Letters in Nonlinear Analysis and its Applications **3** (2025) No. 3, 160-167
Available online at www.lettersinnonlinearanalysis.com
Research Article

Letters in Nonlinear Analysis and its Applications
Peer Review Scientific Journal
ISSN: 2958-874x

Caristi's coincidence and common-fixed point theorems in Hausdorff spaces

Maher Berzig

Université de Tunis, École Nationale Supérieure d'Ingénieurs de Tunis, Département de Mathématiques, 1008 Tunis, Tunisia

Abstract

In this paper, we present a Caristi-type coincidence and common-fixed point theorem and its dual in Hausdorff spaces. We also extend the Caristi-Jachymski and Caristi-Kirk-Saliga fixed point theorems. Moreover, we give a positive answer to a question of Kirk and Shahzad without assuming the standard distance axioms.

Keywords: Caristi fixed point, b-metric, b-suprametric
2010 MSC: 54H25, 47H10, 54E25

1. Introduction

Caristi's fixed point theorem [3] asserts that a mapping $T\colon X \to X$ has a fixed point in a complete metric space (X, d) if there exists a lower semi-continuous function $\varphi\colon X \to \mathbb{R}_+$ such that

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx),$$

for all $x \in X$. This theorem is well-known for being a powerful generalization of the Banach contraction principle. In [10], Kirk demonstrated that the validity of Caristi's theorem characterizes the completeness of a metric space. Jachymski, in [7], demonstrates that certain contraction multifunctions possess a single-valued selection that satisfies the assumptions of the Caristi-Jachymski fixed-point theorem. Moreover, Caristi's theorem is deeply interconnected with several pivotal results in mathematical analysis. It is equivalent to

Email address: `maher.berzig@gmail.com` (Maher Berzig)

Received: May 11, 2025, Accepted: July 22, 2025, Online: July 24, 2025

the Ekeland variational principle, Takahashi's nonconvex minimization theorem, Daneš drop theorem, the petal theorem, and the Oettli-Théra theorem, see for instance [8, 14, 15].

A key difficulty arises from the continuity properties of b-metrics. Unlike standard metrics, b-metrics are generally not continuous with respect to the topology they induce [17]. To address these issues, various enhancements have been proposed, including the concept of strong b-metric spaces [5] and strong b-suprametric [1], which satisfy asymmetric relaxed triangle inequalities and more accommodate continuity assumptions required for fixed point results.

This paper explores this direction and aims to provide meaningful generalizations of Caristi's fixed point theorem for distances that do not satisfy either symmetry or the triangle inequality. We begin by establishing a Caristi coincidence and common-fixed point theorem and its dual in Hausdorff spaces. As a consequence, we extend the Caristi-Jachymski and the Caristi-Kirk-Saliga fixed point theorems. Furthermore, we positively address Kirk and Shahzad's question [12, Remark 12.6] regarding the possibility of extending Caristi's fixed point theorem to b-metric spaces [4] and, more generally, to b-suprametric spaces [2, 9].

2. Common fixed point theorems of Caristi type

Let X be a nonempty set, and let $d\colon X \times X \to \mathbb{R}_+$ be a function. We say that (X, d) is a Hausdorff space (in the sens defined by d) if:

(i) d satisfies the identity of indiscernibles:

$$x = y \iff d(x, y) = 0,$$

(ii) and the topology induced by d (via open balls $B(x, r) := \{y \in X : d(x, r) < r\}$) is Hausdorff.

We say that (X, d) is a sequentially compact Hausdorff space if the topology induced by d makes X a sequentially compact Hausdorff space. Let $S, T\colon X \to X$ be given mappings, then S and T are called weakly compatible if they commute at their coincidence points.

Proposition 2.1. *Let $z \in X$ be a coincidence point of weakly compatible mappings $S, T\colon X \to X$ that satisfy*

$$\min\left\{d(Sx, Ty), d(Sy, Tx)\right\} \leq L \max\left\{|\varphi(Sx) - \varphi(Tx)|, |\varphi(Sy) - \varphi(Ty)|\right\}, \tag{1}$$

for all $x, y \in X$, where L is a positive real constant. Then they have a unique common fixed point.

