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On estimating convergence for Picard sequences in b-metric space

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Abstract

In this paper, we give an estimate of $d(x_n, x^*)$ for a sequence $\{x_n\}$ in a b-metric space that satisfies the contractive condition

$$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}),$$

for all $n \in \mathbb{N}$, where $\lambda \in (0, 1)$. In addition, we give another proof for the convergence of a sequence $\{x_n\}$. Examples of estimation for Banach's, Kannan's, and Reich's fixed point theorems are given. In the end, we give some open problems in which research can be continued.

Keywords: Cauchy sequence, b-metric spaces, fixed point, contraction
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1. Introduction and Preliminaries

One significant consequence of the Banach fixed point theorem[6] is its provision for estimating the error within the Picard iterative sequence. This estimation comes in two valuable forms: a priori estimate, used

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at the outset to predict the number of steps required to achieve a desired level of accuracy, and a posteriori estimate, which can be employed during intermediate steps to assess the convergence rate, allowing for comparison against the initial prediction made by the a priori estimate.

Theorem 1.1. *[6] Let $T : X \to X$ be the contraction mapping in a complete metric space X, x_0 be the initial point of the Picard iterative sequence $x_{n+1} = Tx_n$, and let x^* be the fixed point of the mapping T. The following error estimates hold:*

(i) $d(x_n, x^*) \leq \dfrac{\lambda^n}{1-\lambda} \cdot d(x_0, x_1)$;

(ii) $d(x_{n+1}, x^*) \leq \dfrac{\lambda}{1-\lambda} \cdot d(x_n, x_{n+1})$.

for all $n \in \mathbb{N}$.

While the error estimates for Picard iteration are powerful within standard metric spaces, it's natural to wonder if similar results hold in more generalized settings. This leads us directly to the concept of b-metric spaces. In 1993, Czerwik [11] proposed the terms b-metric and b-metric space. Bakhtin [5] called them "almost metric spaces." But these kinds of spaces were earlier considered under various names (see the Introduction to [10]). In [10], one says that Bakhtin proposed the term "quasi-metric," but the exact translation is that of "almost metric." Also, according to the historical notes in the recent paper of Berinde and Păcurar [7], it appears that the concept of b-metric space (under the name "quasimetric space") was introduced before Bakhtin and Czerwik, by Vulpe et al. [30]. The theory of fixed points in b-metric spaces has expanded in the past ten years (see [1, 3, 4, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]).
We list some well-known facts about b-metric spaces.

Definition 1.2. *Let X be a nonempty set and $s \in [1, +\infty)$. A mapping $d : X \times X \to [0, +\infty)$ is called a b-metric if:*

(1_b) $d(x_1, x_2) = 0$ *if and only if* $x_1 = x_2$,

(2_b) $d(x_1, x_2) = d(x_2, x_1)$,

(3_b) $d(x_1, x_3) \leq s[d(x_1, x_2) + d(x_2, x_3)]$,

for all $x_1, x_2, x_3 \in X$. In this case (X, d, s) is called a b-metric space.

Remark 1.3. *If $s = 1$, then the b-metric space is metric. The notions of convergent sequence, Cauchy sequence, and completeness in b-metric spaces are defined as in metric spaces.*

Remark 1.4. *The space $l^p = \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{+\infty} |x_n|^p < +\infty\}, p \in (0, 1)$, together with the function $d_p : l^p \times l^p \to \mathbb{R}$, defined by*

$$d_p(x, y) = \left(\sum_{n=1}^{+\infty} |x_n - y_n|^p \right)^{\frac{1}{p}},$$

where $x = \{x_n\}, y = \{y_n\} \in l^p$, is not a metric space (the function d_p do not satisfy the triangle inequality), but (l^p, d_p, s) is a b-metric space with $s = 2^{\frac{1}{p}-1}$, [14, 19].

Remark 1.5. *The b-metric need not be continuous (see for example [16]).*

Definition 1.6. *Let (X, d, s) be a b-metric space, $\{x_n\}$ a sequence in X and $x \in X$.*
(a) The sequence $\{x_n\}$ is convergent and converges to x, if for every $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n > n_\epsilon$. We denote this by $\lim_{n\to+\infty} x_n = x$ or $x_n \to x$ as $n \to +\infty$.
(b) The sequence $\{x_n\}$ is called Cauchy if for every $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m > n_\epsilon$.
(c) If every Cauchy sequence in X converges to some $x \in X$ then (X, d, s) is called a complete b-metric space.

