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Remark on some non-uniformly nonlinear elliptic equations

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Abstract

We consider the Dirichlet problem for a class of nonlinear elliptic equations with degenerate coercivity whose model is

$$\operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\right)=\operatorname{div}F,$$

with $0<\theta<1$ and $|F|\in L^s(\Omega)$. When θ is sufficiently close to 1, we prove that the solutions are not in Sobolev spaces.

Keywords: Nonlinear elliptic equations, Degenerate coercivity, Regularity, A priori estimates, Rearrangement.
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1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, with $N\geq 2$, and p a real such that $1<p<N$. We consider the following problem

$$\begin{aligned} -\operatorname{div}\, a(x,u,\nabla u) &= -\operatorname{div}F \quad \text{in } \Omega,\\ u&=0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

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where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function that satisfied the following assumptions for almost every $x \in \Omega$, for every $s \in \mathbb{R}$ and for every ξ, ξ' in $\mathbb{R}^N$ with $\xi \neq \xi'$:

$$a(x,s,\xi) \cdot \xi \geq h^{p-1}(|s|)|\xi|^p \tag{2}$$

where for $t \in \mathbb{R}$, $h(t) = \frac{1}{(1+t)\theta}$ with $0 \leq \theta < 1$,

$$|a(x,s,\xi)| \leq \beta(a_0(x) + |s|^{p-1} + |\xi|^{p-1}) \tag{3}$$

where $\beta > 0$, a_0 is a non negative function in $L^{p'}(\Omega)$ with $p' = \frac{p}{p-1}$, and

$$(a(x,s,\xi) - a(x,s,\xi')) \cdot (\xi - \xi') > 0. \tag{4}$$

Regularity results for problems like (1) have been established by many authors when the function h is constant. The linear case with $p = 2$ has been studied by G. Stampacchia in [14, 15]. The nonlinear case with $1 < p < N$ has been investigated in [7, 8, 13]. For a solution u of (1), it is shown that if $|F|$ belongs to $L^s(\Omega)$ with $s < \frac{N}{p-1}$ then u belongs to $L^{(s(p-1))^*}(\Omega)$, while it is in $L^\infty(\Omega)$ if $s > \frac{N}{p-1}$. The limit case yields u in the Orlicz space $L_\phi(\Omega)$ generated by the N-function $\phi(t) = \exp(|t|^{\frac{N}{N-1}}) - 1$.

When h is not necessarily a constant function ($0 \leq \theta \leq 1$), the problem (1) was studied in [5] and in [19] where an L^∞ result was obtained for solutions of its parabolic counter-part. It has been established in [5] that if $|F|$ belongs to $L^s(\Omega)$, the problem (1) admits a solution u such as :

1. $s > \frac{N}{p-1} \quad \Rightarrow \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$,
2. $s = \frac{N}{p-1} \quad \Rightarrow \quad u \in W_0^{1,p(1-\theta)}(\Omega) \cap L_{\phi_{N,\theta}}(\Omega)$,
3. $\frac{Np'}{N-\theta(N-p)} \leq s < \frac{N}{p-1} \quad \Rightarrow \quad u \in W_0^{1,p}(\Omega) \cap L^r(\Omega)$,
4. $\max\left(p', \frac{Np'}{p((1-\theta)N+\theta)}\right) \leq s < \frac{Np'}{N-\theta(N-p)} \quad \Rightarrow \quad u \in W_0^{1,q}(\Omega) \cap L^r(\Omega)$,

where $L_{\phi_{N,\theta}}(\Omega)$ is the Orlicz space generated by the N-function

$$\phi_{N,\theta}(t) = \exp\left(t^{\frac{N(1-\theta)}{N-1}}\right) - 1, \quad p' = \frac{p}{p-1}, \quad r = \frac{(1-\theta)Ns(p-1)}{N-s(p-1)} \text{ and } q = \frac{(1-\theta)Ns(p-1)}{N-\theta s(p-1)}.$$

The following figure summarizes these different regularity results in light of the different areas to which the pairs (θ, s) belong.

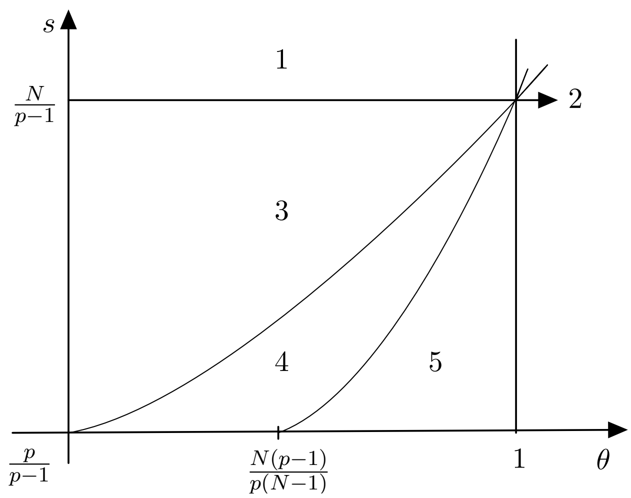

Figure 1: Different areas of regularity of solutions.