Proof. Let $z \in X$ be a coincidence point of weakly compatible mappings $T, S\colon X \to X$ that satisfy (1), so $Sz = Tz$ and thus $STz = TSz$. Then, for some $L > 0$,

$$\min\{d(STz, Tz), d(Sz, T^2z)\} \leq L \max\{|\varphi(STz) - \varphi(T^2z)|, |\varphi(Sz) - \varphi(Tz)|\} = 0.$$

which implies that either $d(STz, Tz) = 0$ or $d(Sz, T^2z) = 0$, so $STz = Tz$ or $Sz = T^2z$. Hence, $STz = TSz = Tz = Sz = T^2z = S^2z$.

Assume now that x and x' are two common fixed points of S and T ($Tx = Sx = x$ and $Tx' = Sx' = x'$), then by applying (1), we get

$$\min\{d(Sx, Tx'), d(Sx', Tx)\} \leq L \max\{|\varphi(Sx) - \varphi(Tx)|, |\varphi(Sx') - \varphi(Tx')|\} = 0,$$

so it follows that either $d(Sx, Tx') = 0$ or $d(Sx', Tx) = 0$. We conclude that x is the unique common fixed point of S and T. □

We present next the Caristi-type common fixed point.

Theorem 2.2. *Let (X, d) be a Hausdorff space such that d is continuous, and $S, T\colon X \to X$ be continuous mappings such that T is sequentially relatively compact. Suppose there exists a continuous function $\varphi\colon X \to \mathbb{R}_+$ such that*

$$\varphi(Sx) \leq \varphi(x), \tag{2}$$

and

$$d(Sx, Tx) \leq \varphi(Sx) - \varphi(Tx), \tag{3}$$

for all $x \in X$. Then S and T have coincidence points. If, in addition, S and T are weakly compatible and satisfy (1), they have a unique common fixed point.

Proof. Let $\xi\colon X \to \mathbb{R}_+$ be a mapping given by

$$\xi(x) = d(Sx, Tx) + \varphi(Tx).$$

If $ST^n x \neq T^{n+1}x$ for all n, that is, $T^n x$ is not a coincidence point of T and S for all n, then by (2) and (3), we get

$$\begin{aligned}\xi(T^{n+1}x) &= d(ST^{n+1}x, T^{n+2}x) + \varphi(T^{n+2}x)\\ &\leq \varphi(ST^{n+1}x)\\ &< \varphi(T^{n+1}x) + d(ST^n x, T^{n+1}x) = \xi(T^n x).\end{aligned}$$

Thus, the sequence $\{\xi(T^n x)\}$ is strictly decreasing and, since it is bounded below, it converges. On the other hand, T is sequentially relatively compact mapping, so the sequence $\{T^n x\}$ possesses a convergent subsequence $\{T^{n(k)}x\}$ that converges to some point z in X. Now, due to the continuity of T, S, φ and d, we have

$$\lim_{k\to\infty} \xi(T^{n(k)+1}x) = d(STz, T^2 z) + \varphi(T^2 z) = \xi(Tz),$$

and

$$\lim_{k\to\infty} \xi(T^{n(k)}x) = d(Sz, Tz) + \varphi(Tz) = \xi(z).$$

In Hausdorff spaces, it is well-established that the limit of a sequence, if it exists, is unique. Given that $\{\xi(T^{n(k)}x)\}$ and $\{\xi(T^{n(k)+1}x)\}$ are two subsequences of the convergent sequence $\{\xi(T^n x)\}$, we conclude that

$$\xi(Tz) = \xi(z).$$

But if $Sz \neq Tz$ then

$$\begin{aligned}\xi(Tz) &= d(STz, T^2 z) + \varphi(T^2 z)\\ &\leq \varphi(STz) - \varphi(T^2 z) + \varphi(T^2 z)\\ &< \varphi(Tz) + d(Sz, Tz) = \xi(z),\end{aligned}$$

which is a contradiction. If, in addition, S and T are weakly compatible and satisfy (1), we conclude by Proposition 2.1 that S and T have a unique common fixed point. □