Recently, Miculescu and Mihail [20] (Lemma 2.2) and Suzuki [29] (Lemma 6), see also Mitrović [21] (Lemma 2.3) obtained following result.

Lemma 1.7. *Let (X, d, s) be a b-metric space and sequence $\{x_n\} \subseteq X$. If there exists $\lambda \in [0, 1)$ such that*

$$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}), \tag{1}$$

for all $n \in \mathbb{N}$, then $\{x_n\}$ is Cauchy.

Remark 1.8. *From Lemma 1.7 we obtain that $\{x_n\}$ is Cauchy if there exist $a \in [0, +\infty)$ and $\lambda \in [0, 1)$ such that $d(x_{n+1}, x_n) \leq \lambda^n a$ for any $n \in \mathbb{N}$.*

In next section we give an estimate for $d(x_n, x^*)$ for a sequence $\{x_n\}$ in a b-metric space that fulfills the condition (1). In addition, we give another proof for the convergence of a sequence $\{x_n\}$.

2. Estimation in b - metric spaces

In this section, we will denote by $\lfloor x \rfloor$ floor function (greatest integer less than or equal to x) and by $\lceil x \rceil$ ceiling function (least integer greater than or equal to x). Firstly, we give an auxiliary lemma that proves the main result.

Lemma 2.1. *Let (X, d, s) be a complete b-metric space and let $\{x_n\}$ be a sequence in X. Assume that there exists $\lambda \in [0, 1)$ such that*

$$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}), \tag{2}$$

for any $n \in \mathbb{N}$. Then the following estimate applies for $i < j$

$$d(x_i, x_j) \leq s\lambda^i(1 + s\lambda + \ldots + s^{j-i-2}\lambda^{j-i-2} + s^{j-i-2}\lambda^{j-i-1})d(x_0, x_1) \tag{3}$$

Proof. Let $i, j \in \mathbb{N}$ and $i < j$. From condition (3_b) we obtain

$$\begin{aligned}
d(x_i, x_j) &\leq s[d(x_i, x_{i+1}) + d(x_{i+1}, x_j)] \\
&\leq s\lambda^i d(x_0, x_1) + s^2[d(x_{i+1}, x_{i+2}) + d(x_{i+2}, x_j)] \\
&\leq \left(s\lambda^i + s^2\lambda^{i+1}\right) d(x_0, x_1) + s^3[d(x_{i+2}, x_{i+3}) + d(x_{i+3}, x_j)] \\
&\leq \left(s\lambda^i + s^2\lambda^{i+1} + s^3\lambda^{i+2}\right) d(x_0, x_1) + s^4[d(x_{i+3}, x_{i+4}) + d(x_{i+4}, x_j)] \\
&\vdots \\
&\leq s\lambda^i(1 + s\lambda + \ldots + s^{j-i-2}\lambda^{j-i-2} + s^{j-i-2}\lambda^{j-i-1})d(x_0, x_1).
\end{aligned}$$

For $s\lambda \neq 1$, since $s > 1$ we can estimate distance

$$d(x_i, x_j) \leq s\lambda^i \frac{1 - (s\lambda)^{j-i}}{1 - s\lambda} d(x_0, x_1).$$

□

Our main result is the following theorem.

Theorem 2.2. *Let (X, d, s) be a complete b-metric space and let $\{x_n\}$ be a sequence in X. Assume that there exists $\lambda \in [0, 1)$ such that*

$$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}),$$

for any $n \in \mathbb{N}$. Let

$$n_0 = \min\{j \in \mathbb{N} \mid s\lambda^j < 1\}. \tag{4}$$

Then for $s\lambda \neq 1$
$$d(x_n, x^*) \leq Cs^2 \lambda^{n_0 \lfloor \frac{n}{n_0} \rfloor} \left(1 + \frac{s^2}{1 - s\lambda^{n_0}}\right), \tag{5}$$
where $x^* = \lim_{n\to+\infty} x_n$ *and* $C = sd(x_0, x_1)\dfrac{1-(s\lambda)^{n_0}}{1-s\lambda}$.

