in } W_0^{1,p}(\Omega),\ \forall k>0. \tag{14}$$

Proof. Let $k>0$. The use of $T_k(u_n)$ as test function in (7) yields

$$\int_\Omega |\nabla T_k(u_n)|^p dx \le \|F\|^{p'}_{(L^s(\Omega))^N} |\Omega|^{1-\frac{p'}{s}}(1+k)^{\theta p}.$$

So that $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ independently of n. Therefore, there exists (see [3]) a measurable function u such that, up to a subsequence,

$$u_n \to u \quad \text{in measure and a.e. in } \Omega$$

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega).$$

□

4. Almost everywhere convergence of the gradients of solutions

Arguing as in [1], we shall prove the almost everywhere convergence of the gradients. Let $0<\sigma<\dfrac{q}{2p}(<1)$. Consider

$$I_n = \int_\Omega \{(a(x,T_n(u_n),\nabla u_n) - a(x,T_n(u_n),\nabla u))\cdot(\nabla u_n - \nabla u)\}^\sigma dx.$$

We split I_n on the sets

$$A_k = \{x\in\Omega : |u(x)|>k\}$$

and

$$C_k = \{x\in\Omega : |u(x)|\le k\},$$

obtaining

$$I_n = I_1(n,k) + I_2(n,k)$$

where

$$I_1(n,k) = \int_{A_k} \{(a(x,T_n(u_n),\nabla u_n) - a(x,T_n(u_n),\nabla u))\cdot(\nabla u_n - \nabla u)\}^\sigma dx$$

and

$$I_2(n,k) = \int_{C_k} \{(a(x,T_n(u_n),\nabla u_n) - a(x,T_n(u_n),\nabla u))\cdot(\nabla u_n - \nabla u)\}^\sigma dx.$$

Using (3) we deduce that there exists a constant c such that

$$I_1(n,k) \le c\left(\int_{A_k}(a_0(x))^{p'\sigma}dx + \int_{A_k}(|u_n|^{p\sigma} + |\nabla u_n|^{p\sigma} + |\nabla u|^{p\sigma})dx\right).$$

Let $\tau = \dfrac{q}{2p\sigma}$. By Hölder's inequality we obtain

$$\begin{aligned}&I_1(n,k)\\&\leq c\left(\int_\Omega (a_0(x))^{p'}dx\right)^{\sigma}|A_k|^{1-\sigma}\\&+c\left(\left(\int_\Omega |u_n|^{\frac{q}{2}}dx\right)^{\frac{1}{\tau}}+\left(\int_\Omega |\nabla u_n|^{\frac{q}{2}}dx\right)^{\frac{1}{\tau}}+\left(\int_\Omega |\nabla u|^{\frac{q}{2}}dx\right)^{\frac{1}{\tau}}\right)|A_k|^{1-\frac{1}{\tau}}.\end{aligned}$$

Note that Lemma 3.2 and Lemma 2.1 enable us to get $|\nabla u|\in\mathcal{M}^q(\Omega)\subset L^{\frac{q}{2}}(\Omega)$. From (8) and (9) we obtain

$$I_1(n,k)\leq c\left(|A_k|^{1-\sigma}+|A_k|^{1-\frac{1}{\tau}}\right),$$

where c is a constant not depending on n. Therefore, we get

$$\lim_{k\to\infty}\lim_{n\to\infty} I_1(n,k)=0. \tag{15}$$

Observe that on C_k one has $T_k(u)=u$ and since the integrand function is positive we have

$$I_2(n,k)\leq I_3(n,k)$$

where

$$I_3(n,k)=\int_\Omega \Big\{(a(x,T_n(u_n),\nabla u_n)-a(x,T_n(u_n),\nabla T_k(u)))\cdot(\nabla u_n-\nabla T_k(u))\Big\}^{\sigma}dx.$$