Example 2.3. Let $X = \mathbb{R}^n$ with the Euclidean topology $d(x, y) = \|x - y\|_2$, which is continuous. Fix $a \in \mathbb{R}^n$ and $\lambda \in (0, 1)$, and define the mapping T, S and φ by

$$T(x) = a,\ S(x) = a + \lambda(x - a) \ \text{ and } \varphi(x) = \|x - a\|_2.$$

One can check that

$$\varphi(Sx) = \lambda\|x - a\|_2 \leq \|x - a\|_2 = \varphi(x),$$

and

$$d(Sx, Tx) = \lambda\|x - a\|_2 = \varphi(Sx) - \varphi(Tx).$$

Then according to Theorem 2.2, S and T have a coincidence point, which is a. Clearly, S and T are weakly compatible, so they have common fixed points, since

$$\min\{d(Sx, Ty), d(Sy, Tx)\} = L \max\{|\varphi(Sx) - \varphi(Tx)|, |\varphi(Sy) - \varphi(Ty)|\},$$

for all $x, y \in X$. Note that a is the unique common-fixed point of S and T.

We next present the dual of the Caristi-type common fixed point theorem.

Theorem 2.4. *Let (X, d) be a Hausdorff space such that d is continuous and bounded above, and $S, T\colon X \to X$ be continuous mappings such that S is sequentially relatively compact. Suppose there exists a bounded above continuous function $\varphi\colon X \to \mathbb{R}_+$ such that*

$$\varphi(x) \leq \varphi(Tx), \tag{4}$$

and

$$d(Sx, Tx) \leq \varphi(Sx) - \varphi(Tx), \tag{5}$$

for all $x \in X$. Then S and T have coincidence points. If, in addition, S and T are weakly compatible and they satisfy the Caristi contraction, then they have a unique common fixed point.

Proof. Let $\xi\colon X \to \mathbb{R}_+$ be a mapping given by

$$\xi(x) = d(Sx, Tx) + \varphi(Tx).$$

If $TS^n x \neq S^{n+1}x$ for all n, that is, $S^n x$ is not a coincidence point of T and S for all n, then by (4) and (5), we get

$$\begin{aligned}\xi(S^n x) &= d(S^{n+1}x, TS^n x) + \varphi(TS^n x)\\ &\leq \varphi(S^{n+1}x)\\ &< \varphi(TS^{n+1}x) + d(S^{n+2}x, TS^{n+1}x) = \xi(S^{n+1}x).\end{aligned}$$

Thus, the sequence $\{\xi(S^n x)\}$ is strictly increasing and, since it is bounded above, it converges. On the other hand, S is sequentially relatively compact mapping, so the sequence $\{S^n x\}$ possesses a convergent subsequence $\{S^{n(k)}x\}$ that converges to some point z in X. Now, due to the continuity of T, S, φ and d, we have

$$\lim_{k\to\infty} \xi(S^{n(k)+1}x) = d(S^2 z, STz) + \varphi(TSz) = \xi(Sz),$$

and

$$\lim_{k\to\infty} \xi(S^{n(k)}x) = d(Sz, Tz) + \varphi(Tz) = \xi(z).$$

In Hausdorff spaces, it is well-established that the limit of a sequence, if it exists, is unique. Given that $\{\xi(S^{n(k)}x)\}$ and $\{\xi(S^{n(k)+1}x)\}$ are two subsequences of the convergent sequence $\{\xi(S^n x)\}$, we conclude that

$$\xi(Sz) = \xi(z).$$

But if $Sz \neq Tz$ then

$$\begin{aligned}\xi(z) &= d(Sz, Tz) + \varphi(Tz)\\ &\leq \varphi(Sz) - \varphi(Tz) + \varphi(Tz)\\ &< \varphi(TSz) + d(S^2 z, TSz) = \xi(Sz),\end{aligned}$$

which is a contradiction. If, in addition, S and T are weakly compatible and satisfy (1), we conclude by Proposition 2.1 that S and T have a unique common fixed point. □