Notice that when θ is close to 1, precisely $\frac{N(p-1)}{p(N-1)} < \theta < 1$, we can decrease the summability of $|F|$, to cover the pairs (θ, s) belonging to the zone 5 which completes the study of the regularity of solutions of the problem (1) started in [5].

For $k > 0$, let $T_k : \mathbb{R} \to \mathbb{R}$ be the truncation function at levels $\pm k$ defined by

$$T_k(s) = \max(-k, \min(k, s)).$$

The main result is stated as follows.

Theorem 1.1. *Suppose that* $\frac{N(p-1)}{p(N-1)} < \theta < 1$ *and* (2), (3) *and* (4) *hold true. Let* $|F| \in L^s(\Omega)$ *with* $p' < s < \frac{Np'}{p((1-\theta)N+\theta)}$. *Then, there exists a measurable function* u *such that:*

$$u \in \mathcal{M}^r(\Omega) \text{ and } |\nabla u| \in \mathcal{M}^q(\Omega)$$

with

$$r = \frac{(1-\theta)Ns(p-1)}{N-s(p-1)} \text{ and } q = \frac{(1-\theta)Ns(p-1)}{N-\theta s(p-1)},$$

solution of (1) *in the sense that*

$$\begin{cases} T_k(u) \in W_0^{1,p}(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla T_k(u-v)\,dx \le \int_\Omega F\cdot\nabla T_k(u-v)\,dx \end{cases} \tag{5}$$

for every $k > 0$ *and for every* v *in* $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

We point out that the results in Theorem 1.1 cover these in [1, Theorem] and vice-versa. The result we provide here for a datum in divergence form is exactly the one obtained in [1, Theorem 1.9] for a source term in suitable Lebesgue spaces. Indeed, going back to [1, Theorem 1.9], and pick a function f in $L^m(\Omega)$. By duality arguments f can be written as $f = -\mathrm{div}(F)$, where $|F| \in L^s(\Omega)$ with $s = \frac{Nm}{N-m}$. Thus, by [1, Theorem 1.9] we obtain the results in Theorem 1.1. Reciprocally, let us put ourselves in the conditions of Theorem 1.1 with $|F| \in L^s(\Omega)$. Choosing a function $f \in L^m(\Omega)$ with $m = \frac{Ns}{N+s}$. The exponent m is such that $1 < m < \frac{N}{(p-1)(N(1-\alpha)+\alpha)+1}$. So that one can clearly recover the regularity results in [1, Theorem 1.9].

Remark 1.2. *We emphasize that* $\frac{N(p-1)}{p(N-1)} < \theta$ *implies that the range of* s *is nonempty.*

We underline that the gradient of the function u which appears in (5) is defined in [3, lemma 2.1] as the unique measurable function $v : \Omega \to \mathbb{R}^N$ satisfying

$$\nabla T_k(u) = v\chi_{\{|u|<k\}}, \text{ for almost every } x \in \Omega,\ \ \forall k > 0,$$

where χ_E is the characteristic function of a measurable set E of Ω. Moreover, if $u \in W_0^{1,1}(\Omega)$ then v coincides with usual distributional gradient of u. Notice that the use of $T_k(u-v)$ as test function yields a meaning to each term in (5), although ∇u does not belong to $(L^{p'}(\Omega))^N$. In fact, both integrals of (5) are only on the set $|u-v| \le k$, and on this set $|u| \le k + \|v\|_{L^\infty(\Omega)} = M$. Therefore, since assumption (2) implies that $a(x,s,0) = 0$, we have

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla T_k(u-v)\,dx = \int_\Omega a(x,T_M(u),\nabla T_M(u))\cdot\nabla T_k(u-v)\,dx,$$

which finite by the growth condition (3) and

$$\int_\Omega F\cdot\nabla T_k(u-v)\,dx = \int_\Omega F\cdot\nabla T_k(T_M(u)-v)\,dx$$

which is finite thanks to Hölder's inequality since $\nabla T_k(T_M(u)-v)$ belongs to $(L^p(\Omega))^N$.

It is worth recalling that when h is not necessarily a constant function, there is a difficulty in dealing with problem (1). Note that since no bounds are assumed on the function h, the operator $-\operatorname{div} a(x,u,\nabla u)$ acting from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ may degenerates when its second argument u has large values and hence it is not coercive. As a consequence, the classical theory of existence of solutions for (1) can not be applied. To overcome this problem, we consider approximate equations in which we introduce a truncation.

