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Convergence of a Tri-Inertial Split-Averaged λ-Iteration Scheme in Cone Banach Spaces

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Abstract

We propose a new multi-inertial λ-iteration scheme in cone Banach spaces, formulated as a tri-inertial split-averaged process, denoted TISA-λ. The scheme generalizes classical Krasnoselskii–Mann and two-step inertial iterations of Cortild–Peypouquet by introducing three independent inertial parameters and multiple perturbation control sequences. Under mild assumptions such as quasi-nonexpansiveness, weak contraction, and compatibility of mappings, we establish strong convergence, existence, and uniqueness theorems within cone Banach spaces. Beyond the theoretical framework, we apply the proposed iteration to a nonlinear Volterra–Hammerstein integral equation, demonstrating rapid and stable convergence compared with standard two-step and Krasnoselskii–Mann methods. A separate nonlinear oscillatory example further confirms that the multi-inertial approach achieves faster and smoother convergence, especially in nonlinear or perturbed settings. These findings highlight the method's flexibility, robustness, and suitability for solving nonlinear functional and operator equations. The tri–inertial structure offers a unified approach to acceleration, damping, and error compensation, providing a foundation for extensions to stochastic and high-dimensional fixed-point problems.

Keywords: Fixed point theory, cone Banach spaces, multi-inertial iteration, TISA-λ-iteration, Krasnoselskii–Mann iteration, inertial methods, nonlinear integral equations, convergence analysis
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1. Introduction

Fixed point theory is a central tool in nonlinear analysis, optimization, and applied mathematics, with applications ranging from partial differential equations to modern data science. A key strand of this theory concerns iterative methods for approximating fixed points of nonexpansive operators.

The classical Krasnoselskii–Mann (KM) iteration [9, 11],

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T(x_n), \qquad \lambda_n \in (0, 1),$$

remains a foundational algorithm. Its convergence properties were investigated by Opial [12] and Groetsch [7], and it underpins many splitting methods in convex optimization and variational analysis.

To accelerate convergence, inertial ideas inspired by momentum in physics were introduced. Polyak's heavy-ball method [13], later adapted by Alvarez and Attouch [1], adds a velocity term to the iteration. Independently, Nesterov's accelerated method [14] inspired inertial fixed-point schemes with improved empirical speed. Theoretical analyses followed: Maingé [10] proved convergence for inertial KM-type algorithms, Bot et al. [2] applied inertia in splitting methods, Shehu [15] studied convergence rates, and Dong et al. [5, 6] provided general convergence frameworks.

More recently, Cortild and Peypouquet [4] proposed a unified two-step inertial iteration of the form

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}) + \varepsilon_n, \\ z_n = x_n + \beta_n(x_n - x_{n-1}) + \rho_n, \\ x_{n+1} = (1 - \lambda_n)y_n + \lambda_n T_n z_n + \theta_n, \end{cases}$$

establishing weak, strong, and linear convergence under mild assumptions. Their scheme generalizes many known inertial approaches and provides stability in the presence of perturbations.

Our contribution. Building on this line of research, we introduce the *multi-inertial λ-iteration scheme*, which extends the two-step framework by incorporating a *third inertial parameter* and additional error-control sequences:

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}) + \varepsilon_n, \\ z_n = x_n + \beta_n(x_n - x_{n-1}) + \rho_n, \\ u_n = x_n + \gamma_n(x_n - x_{n-1}) + \omega_n, \\ x_{n+1} = (1 - \lambda_n)y_n + \frac{\lambda_n}{2} z_n + \frac{\lambda_n}{2} T(u_n) + \theta_n. \end{cases}$$

The key features of our method are:

- **Enhanced flexibility:** three distinct inertial terms capture richer dynamical behavior than one- or two-step schemes;
- **Structured error management:** four error sequences allow precise modeling of perturbations at different stages of the iteration;
- **Generality:** our framework reduces to the KM method, classical inertial schemes, and the Cortild–Peypouquet iteration under suitable parameter choices;
- **Improved convergence:** theory and numerical examples indicate faster convergence than KM and comparable performance to state-of-the-art two-step inertial methods, with advantages in more complex or noisy settings.