Example 2.5. Let $X = \left[\frac{1}{2}, 1\right]$ equipped with the standard metric $d(x, y) = |x - y|$. Then (X, d) is a compact Hausdorff space, and d is continuous and bounded above by $\frac{1}{2}$. Define the mappings $S, T : X \to X$ by

$$S(x) = \tfrac{x}{2} + \tfrac{1}{4}, \quad T(x) = x^2,$$

which are continuous, but not weakly compatible. Note that $S(X) \subseteq \left[\frac{1}{2}, \frac{3}{4}\right] \subseteq X$, so S is sequentially relatively compact. Consider the continuous function $\varphi : X \to \mathbb{R}_+$ defined by

$$\varphi(x) = 1 - x,$$

which is bounded above by $\frac{1}{2}$ on X. We verify the inequalities:

1. For all $x \in X$,
$$\varphi(x) \leq \varphi(Tx) \iff 1-x \leq 1-x^2 \iff x^2 \leq x,$$
which holds on $\left[\frac{1}{2}, 1\right]$.
2. For all $x \in X$,
$$d(Sx, Tx) = \left|\frac{x}{2} + \frac{1}{4} - x^2\right|,$$
and
$$\begin{aligned}\varphi(Sx) - \varphi(Tx) &= (1 - Sx) - (1 - Tx)\\ &= Tx - Sx = x^2 - \left(\frac{x}{2} + \frac{1}{4}\right).\end{aligned}$$
Since for $x \in \left[\frac{1}{2}, 1\right]$,
$$x^2 \geq \frac{x}{2} + \frac{1}{4},$$
we have
$$d(Sx, Tx) = x^2 - \left(\frac{x}{2} + \frac{1}{4}\right) = \varphi(Sx) - \varphi(Tx),$$
so
$$d(Sx, Tx) \leq \varphi(Sx) - \varphi(Tx).$$

Thus, according to Theorem 2.4, S and T have a coincidence point.

3. Fixed point theorems of Caristi type

In this section, we will use following inequalities:
$$\min\left\{d(x, Ty), d(y, Tx)\right\} \leq L \max\left\{|\varphi(x) - \varphi(Tx)|, |\varphi(y) - \varphi(Ty)|\right\}, \tag{6}$$
and
$$\min\left\{d(Sx, y), d(Sy, x)\right\} \leq L \max\left\{|\varphi(Sx) - \varphi(x)|, |\varphi(Sy) - \varphi(y)|\right\}, \tag{7}$$
for all $x, y \in X$, where L is a positive real constant.

The first corollary is the following:

Corollary 3.1. *Let (X, d) be a Hausdorff space such that d is continuous, and $T\colon X \to X$ be a sequentially relatively compact continuous mapping. Suppose there exists a continuous function $\varphi\colon X \to \mathbb{R}_+$ such that*
$$d(x, Tx) \leq \varphi(x) - \varphi(Tx), \quad \text{for all } x \in X. \tag{8}$$
Then T has a fixed point. If, in addition, T satisfy (6), the fixed point is unique.

Proof. Follows from Theorem 2.2, by taking S the identity mapping. □

The dual of Caristi's fixed point theorem is stated as follows.

Corollary 3.2. *Let (X, d) be a sequentially compact Hausdorff space such that d is continuous and bounded above, and $S\colon X \to X$ be a continuous mapping. Suppose there exists a bounded above continuous function $\varphi\colon X \to \mathbb{R}_+$ such that*
$$d(Sx, x) \leq \varphi(Sx) - \varphi(x), \quad \text{for all } x \in X. \tag{9}$$
Then S has a fixed point. If, in addition, S satisfy (7), the fixed point is unique.

Proof. Follows from Theorem 2.4, by taking T the identity mapping. □

We next extend the Caristi-Kirk-Saliga fixed point theorem [11, Theorem 2.3].

Corollary 3.3. *Let (X, d) be a Hausdorff space such that d is continuous, and let $T\colon X \to X$ be a sequentially relatively compact continuous mapping. Suppose there exist a positive constant $p \in \mathbb{N}$ and a continuous function $\psi\colon X \to \mathbb{R}_+$ such that*

$$d(x, Tx) \leq \psi(x) - \psi(T^p x), \text{ for all } x \in X. \tag{10}$$

Then T has a fixed point. If, in addition, T satisfy (6), the fixed point is unique.