2. Preliminary results

Let Ω be a bounded open set of $\mathbb{R}^N$. If u is a measurable function in Ω, we denote by $\mu_u(t)$ its distribution function, that is

$$\mu_u(t)=|\{x\in\Omega:|u(x)|>t\}|,\qquad t\geq 0,$$

where $|E|$ denotes the Lebesgue measure of a measurable subset E of $\mathbb{R}^N$.
The decreasing rearrangement u^* of u is defined by

$$u^*(s)=\inf\{t\geq 0:\mu_u(t)\leq s\}\qquad \text{for}\ \ s\in[0,|\Omega|].$$

We refer to [4, 16, 17, 18] for a detailed exposition of basic facts on rearrangements.

For $0<q<+\infty$, the Marcinkiewicz space $\mathcal{M}^q(\Omega)$ consists of all measurable functions $u:\Omega\to\mathbb{R}$ such that for all $t>0$

$$\|u_n\|_{\mathcal{M}^q(\Omega)}:=t^q\mu_u(t)\leq c,$$

for some constant $c>0$. We observe that this condition is equivalent to say that

$$\tau^{\frac{1}{q}}u^*(\tau)\leq c$$

for all $\tau\in]0,|\Omega|[$ and for some constant $c>0$. Recall that the quantity $\|.\|_{\mathcal{M}^q(\Omega)}$ does not define a norm on $\mathcal{M}^q(\Omega)$ since the triangle inequality is not satisfied (see [11]). We also recall the following connection between Marcinkiewicz and Lebesgue spaces (see, for instance, [11])

$$L^q(\Omega)\subset\mathcal{M}^q(\Omega)\subset L^r(\Omega)$$

for $0<r<q$. Indeed for the last inclusion, if $|\{x\in\Omega:|u(x)|>t\}|\leq Ct^{-q}$ for all $t\geq 1$, then we have

$$\begin{aligned}\int_\Omega |u|^r dx &= \int_0^\infty |\{x\in\Omega:|u(x)|^r>t\}|dt\\ &\leq |\Omega|+C\int_1^\infty t^{-\frac{q}{r}}dt\\ &= |\Omega|+\frac{rC}{q-r}.\end{aligned}$$

Let us point out that in the literature, Marcinkiewicz spaces are also known as weak-Lebesgue spaces. The following lemma (see [1]) provides a sufficient condition for a measurable function to be in a Marcinkiewicz space.

Lemma 2.1. *Let u be a measurable function belonging to $\mathcal{M}^r(\Omega)$ for some $r>0$, such that, for every $k>0$, $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$, $p>1$. Suppose that*

$$\int_\Omega |\nabla T_k(u)|^p dx\leq ck^\lambda,\qquad \forall k>k_0,$$

for some non-negative λ, c and k_0. Then $|\nabla u|$ belongs to $\mathcal{M}^{\frac{rp}{r+\lambda}}(\Omega)$.

3. A priori estimates

The proof is based on approximation introducing truncations. Let $n \in \mathbb{N}$, we define the operator A_n by

$$A_n(u) := -\mathrm{div}\, a(\cdot, T_n(u), \nabla u).$$

From (2), we have

$$\int_\Omega a(x, T_n(u), \nabla u) \cdot \nabla u \, dx \geq h(n) \int_\Omega |\nabla u|^p \, dx,$$

so that A_n is coercive and satisfies the classical Leray-Lions conditions. It follows from [12], that A_n is surjective from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$. Since $s > p'$, the term $\mathrm{div}F$ is an element of $W^{-1,p'}(\Omega)$. Therefore, there exists at least one solution u_n in $W_0^{1,p}(\Omega)$ of

$$\begin{aligned} -\,\mathrm{div}\ a(x, T_n(u_n), \nabla u_n) &= -\mathrm{div}F \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{6}$$

in the sense that

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla v \, dx = \int_\Omega F \cdot \nabla v \, dx \tag{7}$$

for every v in $W_0^{1,p}(\Omega)$.