Proof. Let $n, p \in \mathbb{N}$. From condition (3_b) we obtain
$$d(x_n, x_{n+p}) \leq sd(x_n, x_{n_0\lfloor \frac{n}{n_0}\rfloor}) + s^2[d(x_{n_0\lfloor \frac{n}{n_0}\rfloor}, x_{n_0\lfloor \frac{n+p}{n_0}\rfloor}) + d(x_{n_0\lfloor \frac{n+p}{n_0}\rfloor}, x_{n+p})]. \tag{6}$$
From $n_0(\frac{n}{n_0} - 1) < n_0\lfloor \frac{n}{n_0}\rfloor \leq n_0\frac{n}{n_0}$ we obtain $0 \leq n - n_0\lfloor \frac{n}{n_0}\rfloor < n_0$. Applying Lemma 2.1 we have
$$d(x_n, x_{n_0\lfloor \frac{n}{n_0}\rfloor}) \leq s\lambda^{n_0\lfloor \frac{n}{n_0}\rfloor} d(x_0, x_1)\frac{1-(s\lambda)^{n-n_0\lfloor \frac{n}{n_0}\rfloor}}{1-s\lambda}.$$
So,
$$d(x_n, x_{n_0\lfloor \frac{n}{n_0}\rfloor}) \leq C\lambda^{n_0\lfloor \frac{n}{n_0}\rfloor}. \tag{7}$$
In the same way, we have
$$d(x_{n_0\lfloor \frac{n+p}{n_0}\rfloor}, x_{n+p}) \leq C\lambda^{n_0\lfloor \frac{n+p}{n_0}\rfloor}. \tag{8}$$
From (8) we conclude that
$$\lim_{p\to+\infty} d(x_{n_0\lfloor \frac{n+p}{n_0}\rfloor}, x_{n+p}) = 0. \tag{9}$$
Further, we have
$$\begin{aligned} d(x_{(n+1)n_0}, x_{nn_0}) &\leq s^{n_0}[d(x_{(n+1)n_0}, x_{(n+1)n_0-1}) + \cdots + d(x_{nn_0+1}, x_{nn_0})] \\ &\leq s^{n_0}(\lambda^{(n+1)n_0-1} + \cdots + \lambda^{nn_0})d(x_1, x_0) \\ &\leq s^{n_0}\lambda^{nn_0}\frac{d(x_1, x_0)}{1-\lambda}. \end{aligned}$$
So,
$$d(x_{(n+1)n_0}, x_{nn_0}) \leq C\mu^n, \tag{10}$$
where, $\mu = \lambda^{n_0}$. Using (4) we have that $\mu s < 1$. From (3_b) and (10) we obtain
$$\begin{aligned} d(x_{n_0\lfloor \frac{n}{n_0}\rfloor}, x_{n_0\lfloor \frac{n+p}{n_0}\rfloor}) &\leq \sum_{j=n_0\lfloor \frac{n}{n_0}\rfloor}^{n_0\lfloor \frac{n+p}{n_0}\rfloor-2} s^{j+1-n_0\lfloor \frac{n}{n_0}\rfloor} d(x_{n_0 j}, x_{n_0(j+1)}) \\ &+ s^{n_0\lfloor \frac{n+p}{n_0}\rfloor-1-n_0\lfloor \frac{n}{n_0}\rfloor} d(x_{n_0(n_0\lfloor \frac{n+p}{n_0}\rfloor-1)}, x_{n_0\lfloor \frac{n+p}{n_0}\rfloor}) \\ &\leq \sum_{j=n_0\lfloor \frac{n}{n_0}\rfloor}^{n_0\lfloor \frac{n+p}{n_0}\rfloor-2} s^{j+1-n_0\lfloor \frac{n}{n_0}\rfloor} C\mu^j \\ &+ s^{n_0\lfloor \frac{n+p}{n_0}\rfloor-1-n_0\lfloor \frac{n}{n_0}\rfloor} C\mu^{n_0(\lfloor \frac{n+p}{n_0}\rfloor-1)} \\ &= Cs^{1-n_0\lfloor \frac{n}{n_0}\rfloor} \sum_{j=n_0\lfloor \frac{n}{n_0}\rfloor}^{n_0\lfloor \frac{n+p}{n_0}\rfloor-2} (s\mu)^j \\ &+ Cs^{n_0-1-n_0\lfloor \frac{n}{n_0}\rfloor}(s\mu)^{n_0(\lfloor \frac{n+p}{n_0}\rfloor-1)}, \end{aligned}$$