Proof. Let

$$\varphi(x) = \sum_{i=0}^{p-1} \psi(T^i x), \text{ for all } x \in X. \tag{11}$$

Then (10) reduces to (8). The result follows then by Corollary 3.1. □

Corollary 3.4. *Let (X, d) be a sequentially compact Hausdorff space such that d is continuous and bounded above, and let $S\colon X \to X$ be a continuous mapping. Suppose there exist a positive constant $p \in \mathbb{N}$ and a bounded above continuous function $\psi\colon X \to \mathbb{R}_+$ such that*

$$d(Sx, x) \leq \psi(S^p x) - \psi(x), \quad \text{for all } x \in X. \tag{12}$$

Then S has a fixed point. If, in addition, S satisfy (7), the fixed point is unique.

Proof. Similar to the previous corollary, the result follows by Corollary 3.2. □

Remark 3.5. Note that in Corollaries 3.3 and 3.4, we do not assume the condition $\psi(Tx) \leq \psi(x)$, which is required in [11, Theorem 2.3]. However, our results require additional regularity assumptions on the involved functions.

We also obtain Caristi-Jachymski fixed point theorems as follows.

Corollary 3.6. *Let (X, d) be a Hausdorff space such that d is continuous, and let $T\colon X \to X$ be a sequentially relatively compact continuous mapping. Suppose there exist a continuous functions $\varphi, \eta\colon X \to \mathbb{R}_+$ with $\eta^{-1}(0) = \{0\}$, such that*

$$\eta(d(x, Tx)) \leq \varphi(x) - \varphi(Tx), \quad \text{for all } x \in X. \tag{13}$$

Then T has a fixed point. If, in addition, T satisfy (6), the fixed point is unique.

Proof. First, observe that the topologies induced by d and d_η are the same, where $d_\eta := \eta \circ d$. Therefore, (X, d_η) is a sequentially compact Hausdorff space, and the result follows from Corollary 8. □

Corollary 3.7. *Let (X, d) be a sequentially compact Hausdorff space such that d is continuous and bounded above, and let $S\colon X \to X$ be a continuous mapping. Suppose there exist a bounded above continuous functions $\varphi, \eta\colon X \to \mathbb{R}_+$ with $\eta^{-1}(0) = \{0\}$, such that*

$$\eta(d(Sx, x)) \leq \varphi(Sx) - \varphi(x), \quad \text{for all } x \in X. \tag{14}$$

Then S has a fixed point. If, in addition, S satisfy (7), the fixed point is unique.

Proof. Similarly, observe that the topologies induced by d and d_η are the same. Hence, (X, d_η) is a sequentially compact Hausdorff space, and the result follows by applying Corollary 9. □

Remark 3.8. Note that in Corollaries 3.6 and 3.7, we do not assume the subadditive condition $\eta(s + t) \leq \eta(s) + \eta(t)$ for all $s, t \in \mathbb{R}_+$, as stated in [7, Theorem 6]. Instead, we consider additional regularity assumptions on the functions involved.

It is clear that every b-suprametric space is Hausdorff and we have the following result.

Proposition 3.9. *Ever sequentially compact b-suprametric is bounded. In particular, every sequentially compact b-metric is bounded.*

Proof. Suppose by contradiction that X is not bonded. Then, for every $n \in \mathbb{N}$, there is $x_n \in X$ such that

$$d(x_0, x_n) \geq n \text{ for some } x_0 \in X.$$

But since X is sequentially compact b-suprametric, there must exist a convergent subsequence $\{x_{n(k)}\}$ to some $x \in X$. That means,

$$\lim_{k\to\infty} d(x_{n(k)}, x) = 0.$$

Now, by using the relaxed triangle inequality of the b-suprametric space, we get

$$d(x_0, x_{n(k)}) \leq b(d(x_0, x) + d(x, x_{n(k)})) + cd(x_0, x)d(x, x_{n(k)}),$$

where $b \geq 1$ and $c \geq 0$. Hence, by taking k to infinity, the right-hand side becomes bounded. Thus, the left-hand side is also bounded, which is a contradiction. □

Next, we immediately derive the following corollaries.