Theorem 3.1. *Let $|F| \in L^s(\Omega)$ with $p' < s < \frac{Np'}{p((1-\theta)N+\theta)}$ and let u_n be a solution of (6). Then*

$$u_n \in \mathcal{M}^r(\Omega) \ \ with \ \ r = \frac{(1-\theta)Ns(p-1)}{N - s(p-1)}, \tag{8}$$

and

$$\|u_n\|_{\mathcal{M}^r(\Omega)} \leq \left(\frac{1-\theta}{NC_N^{\frac{1}{N}}} \|F\|^{\frac{p'}{p}}_{(L^s(\Omega))^N} \left(\frac{N(s(p-1)-1)}{N - s(p-1)} \right)^{1-\frac{p'}{ps}} + |\Omega|^{\frac{1-\theta}{r}} \right)^{\frac{r}{1-\theta}},$$

where C_N is the measure of the unit ball in $\mathbb{R}^N$. Furthermore we have

$$|\nabla u_n| \in \mathcal{M}^q(\Omega) \ \ with \ \ q = \frac{(1-\theta)Ns(p-1)}{N - \theta s(p-1)}, \tag{9}$$

and

$$\begin{aligned} \|\nabla u_n\|_{\mathcal{M}^q(\Omega)} \ &\leq \ \|F\|^{p'}_{(L^s(\Omega))^N} (|\Omega|^{\frac{1}{r}} + c)^{\theta p} \left(\frac{r(s-p')}{r(s-p') - \theta ps} \right)^{\frac{s-p'}{s}} \\ &\quad \times \left(|\Omega|^{\frac{r(s-p')-\theta ps}{rs}} + c^{\frac{r(s-p')-\theta ps}{rs}} \right). \end{aligned}$$

Proof. For $\epsilon > 0$ and $t > 0$, we use in the formulation of solution (7) the test function $v = T_\epsilon(u_n - T_t(u_n))$, which belongs to $W_0^{1,p}(\Omega)$, obtaining

$$\int_{\{t<|u_n|\leq t+\epsilon\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx = \int_{\{t<|u_n|\leq t+\epsilon\}} F \cdot \nabla u_n \, dx,$$

where $\{t < |u_n| \leq t+\epsilon\}$ denotes the set $\{x \in \Omega : t < |u_n(x)| \leq t+\epsilon\}$.
Assumption (2) yields

$$h^{p-1}(t+\epsilon) \int_{\{t<|u_n|\leq t+\epsilon\}} |\nabla u_n|^p dx \leq \int_{\{t<|u_n|\leq t+\epsilon\}} F \cdot \nabla u_n dx.$$

Since F belongs at least to $(L^{p'}(\Omega))^N$, Hölder's inequality gives

$$h^{p-1}(t+\epsilon)\int_{\{t<|u_n|\leq t+\epsilon\}}|\nabla u_n|^p dx \leq \left(\int_{\{t<|u_n|\leq t+\epsilon\}}|F|^{p'}dx\right)^{\frac{1}{p'}}\left(\int_{\{t<|u_n|\leq t+\epsilon\}}|\nabla u_n|^p dx\right)^{\frac{1}{p}}.$$

Since $s>p'$, we apply again Hölder's inequality to get

$$h^{p}(t+\epsilon)\int_{\{t<|u_n|\leq t+\epsilon\}}|\nabla u_n|^p dx \leq \left(\int_{\{t<|u_n|\leq t+\epsilon\}}|F|^{s}dx\right)^{\frac{p'}{s}}|\{t<|u_n|\leq t+\epsilon\}|^{1-\frac{p'}{s}}.$$

Dividing both sides of the previous inequality by ϵ and then letting ϵ tends to 0^+, being h continuous, it follows that for almost every $t>0$

$$h^p(t)\left(-\frac{d}{dt}\int_{\{|u_n|>t\}}|\nabla u_n|^p dx\right) \leq \left(-\frac{d}{dt}\int_{\{|u_n|>t\}}|F|^s dx\right)^{\frac{p'}{s}}(-\mu_n'(t))^{1-\frac{p'}{s}}, \tag{10}$$

where $\{|u_n|>t\}$ denotes the set $\{x\in\Omega : |u_n(x)|>t\}$ and $\mu_n(t)$ stands for the distribution function of u_n, that is $\mu_n(t)=|\{x\in\Omega : |u_n(x)|>t\}|$.
On the other hand, from Fleming-Rishel coarea formula (see [10]) and isoperimetric inequality (see [9]), we have for almost every $t>0$

$$NC_N^{\frac{1}{N}}(\mu_n(t))^{\frac{N-1}{N}} \leq -\frac{d}{dt}\int_{\{|u_n|>t\}}|\nabla u_n|dx, \tag{11}$$

where C_N is the measure of the unit ball in $\mathbb{R}^N$. Using the Hölder inequality we obtain that for almost every $t>0$

$$-\frac{d}{dt}\int_{\{|u_n|>t\}}|\nabla u_n|dx \leq \left(-\frac{d}{dt}\int_{\{|u_n|>t\}}|\nabla u_n|^p dx\right)^{\frac{1}{p}}(-\mu_n'(t))^{1-\frac{1}{p}}. \tag{12}$$