Now, we have

$$\begin{aligned}\lim_{p\to+\infty} d(x_{n_0\lfloor\frac{n}{n_0}\rfloor}, x_{n_0\lfloor\frac{n+p}{n_0}\rfloor}) &\leq Cs^{1-n_0\lfloor\frac{n}{n_0}\rfloor}\frac{(s\mu)^{n_0\lfloor\frac{n}{n_0}\rfloor}}{1-s\mu}\\ &= Cs\frac{\mu^{n_0\lfloor\frac{n}{n_0}\rfloor}}{1-s\mu}.\end{aligned}$$

Therefore,

$$\lim_{p\to+\infty} d(x_{n_0\lfloor\frac{n}{n_0}\rfloor}, x_{n_0\lfloor\frac{n+p}{n_0}\rfloor}) \leq Cs\frac{\lambda^{\lfloor\frac{n}{n_0}\rfloor}}{1-s\lambda^{n_0}}. \tag{11}$$

From (6), (7), (8) and (11) we obtain

$$\lim_{p\to+\infty} d(x_n, x_{n+p}) \leq Cs\lambda^{n_0\lfloor\frac{n}{n_0}\rfloor}\left(1+\frac{s^2}{1-s\lambda^{n_0}}\right). \tag{12}$$

On the other hand,

$$d(x_n, x^*) \leq s[d(x_n, x_{n+p}) + d(x_{n+p}, x^*)], \tag{13}$$

now, from (12) and (13), if $p\to+\infty$ we obtain (5). □

Example 2.3. *Consider the sequence $x_0 = (1, \frac{1}{2}, \frac{1}{2^2}, \dots)$ in $l_{\frac{1}{2}}$ space. It follows from the Remark 1.4 that $s = 2$. Also, after basic calculations and using the definition of the distance function from Remark 1.4, we have*

$$d(x_{n+1}, x_n) = d\left(\frac{1}{2}x_n, x_n\right) \leq \frac{1}{2}d(x_n, x_{n-1}).$$

Therefore, we have a special case where $s\lambda = 1$.

As we can see in Example 2.3 and Theorem 2.2 there is a minor issue with $s\lambda = 1$ so we will now provide a proof where that assumption is not necessary. While this new proof overcomes the previous issue, there is a reduction in the precision of the inequality compared to the first one.

Theorem 2.4. *Let (X, d, s) be a complete b-metric space and let $\{x_n\}$ be a sequence in X. Assume that there exists $\lambda \in (0,1)$ such that*

$$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}), \tag{14}$$

for any $n \in \mathbb{N}$. Let

$$n_0 = \min\{j \in \mathbb{N} \mid s\lambda^j < 1\}. \tag{15}$$

Then

$$d(x_n, x^*) \leq Cs^2\lambda^{n_0\lfloor\frac{n}{n_0}\rfloor}\left(1+\frac{s^2}{1-s\lambda^{n_0}}\right), \tag{16}$$

where $x^* = \lim_{n\to+\infty} x_n$ and $C = s^{n_0}\frac{d(x_1,x_0)}{1-\lambda}$.

Proof. Let $n, p \in \mathbb{N}$. From condition (3_b) we obtain

$$d(x_n, x_{n+p}) \leq sd(x_n, x_{n_0\lfloor\frac{n}{n_0}\rfloor}) + s^2[d(x_{n_0\lfloor\frac{n}{n_0}\rfloor}, x_{n_0\lfloor\frac{n+p}{n_0}\rfloor}) + d(x_{n_0\lfloor\frac{n+p}{n_0}\rfloor}, x_{n+p})]. \tag{17}$$

From $n_0(\frac{n}{n_0}-1) < n_0\lfloor\frac{n}{n_0}\rfloor \leq n_0\frac{n}{n_0}$ we obtain $0 \leq n - n_0\lfloor\frac{n}{n_0}\rfloor < n_0$. So, we have

$$\begin{aligned}d(x_n, x_{n_0\lfloor\frac{n}{n_0}\rfloor}) &\leq s^{n_0}[d(x_n, x_{n-1}) + \cdots + d(x_{n_0\lfloor\frac{n}{n_0}\rfloor+1}, x_{n_0\lfloor\frac{n}{n_0}\rfloor})]\\ &\leq s^{n_0}(\lambda^{n-1} + \cdots + \lambda^{n_0\lfloor\frac{n}{n_0}\rfloor})d(x_1, x_0)\\ &\leq s^{n_0}\lambda^{n_0\lfloor\frac{n}{n_0}\rfloor}\frac{d(x_1,x_0)}{1-\lambda}.\end{aligned}$$