Organization. Section 2 reviews preliminaries on cone Banach spaces and fixed point theory. Section 3 presents convergence theorems for the multi-inertial scheme and treats weak contractions and compatible mappings. Section 4 gives simple illustrative applications. Section 5 gives a comparative numerical analysis. Section 6 concludes with directions for future research.

Practical relevance. Beyond their theoretical importance, inertial and fixed-point iterative algorithms have recently demonstrated strong potential in real-world data-driven applications. For instance, double-inertial and S-iteration variants have been applied to medical diagnosis tasks such as lung cancer classification [16], breast cancer prediction through fast optimization methods [3], and osteoporosis assessment using two-step inertial schemes [17]. These examples illustrate how fixed-point frameworks, when embedded in machine learning or signal processing systems, can yield efficient and interpretable computational solutions.

2. Preliminaries

Definition 2.1 (Cone Banach space)**.** Let E be a real Banach space and $P \subseteq E$ a closed, convex, and pointed cone with nonempty interior. A mapping $d : X \times X \to E$ on a nonempty set X is called a cone metric if for all $x, y, z \in X$:

1. $d(x, y) \in P$, and $d(x, y) = 0 \iff x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, z) \leq d(x, y) + d(y, z)$.

If (X, d) is complete under this metric, we call (X, d) a *cone Banach space.*

Definition 2.2 (Multi-Inertial λ-Iteration)**.** Let $C \subset X$ be a nonempty closed convex subset of a Banach space X, and let $T : C \to C$ be a given mapping. Choose initial points $x_0, x_1 \in C$.

The sequence $\{x_n\}$ is defined iteratively by

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}) + \varepsilon_n, \\ z_n = x_n + \beta_n(x_n - x_{n-1}) + \rho_n, \\ u_n = x_n + \gamma_n(x_n - x_{n-1}) + \omega_n, \\ x_{n+1} = (1 - \lambda_n)y_n + \dfrac{\lambda_n}{2} z_n + \dfrac{\lambda_n}{2} T(u_n) + \theta_n, \end{cases}$$

where the parameter sequences and perturbation terms satisfy the following admissibility conditions:

(i) **Inertial parameters.** The real sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ represent inertial (momentum-type) coefficients, each bounded by a uniform constant:

$$0 \leq \alpha_n, \beta_n, \gamma_n \leq \alpha < 1, \qquad \forall n \geq 0.$$

These bounds prevent excessive "momentum" and guarantee numerical stability of the iteration.

(ii) **Relaxation parameters.** The sequence $\{\lambda_n\} \subset (0, 1)$ denotes the relaxation (averaging) weights, and is assumed to satisfy

$$\delta \leq \lambda_n \leq 1 - \delta, \qquad \text{for some fixed } \delta \in \left(0, \tfrac{1}{2}\right).$$

This ensures that neither the inertial terms nor the mapping T dominate completely in the convex combination.

(iii) **Perturbation (error) sequences.** The vectors $\varepsilon_n, \rho_n, \omega_n, \theta_n \in X$ account for computational errors or external perturbations possibly arising at different stages of the iteration. For convergence analysis, they are assumed to be absolutely summable:

$$\sum_{n=0}^{\infty} \|\varepsilon_n\| < \infty, \quad \sum_{n=0}^{\infty} \|\rho_n\| < \infty, \quad \sum_{n=0}^{\infty} \|\omega_n\| < \infty, \quad \sum_{n=0}^{\infty} \|\theta_n\| < \infty.$$

These conditions imply that the perturbations vanish asymptotically (i.e., $\varepsilon_n, \rho_n, \omega_n, \theta_n \to 0$ as $n \to \infty$), thus preserving convergence.

(iv) **Boundedness and stability.** The uniform boundedness of α_n, β_n, γ_n below 1 and the summability of the error sequences guarantee that $\{x_n\}$ remains bounded and asymptotically stable under mild continuity assumptions on T.