Then, combining (10), (11) and (12) we obtain that for almost every $t>0$

$$h(t) \leq \frac{1}{NC_N^{\frac{1}{N}}}\left(-\frac{d}{dt}\int_{\{|u_n|>t\}}|F|^s dx\right)^{\frac{p'}{ps}}\frac{(-\mu_n'(t))^{1-\frac{p'}{ps}}}{(\mu_n(t))^{\frac{N-1}{N}}},$$

Integrating between 0 and $\tau>0$ both sides of this inequality and then using Hölder's inequality (since $ps>p'$) with exponents $\frac{ps}{p'}$ and $\frac{ps}{ps-p'}$, we obtain

$$\begin{aligned}&H(\tau)\\ &\leq \frac{1}{NC_N^{\frac{1}{N}}}\left(\int_0^\tau\left(-\frac{d}{dt}\int_{\{|u_n|>t\}}|F|^s dx\right)dt\right)^{\frac{p'}{ps}}\left(\int_0^\tau \frac{-\mu_n'(t)}{\mu_n(t)^{\frac{N-1}{N}\frac{ps}{ps-p'}}}dt\right)^{1-\frac{p'}{ps}},\end{aligned}$$

where $H(s)=\displaystyle\int_0^s h(t)dt$. Taking into account that $|F|$ belongs to $L^s(\Omega)$, and making a change of variables in the last integral, we get

$$H(\tau) \leq \frac{1}{NC_N^{\frac{1}{N}}}\|F\|^{\frac{p'}{p}}_{(L^s(\Omega))^N}\left(\int_{\mu_n(\tau)}^{|\Omega|}\frac{d\sigma}{\sigma^{\frac{N-1}{N}\frac{ps}{ps-p'}}}\right)^{1-\frac{p'}{ps}}.$$

A straightforward calculation of the integral in the right-hand side and the fact that $\tau^{1-\theta} \leq (1-\theta)H(\tau)+1$, for $\tau > 0$, allow us to have

$$\tau^{1-\theta} \leq \frac{1-\theta}{NC_N^{\frac{1}{N}}} \|F\|^{\frac{p'}{p}}_{(L^s(\Omega))^N} \left(\frac{N(s(p-1)-1)}{N-s(p-1)}\right)^{1-\frac{p'}{ps}} \mu_n(\tau)^{-\frac{N-s(p-1)}{Ns(p-1)}} + 1,$$

which gives

$$\begin{aligned}&\tau^{\frac{(1-\theta)Ns(p-1)}{N-s(p-1)}} \mu_n(\tau) \\ &\leq \left(\frac{1-\theta}{NC_N^{\frac{1}{N}}} \|F\|^{\frac{p'}{p}}_{(L^s(\Omega))^N} \left(\frac{N(s(p-1)-1)}{N-s(p-1)}\right)^{1-\frac{p'}{ps}} + |\Omega|^{\frac{N-s(p-1)}{Ns(p-1)}}\right)^{\frac{Ns(p-1)}{N-s(p-1)}}.\end{aligned}$$

This means that

$$u_n \in \mathcal{M}^r(\Omega) \text{ with } r = \frac{(1-\theta)Ns(p-1)}{N-s(p-1)}.$$

We now turn to the estimation of the gradients of solutions u_n. Going back to (10), dividing by $h^p(t)$, integrating both sides of the inequality between 0 and k, ($k \geq 1$), and then using Hölder's inequality we obtain

$$\begin{aligned}\int_{\{|u_n|\leq k\}} |\nabla u_n|^p dx &\leq \left(\int_{\{|u_n|\leq k\}} |F|^s dx\right)^{\frac{p'}{s}} \left(\int_0^k \frac{-\mu_n'(t)}{h^{\frac{ps}{s-p'}}(t)} dt\right)^{1-\frac{p'}{s}} \\ &\leq \|F\|^{p'}_{(L^s(\Omega))^N} \left(\int_{\mu_n(k)}^{|\Omega|} (1+u_n^*(\sigma))^{\frac{\theta ps}{s-p'}} d\sigma\right)^{1-\frac{p'}{s}}.\end{aligned}$$

Since $u_n \in \mathcal{M}^r(\Omega)$, there exists a constant $c > 0$ such that $t^{\frac{1}{r}} u_n^*(t) \leq c$. Hence,

$$\begin{aligned}\int_{\{|u_n|\leq k\}} |\nabla u_n|^p dx &\leq \|F\|^{p'}_{(L^s(\Omega))^N} \left(\int_{\mu_n(k)}^{|\Omega|} (1+c\sigma^{-\frac{1}{r}})^{\frac{\theta ps}{s-p'}} d\sigma\right)^{1-\frac{p'}{s}} \\ &\leq \|F\|^{p'}_{(L^s(\Omega))^N} (|\Omega|^{\frac{1}{r}}+c)^{\theta p} \left(\int_{\mu_n(k)}^{|\Omega|} \sigma^{-\frac{\theta ps}{s-p'}} d\sigma\right)^{1-\frac{p'}{s}} \\ &= C_1 \left(|\Omega|^{\frac{r(s-p')-\theta ps}{rs}} - \mu_n^{\frac{r(s-p')-\theta ps}{rs}}(k)\right).\end{aligned}$$