We fix $j>0$ and split the integral in $I_3(n,k)$ on the sets $\{|u_n-T_k(u)|>j\}$ and $\{|u_n-T_k(u)|\leq j\}$, obtaining

$$I_3(n,k)=I_4(n,k,j)+I_5(n,k,j),$$

where

$$\begin{aligned}I_4(n,k,j)= &\int_{\{|u_n-T_k(u)|>j\}}\Big\{(a(x,T_n(u_n),\nabla u_n)-a(x,T_n(u_n),\nabla T_k(u)))\\&\cdot(\nabla u_n-\nabla T_k(u))\Big\}^{\sigma}dx\end{aligned}$$

and

$$\begin{aligned}I_5(n,k,j)= &\int_{\{|u_n-T_k(u)|\leq j\}}\Big\{(a(x,T_n(u_n),\nabla u_n)-a(x,T_n(u_n),\nabla T_k(u)))\\&\cdot(\nabla u_n-\nabla T_k(u))\Big\}^{\sigma}dx\end{aligned}$$

By (8) the measure of the set $\{|u_n-T_k(u)|>j\}$ tends to zero as j tends to ∞ uniformly in n and k. So that, reasoning as for $I_1(n,k)$ we obtain

$$\lim_{j\to\infty}\lim_{k\to\infty}\lim_{n\to\infty} I_4(n,k,j)=0. \tag{16}$$

Applying the Hölder inequality with exponents $\dfrac{1}{\sigma}$ and $\dfrac{1}{1-\sigma}$, we get

$$\begin{aligned}I_5(n,k,j)= &\int_\Omega \Big\{(a(x,T_n(u_n),\nabla u_n)-a(x,T_n(u_n),\nabla T_k(u)))\cdot\nabla T_j(u_n-T_k(u))\Big\}^{\sigma}dx\\&\leq |\Omega|^{1-\sigma}\left(I_6(n,k,j)-I_7(n,k,j)\right)^{\sigma},\end{aligned}$$

where

$$I_6(n,k,j) = \int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_j(u_n - T_k(u))dx$$

and

$$I_7(n,k,j) = \int_\Omega a(x,T_n(u_n),\nabla T_k(u))\cdot\nabla T_j(u_n - T_k(u))dx.$$

Using $T_j(u_n - T_k(u))$ as test function in (7), we obtain

$$I_6(n,k,j) = \int_\Omega F\cdot\nabla T_j(u_n - T_k(u))dx.$$

Note that on the set $\{|u_n - T_k(u)| \le j\}$ we have $|u_n| \le k+j$. Thus, by (14) we get

$$\begin{aligned}\lim_{n\to\infty} I_6(n,k,j) &= \int_\Omega F\cdot\nabla T_j(u - T_k(u))dx\\ &= \int_{\{|u|>k\}} F\cdot\nabla T_j(u)dx.\end{aligned}$$

Thus

$$\lim_{k\to\infty}\lim_{n\to\infty} I_6(n,k,j) = 0. \tag{17}$$

Let $M = k+j$. For $n > M$ one has

$$I_7(n,k,j) = \int_\Omega a(x,T_M(u_n),\nabla T_k(u))\cdot\nabla T_j(u_n - T_k(u))dx.$$

By (13) and Viltali's theorem we obtain that

$$a(x,T_M(u_n),\nabla T_k(u)) \to a(x,T_M(u),\nabla T_k(u))$$

strongly in $(L^{p'}(\Omega))^N$ as n tends to $+\infty$. It follows by (14) that

$$\begin{aligned}\lim_{n\to\infty} I_7(n,k,j) &= \int_\Omega a(x,T_M(u),\nabla T_k(u))\cdot\nabla T_j(u - T_k(u))dx\\ &= \int_{\{|u|>k\}} a(x,u,0)\cdot\nabla T_j(u)dx\\ &= 0.\end{aligned} \tag{18}$$