So,

$$d(x_n, x_{n_0\lfloor \frac{n}{n_0} \rfloor}) \leq C\lambda^{n_0\lfloor \frac{n}{n_0} \rfloor}. \tag{18}$$

In the same way, we have

$$d(x_{n_0\lfloor \frac{n+p}{n_0} \rfloor}, x_{n+p}) \leq C\lambda^{n_0\lfloor \frac{n+p}{n_0} \rfloor}. \tag{19}$$

From (19) we conclude that

$$\lim_{p\to+\infty} d(x_{n_0\lfloor \frac{n+p}{n_0} \rfloor}, x_{n+p}) = 0. \tag{20}$$

Further, we have

$$\begin{aligned} d(x_{(n+1)n_0}, x_{nn_0}) &\leq s^{n_0}[d(x_{(n+1)n_0}, x_{(n+1)n_0-1}) + \cdots + d(x_{nn_0+1}, x_{nn_0})] \\ &\leq s^{n_0}(\lambda^{(n+1)n_0-1} + \cdots + \lambda^{nn_0})d(x_1, x_0) \\ &\leq s^{n_0}\lambda^{nn_0}\frac{d(x_1, x_0)}{1-\lambda}. \end{aligned}$$

So,

$$d(x_{(n+1)n_0}, x_{nn_0}) \leq C\mu^n, \tag{21}$$

where, $\mu = \lambda^{n_0}$. Using (15) we have that $\mu b < 1$. From (3_b) and (21) we obtain

$$\begin{aligned} d(x_{n_0\lfloor \frac{n}{n_0} \rfloor}, x_{n_0\lfloor \frac{n+p}{n_0} \rfloor}) &\leq \sum_{j=n_0\lfloor \frac{n}{n_0} \rfloor}^{n_0\lfloor \frac{n+p}{n_0} \rfloor - 2} s^{j+1-n_0\lfloor \frac{n}{n_0} \rfloor} d(x_{n_0 j}, x_{n_0(j+1)}) \\ &+ s^{n_0\lfloor \frac{n+p}{n_0} \rfloor - 1 - n_0\lfloor \frac{n}{n_0} \rfloor} d(x_{n_0(n_0\lfloor \frac{n+p}{n_0} \rfloor - 1)}, x_{n_0\lfloor \frac{n+p}{n_0} \rfloor}) \\ &\leq \sum_{j=n_0\lfloor \frac{n}{n_0} \rfloor}^{n_0\lfloor \frac{n+p}{n_0} \rfloor - 2} s^{j+1-n_0\lfloor \frac{n}{n_0} \rfloor} C\mu^j \\ &+ s^{n_0\lfloor \frac{n+p}{n_0} \rfloor - 1 - n_0\lfloor \frac{n}{n_0} \rfloor} C\mu^{n_0(\lfloor \frac{n+p}{n_0} \rfloor - 1)} \\ &= Cs^{1-n_0\lfloor \frac{n}{n_0} \rfloor} \sum_{j=n_0\lfloor \frac{n}{n_0} \rfloor}^{n_0\lfloor \frac{n+p}{n_0} \rfloor - 2} (s\mu)^j \\ &+ Cs^{n_0-1-n_0\lfloor \frac{n}{n_0} \rfloor}(s\mu)^{n_0(\lfloor \frac{n+p}{n_0} \rfloor - 1)}, \end{aligned}$$

Now, we have

$$\begin{aligned} \lim_{p\to+\infty} d(x_{n_0\lfloor \frac{n}{n_0} \rfloor}, x_{n_0\lfloor \frac{n+p}{n_0} \rfloor}) &\leq Cs^{1-n_0\lfloor \frac{n}{n_0} \rfloor}\frac{(s\mu)^{n_0\lfloor \frac{n}{n_0} \rfloor}}{1-s\mu} \\ &= Cs\frac{\mu^{n_0\lfloor \frac{n}{n_0} \rfloor}}{1-s\mu}. \end{aligned}$$

Therefore,

$$\lim_{p\to+\infty} d(x_{n_0\lfloor \frac{n}{n_0} \rfloor}, x_{n_0\lfloor \frac{n+p}{n_0} \rfloor}) \leq Cs\frac{\lambda^{\lfloor \frac{n}{n_0} \rfloor}}{1-s\lambda^{n_0}}. \tag{22}$$