where $C_1 = \|F\|^{p'}_{(L^s(\Omega))^N} (|\Omega|^{\frac{1}{r}}+c)^{\theta p} \left(\frac{r(s-p')}{r(s-p')-\theta ps}\right)^{\frac{s-p'}{s}}$. Again, since $u_n \in \mathcal{M}^r(\Omega)$ we have $\mu_n(k) \leq \dfrac{c}{k^r}$. Therefore, we obtain

$$\int_{\{|u_n|\leq k\}} |\nabla u_n|^p dx \leq C_1 \left(|\Omega|^{\frac{r(s-p')-\theta ps}{rs}} + c^{\frac{r(s-p')-\theta ps}{rs}} k^{\lambda}\right),$$

where $\lambda = \theta p - r\left(1-\dfrac{p'}{s}\right)$. The assumption made on s ensures that $\lambda > 0$.
Then, for every $k \geq 1$ we obtain

$$\int_{\{|u_n|\leq k\}} |\nabla u_n|^p dx \leq C_2 k^{\lambda},$$

where

$$C_2 = C_1 \left(|\Omega|^{\frac{r(s-p')-\theta ps}{rs}} + c^{\frac{r(s-p')-\theta ps}{rs}}\right).$$

A straightforward calculation shows that

$$r+\lambda = p\frac{r}{q}.$$

An application of Lemma 2.1 implies that

$$|\nabla u_n| \in \mathcal{M}^q(\Omega).$$

□

Lemma 3.2. *Let u_n be a solution of (6). Then, there exists a measurable function u such that:*

$$u_n \to u \quad \text{in measure and a.e. in } \Omega \tag{13}$$

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega),\ \forall k>0. \tag{14}$$

Proof. Let $k>0$. The use of $T_k(u_n)$ as test function in (7) yields

$$\int_\Omega |\nabla T_k(u_n)|^p dx \le \|F\|^{p'}_{(L^s(\Omega))^N} |\Omega|^{1-\frac{p'}{s}} (1+k)^{\theta p}.$$

So that $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ independently of n. Therefore, there exists (see [3]) a measurable function u such that, up to a subsequence,

$$u_n \to u \quad \text{in measure and a.e. in } \Omega$$

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega).$$

□

4. Almost everywhere convergence of the gradients of solutions

Arguing as in [1], we shall prove the almost everywhere convergence of the gradients. Let $0<\sigma<\frac{q}{2p}(<1)$. Consider

$$I_n = \int_\Omega \{(a(x,T_n(u_n),\nabla u_n) - a(x,T_n(u_n),\nabla u))\cdot(\nabla u_n - \nabla u)\}^\sigma dx.$$

We split I_n on the sets

$$A_k = \{x\in\Omega : |u(x)|>k\}$$

and

$$C_k = \{x\in\Omega : |u(x)|\le k\},$$

obtaining

$$I_n = I_1(n,k) + I_2(n,k)$$

where

$$I_1(n,k) = \int_{A_k} \{(a(x,T_n(u_n),\nabla u_n) - a(x,T_n(u_n),\nabla u))\cdot(\nabla u_n - \nabla u)\}^\sigma dx$$

and

$$I_2(n,k) = \int_{C_k} \{(a(x,T_n(u_n),\nabla u_n) - a(x,T_n(u_n),\nabla u))\cdot(\nabla u_n - \nabla u)\}^\sigma dx.$$

Using (3) we deduce that there exists a constant c such that

$$I_1(n,k) \le c\left(\int_{A_k} (a_0(x))^{p'\sigma}dx + \int_{A_k} (|u_n|^{p\sigma} + |\nabla u_n|^{p\sigma} + |\nabla u|^{p\sigma})dx\right).$$

Let $\tau = \dfrac{q}{2p\sigma}$. By Hölder's inequality we obtain

$$\begin{aligned}
& I_1(n,k) \\
& \leq c\left(\int_\Omega (a_0(x))^{p'}dx\right)^\sigma |A_k|^{1-\sigma} \\
& + c\left(\left(\int_\Omega |u_n|^{\frac{q}{2}}dx\right)^{\frac{1}{\tau}} + \left(\int_\Omega |\nabla u_n|^{\frac{q}{2}}dx\right)^{\frac{1}{\tau}} + \left(\int_\Omega |\nabla u|^{\frac{q}{2}}dx\right)^{\frac{1}{\tau}}\right)|A_k|^{1-\frac{1}{\tau}}.
\end{aligned}$$