Combining (15), (16), (17) and (18) we obtain

$$\lim_{n\to\infty} I_n = 0.$$

Since the integrand function in I_n is positive, we have

$$\{(a(x,T_n(u_n),\nabla u_n) - a(x,T_n(u_n),\nabla u))\cdot(\nabla u_n - \nabla u)\}^\sigma \to 0$$

strongly in $L^1(\Omega)$. Hence, there exists a subsequence still indexed by n, such that

$$(a(x,T_n(u_n(x)),\nabla u_n(x)) - a(x,T_n(u_n(x)),\nabla u(x)))\cdot(\nabla u_n(x) - \nabla u(x)) \to 0.$$

for almost every x in Ω. As in [1] we conclude that

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega. \tag{19}$$

5. Passage to the limit

Let $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and let $k > 0$. Taking $T_k(u_n - v)$ as test function in (7) we obtain

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u_n - v)\, dx = \int_\Omega F \cdot \nabla T_k(u_n - v)\, dx.$$

By (14) we have

$$\lim_{n\to\infty} \int_\Omega F \cdot \nabla T_k(u_n - v)\, dx = \int_\Omega F \cdot \nabla T_k(u - v)\, dx.$$

Taking into account that on the set $\{|u_n - v| < k\}$ we have $|u_n| \le k + \|v\|_\infty := M$, we split the left-hand side, for $n > M$, as the sum

$$\begin{aligned}&\int_{\{|u_n - v|\le k\}} a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla T_M(u_n)\, dx\\ &- \int_{\{|u_n - v|\le k\}} a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla v\, dx.\end{aligned}$$

Since $\{a(x, T_M(u_n), \nabla T_M(u_n))\}$ is bounded in $(L^{p'}(\Omega))^N$, by (13) and (19) we have that

$$a(x, T_M(u_n), \nabla T_M(u_n)) \rightharpoonup a(x, T_M(u), \nabla T_M(u))$$

weakly in $(L^{p'}(\Omega))^N$. Thus, we get

$$\begin{aligned}&\lim_{n\to\infty} \int_{\{|u_n - v|\le k\}} a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla T_M(u_n)\, dx\\ &= \int_{\{|u - v|\le k\}} a(x, u, \nabla u) \cdot \nabla v\, dx.\end{aligned}$$

By (2), the integrand function $a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla T_M(u_n)$ is non negative. Therefore, Fatou's lemma allows us to have

$$\begin{aligned}&\int_{\{|u - v|\le k\}} a(x, u, \nabla u) \cdot \nabla u\, dx\\ &\le \liminf_{n\to\infty} \int_{\{|u_n - v|\le k\}} a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla T_M(u_n)\, dx.\end{aligned}$$

Finally we get

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla T_k(u - v)\, dx \le \int_\Omega F \cdot \nabla T_k(u - v)\, dx.$$

Furthermore, we have $u \in \mathcal{M}^r(\Omega)$ and $|\nabla u| \in \mathcal{M}^q(\Omega)$. The proof of the Theorem 1.1 is then achieved.

Remark 5.1. *The case $s = p'$ which can not be considered in Theorem 1.1, has been treated in [1, Theorem 5.1]. The authors proved that the approximation solutions u_n of (6) belong to $L^{\frac{(1-\theta)Np}{N-p}}(\Omega) \subset \mathcal{M}^{\frac{(1-\theta)Np}{N-p}}(\Omega)$.*

To get information about the gradient of u_n we use of $T_k(u_n)$, for $k \ge 1$, as test function in (7) obtaining

$$\frac{1}{(1+k)^{\theta(p-1)}} \int_\Omega |\nabla T_k(u_n)|^p dx \le \int_\Omega F \cdot \nabla T_k(u_n) dx.$$

Hence, by the Hölder inequality we get

$$\int_\Omega |\nabla T_k(u_n)|^p dx \le 2^{\theta p} \|F\|_{p'}^{p'} k^{\theta p}.$$

Therefore, an application of Lemma 2.1 gives

$$|\nabla u_n| \in \mathcal{M}^{\frac{(1-\theta)Np}{N-\theta p}}(\Omega).$$

As above, we get the existence of a measurable function u with the properties (13), (14) and (19). Passing to the limit as previously done, we obtain that u is a solution of (1) in the sense of (5). Moreover, we have $u \in \mathcal{M}^{\frac{(1-\theta)Np}{N-p}}(\Omega)$ and $|\nabla u| \in \mathcal{M}^{\frac{(1-\theta)Np}{N-\theta p}}(\Omega)$.