From (6), (7), (8) and (11) we obtain

$$\lim_{p\to+\infty} d(x_n, x_{n+p}) \leq Cs\lambda^{n_0\lfloor \frac{n}{n_0} \rfloor}\left(1 + \frac{s^2}{1-s\lambda^{n_0}}\right). \tag{23}$$

On the other hand,

$$d(x_n, x^*) \leq s[d(x_n, x_{n+p}) + d(x_{n+p}, x^*)], \tag{24}$$

now, from (23) and (24), if $p \to +\infty$ we obtain (16). □

Remark 2.5. *Note that from (12) we obtain another proof of the result of Miculescu and Mihail [20] (Lemma 2.2) and Suzuki [29] (Lemma 6).*

Remark 2.6. *Theorem 2.2 is also valid for b-metric like spaces [2], that is, when we replace condition* (1_b) *in Definition 1.2 with condition*

$$d(x_1, x_2) = 0 \text{ implies } x_1 = x_2. \tag{25}$$

Remark 2.7. *Let* (X, d, s) *be a complete b-metric space and let a mapping* $T : X \to X$*. If the sequence of iterates* $\{T^n x\}$ *converges to fixed point* x^* *for any* $x \in X$ *then from (5) for a given* $\epsilon > 0$ *we can determine* $n_\varepsilon \in \mathbb{N}$ *such that it is*

$$d(x_{n_\epsilon}, x^*) \leq \varepsilon. \tag{26}$$

Thus, the estimate (5) can be used in a large number of results on fixed points in b metric spaces, where the sequence iterates $\{T^n x\}$ *converges to a fixed point* x^**. We can use it for theorems like Banach, Kannan, Chatterje, Reich, Ćirić, Gusseman, Hardy-Rogers, etc. Also, the estimation of (5) can be used for results about common fixed points, for example, Jungck, Fisher type, and for multivalued mappings, for example, Nadler type results.*

Example 2.8. *(Estimate of Banach, Kannan and Reich type contraction) As we mentioned in Remark 2.7 estimate can be used in a large number of results about fixed points in b-metric spaces. First we determine* n_ε *for the Banach contraction with parameter* $\lambda \in (0, 1)$*. Set* $\varepsilon > 0$*, then, from (5), we get*

$$Cs^2 \lambda^{n_0 \lfloor \frac{n}{n_0} \rfloor} \left(1 + \frac{s^2}{1 - s\lambda^{n_0}}\right) < \varepsilon. \tag{27}$$

For the last inequality to hold, from basic calculations, it is necessary to choose $n \geq n_\varepsilon$*, where*

$$n_\varepsilon = \left\lceil \log_\lambda \frac{\varepsilon(1 - s\lambda^{n_0})}{Cs^2(1 + s^2 - s\lambda^{n_0})} \right\rceil. \tag{28}$$

For Kannan contraction $T : X \to X$ *satisfying* $d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)]$*, where* $0 < \alpha < \frac{1}{2}$*, we can determine* n_ε *as we can determine* n_ε *by replacing* λ *with* $\frac{\alpha}{1-\alpha}$ *in (28). For Reich contraction* $T : X \to X$*, satisfying* $d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty)$*, where* $\alpha, \beta, \gamma \in (0, 1)$ *is such that h* $\alpha + \beta + \gamma < 1$*, we can determine* n_ε *by replacing* λ *with* $\frac{\alpha+\gamma}{1-\beta}$ *in (28).*

Remark 2.9. *In metric space* (X, d) *we have that* $s = 1$ *and* $n_0 = 1$*. So, from Theorem 2.4 we obtain*

$$d(x_n, x^*) \leq \frac{\lambda^n}{1 - \lambda}\left(1 + \frac{1}{1 - \lambda}\right) d(x_0, x_1), \tag{29}$$

and from Theorem 2.2 we obtain

$$d(x_n, x^*) \leq \frac{\lambda^n(2 - \lambda)}{1 - \lambda} d(x_0, x_1), \tag{30}$$

which is still a weaker estimate than the known estimate in metric spaces

$$d(x_n, x^*) \leq \frac{\lambda^n}{1 - \lambda} d(x_0, x_1).$$

Therefore, the natural question is:
Can the inequality (30) be improved?

Remark 2.10. *In paper [15] George et al introduced rectangular b-metric space. Rectangular b-metric spaces are a generalization of b-metric spaces. Therefore, an idea for further research is to obtain an estimate for* $d(x_n, x^*)$ *in rectangular b metric spaces.*

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