Note that Lemma 3.2 and Lemma 2.1 enable us to get $|\nabla u| \in \mathcal{M}^q(\Omega) \subset L^{\frac{q}{2}}(\Omega)$. From (8) and (9) we obtain

$$I_1(n,k) \leq c\left(|A_k|^{1-\sigma} + |A_k|^{1-\frac{1}{\tau}}\right),$$

where c is a constant not depending on n. Therefore, we get

$$\lim_{k\to\infty}\lim_{n\to\infty} I_1(n,k) = 0. \tag{15}$$

Observe that on C_k one has $T_k(u) = u$ and since the integrand function is positive we have

$$I_2(n,k) \leq I_3(n,k)$$

where

$$I_3(n,k) = \int_\Omega \Big\{(a(x,T_n(u_n),\nabla u_n) - a(x,T_n(u_n),\nabla T_k(u)))\cdot(\nabla u_n - \nabla T_k(u))\Big\}^\sigma dx.$$

We fix $j > 0$ and split the integral in $I_3(n,k)$ on the sets $\{|u_n - T_k(u)| > j\}$ and $\{|u_n - T_k(u)| \leq j\}$, obtaining

$$I_3(n,k) = I_4(n,k,j) + I_5(n,k,j),$$

where

$$\begin{aligned}
I_4(n,k,j) = & \int_{\{|u_n - T_k(u)|>j\}} \Big\{(a(x,T_n(u_n),\nabla u_n) - a(x,T_n(u_n),\nabla T_k(u))) \\
& \cdot(\nabla u_n - \nabla T_k(u))\Big\}^\sigma dx
\end{aligned}$$

and

$$\begin{aligned}
I_5(n,k,j) = & \int_{\{|u_n - T_k(u)|\leq j\}} \Big\{(a(x,T_n(u_n),\nabla u_n) - a(x,T_n(u_n),\nabla T_k(u))) \\
& \cdot(\nabla u_n - \nabla T_k(u))\Big\}^\sigma dx
\end{aligned}$$

By (8) the measure of the set $\{|u_n - T_k(u)| > j\}$ tends to zero as j tends to ∞ uniformly in n and k. So that, reasoning as for $I_1(n,k)$ we obtain

$$\lim_{j\to\infty}\lim_{k\to\infty}\lim_{n\to\infty} I_4(n,k,j) = 0. \tag{16}$$

Applying the Hölder inequality with exponents $\dfrac{1}{\sigma}$ and $\dfrac{1}{1-\sigma}$, we get

$$\begin{aligned}
I_5(n,k,j) = & \int_\Omega \Big\{(a(x,T_n(u_n),\nabla u_n) - a(x,T_n(u_n),\nabla T_k(u)))\cdot\nabla T_j(u_n - T_k(u))\Big\}^\sigma dx \\
& \leq |\Omega|^{1-\sigma}\left(I_6(n,k,j) - I_7(n,k,j)\right)^\sigma,
\end{aligned}$$

where

$$I_6(n,k,j) = \int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_j(u_n - T_k(u))dx$$

and

$$I_7(n,k,j) = \int_\Omega a(x, T_n(u_n), \nabla T_k(u)) \cdot \nabla T_j(u_n - T_k(u))dx.$$

Using $T_j(u_n - T_k(u))$ as test function in (7), we obtain

$$I_6(n,k,j) = \int_\Omega F \cdot \nabla T_j(u_n - T_k(u))dx.$$

Note that on the set $\{|u_n - T_k(u)| \le j\}$ we have $|u_n| \le k + j$. Thus, by (14) we get

$$\begin{aligned} \lim_{n\to\infty} I_6(n,k,j) &= \int_\Omega F \cdot \nabla T_j(u - T_k(u))dx \\ &= \int_{\{|u|>k\}} F \cdot \nabla T_j(u)dx. \end{aligned}$$

Thus

$$\lim_{k\to\infty} \lim_{n\to\infty} I_6(n,k,j) = 0. \tag{17}$$

Let $M = k + j$. For $n > M$ one has

$$I_7(n,k,j) = \int_\Omega a(x, T_M(u_n), \nabla T_k(u)) \cdot \nabla T_j(u_n - T_k(u))dx.$$

By (13) and Viltali's theorem we obtain that

$$a(x, T_M(u_n), \nabla T_k(u)) \to a(x, T_M(u), \nabla T_k(u))$$

strongly in $(L^{p'}(\Omega))^N$ as n tends to $+\infty$. It follows by (14) that

$$\begin{aligned} \lim_{n\to\infty} I_7(n,k,j) &= \int_\Omega a(x, T_M(u), \nabla T_k(u)) \cdot \nabla T_j(u - T_k(u))dx \\ &= \int_{\{|u|>k\}} a(x, u, 0) \cdot \nabla T_j(u)dx \\ &= 0. \end{aligned} \tag{18}$$