Corollary 3.10. *Let (X, d) be a b-suprametric space such that d is continuous, and let $T: X \to X$ be a sequentially relatively compact continuous mapping. Suppose that there exists a continuous function $\varphi: X \to \mathbb{R}_+$ such that (8) holds. Then T has a fixed point. If, in addition, T satisfy (6), the fixed point is unique.*

Corollary 3.11. *Let (X, d) be a sequentially compact b-suprametric space such that d is continuous, and let $S: X \to X$ be a continuous mapping. Suppose that there exists a bounded above continuous function $\varphi: X \to \mathbb{R}_+$ such that (9) holds. Then S has a fixed point. If, in addition, S satisfy (7), the fixed point is unique.*

Corollary 3.12. *Let (X, d) be a b-metric space such that d is continuous, and let $T: X \to X$ be a sequentially relatively compact continuous mapping. Suppose that there exists a continuous function $\varphi: X \to \mathbb{R}_+$ such that (8) holds. Then T has a fixed point. If, in addition, T satisfy (6), the fixed point is unique.*

Corollary 3.13. *Let (X, d) be a sequentially compact b-metric space such that d is continuous, and let $S: X \to X$ be a continuous mapping. Suppose that there exists a bounded above continuous function $\varphi: X \to \mathbb{R}_+$ such that (9) holds. Then S has a fixed point. If, in addition, S satisfy (7), the fixed point is unique.*

Finally, it is natural to inquire whether the hypotheses of the main theorems align with those of Caristi's fixed point theorem in metric spaces. Specifically, we pose the following questions:

Q1. Under what conditions can Theorems 2.2 and 2.4 be extended to non-continuous mappings.

Q2. Is it possible to relax the continuity of φ and instead use lower semi-continuity?

References

[1] M. Berzig. *Strong b-Suprametric Spaces and Fixed Point Principles.* Complex Anal. Oper. Theory **18** (2024), 1–26.

[2] M. Berzig. *Nonlinear contraction in b-suprametric spaces.* J. Anal. **32** (2024), 2401–2414.

[3] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions.* Trans. Amer. Math. Soc. **215** (1976), 241–251.

[4] S. Czerwik. *Contraction mappings in b-metric spaces.* Acta Math. Univ. Ostrav. **1** (1993), 5–11.

[5] S. Czerwik. *On b-metric spaces and Brower and Schauder fixed point principles.* In: Approximation Theory and Analytic Inequalities, pp. 71–86. Springer 2021.

[6] N.V. Dung and V.T.L Hang. *On relaxations of contraction constants and Caristi's theorem in b-metric spaces.* J. Fixed Point Theory Appl. **18** (2016), 267–284.

[7] J.R. Jachymski. *Caristi's fixed point theorem and selections of set-valued contractions.* J. Math. Anal. Appl. **227** (1998), 55–67.

[8] E. Karapınar. *Generalizations of Caristi Kirk's theorem on partial metric spaces.* Fixed Point Theory and Appl. (2011), 1–7.

[9] H. Khandani. *Interpolative metrics are not new: a study of generalized contractions in b-suprametric spaces.* arXiv:2506.-11205 (2025), 1–11.

[10] W.A. Kirk. *Caristi's fixed point theorem and metric convexit*y. Colloq. Math. **361**, (1976), 81–86.

[11] W.A. Kirk and L.M. Saliga. *The Brézis-Browder order principle and extensions of Caristi's theorem.* Nonlinear Anal. **47** (2001), 2765–2778.

[12] W. Kirk and N. Shahzad. Fixed Point Theory in Distance Spaces. Springer, Cham, 2014.

[13] R. Miculescu, A. Mihail, *Caristi-Kirk type and Boyd & Wong–Browder-Matkowski-Rus type fixed point results in b-metric spaces,* Filomat **31** (2017), 4331–4340.

[14] W. Oettli and M. Théra. *Equivalents of Ekeland's principle.* Bull. Aust. Math. Soc. **48** (1993), 385–392.

[15] J.P. Penot. *The drop theorem, the petal theorem and Ekeland's variational principle.* Nonlinear Anal. **10** (1986), 813–822.

[16] S. Romaguera. *A theorem of Caristi-Kirk type for b-metric spaces.* Fixed Point Theory **25** (2024), 371–378.

[17] T. Van An, L.Q. Tuyen and N. Van Dung. *Stone-type theorem on b-metric spaces and applications.* Topology and its Appl. **185** (2015), 50–64.