References

[1] A. Alvino, L. Boccardo, V. Ferone, L. Orsina, G. Trombetti; *Existence results for nonlinear elliptic equations with degenerate coercivity*, Ann. Mat. Pura Appl., IV. Ser. **182**, No.1, (2003), 53-79.

[2] A. Alvino, V. Ferone, G. Trombetti; *A priori estimates for a class of non uniformly elliptic equations*, Atti Semin. Mat. Fis. Univ. Modena **46**-suppl., (1998), 381-391.

[3] P. Bénilan, T. Gallouet, R. Gariepy, M. Pierre, J. L. Vazquez; *An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa, Cl. Sci., IV. Ser. **22**, (1995), 240-273.

[4] C. Bennett, R. Sharpley; *Interpolation of operators*, Academic press, Boston, (1988).

[5] A. Benkirane, A. Youssfi; *Regularity for solutions of nonlinear elliptic equations with degenerate coercivity*, Ricerche Mat., **56**-2, (2007), 241-275.

[6] L. Boccardo, A. Dall'Aglio, L. Orsina; *Existence and regularity results for some elliptic equations with degenerate coercivity*, Atti Semin. Mat. Fis. Univ. Modena **46**-suppl., (1998), 51-81.

[7] L. Boccardo, D. Giachetti; *Alcune osservazioni sulla regolarita delle soluzioni di problemi fortemente non lineari e applicazioni*, Ricerche Mat., **34**, (1985), 309-323.

[8] L. Boccardo, D. Giachetti; *Existence results via regularity for some nonlinear elliptic problems*, Comm. P.D.E., **14**, 5, (1989), 663-680.

[9] E. De Giorgi; *Su una teoria generale della misura $(r-1)-$dimensionale in uno spazio ad r dimensioni*, Ann. Mat. Pura Appl., IV. Ser. **36**, (1954), 191-213.

[10] W. Fleming, R. Rishel; *An integral formula for total gradient variation*, Arch. Math., **11**, (1960), 218-222.

[11] A. Kufner, O. John, B. Opic; *Function spaces*, Academia, Praha, (1977).

[12] J. Leray, J. L. Lions; *Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder*, Bull. Soc. Mat. France, **93**, (1965), 97-107.

[13] E. Giarrusso, D. Nunziante; *Regularity theorems in limit cases for solutions of linear and nonlinear elliptic equations*, Rend. Inst. Mat. Univ. Trieste, **20**, (1988), 39-58.

[14] G. Stampacchia; *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Ist. Fourier, (Grenoble), **15**, (1965), 189-258.

[15] G. Stampacchia; *Équations elliptiques du second ordre à coefficients discontinus*, Montréal, Presses Univ. Montréal, (1966), (Séminaire de Mathématique supérieure, 16).

[16] G. Talenti; *Linear elliptic P.D.E's: Level sets, rearrangements and a priori estimates of solutions*, Boll. Un. Mat. Ital., **4**-B(6), (1985), 917-949.

[17] G. Talenti; *Nonlinear elliptic equations, Rearrangements of functions and Orlicz spaces*, Ann. Mat. Pura Appl., **120**, IV. Ser., (1979), 159-184.

[18] G. Talenti; *Elliptic equations and rearrangements*, Ann. Scuola. Norm. Sup. Pisa, **3**, (4), (1976), 697-718.

[19] A. Youssfi, A. Benkirane, Y. El Hadfi, On bounded solutions for nonlinear parabolic equations with degenerate coercivity, Mediterr. J. Math., **13** (5), (2016), 3029–3040