Combining (15), (16), (17) and (18) we obtain

$$\lim_{n\to\infty} I_n = 0.$$

Since the integrand function in I_n is positive, we have

$$\{(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u)) \cdot (\nabla u_n - \nabla u)\}^\sigma \to 0$$

strongly in $L^1(\Omega)$. Hence, there exists a subsequence still indexed by n, such that

$$(a(x, T_n(u_n(x)), \nabla u_n(x)) - a(x, T_n(u_n(x)), \nabla u(x))) \cdot (\nabla u_n(x) - \nabla u(x)) \to 0.$$

for almost every x in Ω. As in [1] we conclude that

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega. \tag{19}$$

5. Passage to the limit

Let $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and let $k > 0$. Taking $T_k(u_n - v)$ as test function in (7) we obtain

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u_n - v)\, dx = \int_\Omega F \cdot \nabla T_k(u_n - v)\, dx.$$

By (14) we have

$$\lim_{n\to\infty} \int_\Omega F \cdot \nabla T_k(u_n - v)\, dx = \int_\Omega F \cdot \nabla T_k(u - v)\, dx.$$

Taking into account that on the set $\{|u_n - v| < k\}$ we have $|u_n| \leq k + \|v\|_\infty := M$, we split the left-hand side, for $n > M$, as the sum

$$\begin{aligned}&\int_{\{|u_n-v|\leq k\}} a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla T_M(u_n)\, dx\\ &-\int_{\{|u_n-v|\leq k\}} a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla v\, dx.\end{aligned}$$

Since $\{a(x, T_M(u_n), \nabla T_M(u_n))\}$ is bounded in $(L^{p'}(\Omega))^N$, by (13) and (19) we have that

$$a(x, T_M(u_n), \nabla T_M(u_n)) \rightharpoonup a(x, T_M(u), \nabla T_M(u))$$

weakly in $(L^{p'}(\Omega))^N$. Thus, we get

$$\begin{aligned}&\lim_{n\to\infty}\int_{\{|u_n-v|\leq k\}} a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla T_M(u_n)\, dx\\ &=\int_{\{|u-v|\leq k\}} a(x, u, \nabla u) \cdot \nabla v\, dx.\end{aligned}$$

By (2), the integrand function $a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla T_M(u_n)$ is non negative. Therefore, Fatou's lemma allows us to have

$$\begin{aligned}&\int_{\{|u-v|\leq k\}} a(x, u, \nabla u) \cdot \nabla u\, dx\\ &\leq \liminf_{n\to\infty}\int_{\{|u_n-v|\leq k\}} a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla T_M(u_n)\, dx.\end{aligned}$$

Finally we get

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla T_k(u - v)\, dx \leq \int_\Omega F \cdot \nabla T_k(u - v)\, dx.$$

Furthermore, we have $u \in \mathcal{M}^r(\Omega)$ and $|\nabla u| \in \mathcal{M}^q(\Omega)$. The proof of the Theorem 1.1 is then achieved.

Remark 5.1. *The case $s = p'$ which can not be considered in Theorem 1.1, has been treated in [1, Theorem 5.1]. The authors proved that the approximation solutions u_n of (6) belong to $L^{\frac{(1-\theta)Np}{N-p}}(\Omega) \subset \mathcal{M}^{\frac{(1-\theta)Np}{N-p}}(\Omega)$.*

To get information about the gradient of u_n we use of $T_k(u_n)$, for $k \geq 1$, as test function in (7) obtaining

$$\frac{1}{(1+k)^{\theta(p-1)}} \int_\Omega |\nabla T_k(u_n)|^p dx \leq \int_\Omega F \cdot \nabla T_k(u_n) dx.$$

Hence, by the Hölder inequality we get

$$\int_\Omega |\nabla T_k(u_n)|^p dx \leq 2^{\theta p} \|F\|_{p'}^{p'} k^{\theta p}.$$

Therefore, an application of Lemma 2.1 gives

$$|\nabla u_n| \in \mathcal{M}^{\frac{(1-\theta)Np}{N-\theta p}}(\Omega).$$

As above, we get the existence of a measurable function u with the properties (13), (14) and (19). Passing to the limit as previously done, we obtain that u is a solution of (1) in the sense of (5). Moreover, we have $u \in \mathcal{M}^{\frac{(1-\theta)Np}{N-p}}(\Omega)$ and $|\nabla u| \in \mathcal{M}^{\frac{(1-\theta)Np}{N-\theta p}}(\Omega)$.

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