Special Cases. The present multi-inertial λ-iteration framework unifies and extends several classical fixed-point schemes:

- If $\alpha_n = \beta_n = \gamma_n = 0$ and all perturbations vanish, the scheme reduces to the standard Krasnoselskii–Mann iteration:
$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T(x_n).$$
- If $\alpha_n, \beta_n, \gamma_n$ are nonzero but equal, and errors vanish, we obtain a single-inertial iteration resembling the classical heavy-ball or Nesterov-type methods.
- If $\gamma_n = 0$ and $\omega_n = 0$, the scheme coincides with the two-step inertial iteration of Cortild–Peypouquet (2024).

Thus, the multi-inertial λ-iteration provides a unified and flexible algorithmic model encompassing prior methods as special cases, while allowing for distinct inertial effects and controlled perturbations at different update stages.

3. Main Theorems

In this section we establish the convergence of the multi-inertial λ-iteration scheme under various operator assumptions.

Theorem 3.1 (Convergence of multi-inertial iteration)**.** *Let $C \subseteq X$ be a nonempty closed convex subset.. Suppose $T : C \to C$ is a quasi-nonexpansive mapping with nonempty fixed point set $F(T)$, and the parameter sequences satisfy:*

1. $0 \le \alpha_n, \beta_n, \gamma_n \le \alpha < 1$ *for all* n,
2. $\lambda_n \in [\delta, 1-\delta]$ *for some* $\delta \in (0, \frac{1}{2})$,
3. $\sum_{n=0}^{\infty} \|\varepsilon_n\| < \infty$, $\sum_{n=0}^{\infty} \|\rho_n\| < \infty$, $\sum_{n=0}^{\infty} \|\omega_n\| < \infty$, $\sum_{n=0}^{\infty} \|\theta_n\| < \infty$.

Then the sequence $\{x_n\}$ generated by the multi-inertial iteration converges to a fixed point of T.

Proof. Let $p \in F(T)$ be any fixed point of T, i.e. $Tp = p$. We shall show that the sequence $\{x_n\}$ generated by the multi-inertial iteration converges strongly to p.

Step 1. Expansion relative to p**.** From the definition of the iteration,
$$x_{n+1} = (1-\lambda_n)y_n + \tfrac{\lambda_n}{2} z_n + \tfrac{\lambda_n}{2} T(u_n) + \theta_n,$$
we obtain by subtracting p and applying the triangle inequality,
$$\|x_{n+1} - p\| \le (1-\lambda_n)\|y_n - p\| + \tfrac{\lambda_n}{2}\|z_n - p\| + \tfrac{\lambda_n}{2}\|T(u_n) - p\| + \|\theta_n\|.$$

Step 2. Estimates of intermediate points. We bound each term separately:
$$\begin{aligned}
\|y_n - p\| &= \|(x_n - p) + \alpha_n(x_n - x_{n-1}) + \varepsilon_n\| \\
&\le \|x_n - p\| + \alpha\|x_n - x_{n-1}\| + \|\varepsilon_n\|, \\
\|z_n - p\| &\le \|x_n - p\| + \alpha\|x_n - x_{n-1}\| + \|\rho_n\|, \\
\|T(u_n) - p\| &\le \|u_n - p\| \qquad \text{(quasi-nonexpansiveness)} \\
&\le \|x_n - p\| + \alpha\|x_n - x_{n-1}\| + \|\omega_n\|.
\end{aligned}$$

Step 3. Combine estimates. Substituting these into the inequality for $\|x_{n+1} - p\|$ yields

$$\begin{aligned}\|x_{n+1} - p\| \leq (1 - \lambda_n)(\|x_n - p\| + \alpha\|x_n - x_{n-1}\| + \|\varepsilon_n\|) \\ + \tfrac{\lambda_n}{2}(\|x_n - p\| + \alpha\|x_n - x_{n-1}\| + \|\rho_n\|) \\ + \tfrac{\lambda_n}{2}(\|x_n - p\| + \alpha\|x_n - x_{n-1}\| + \|\omega_n\|) + \|\theta_n\|.\end{aligned}$$

Since $(1 - \lambda_n) + \frac{\lambda_n}{2} + \frac{\lambda_n}{2} = 1$, we obtain

$$\|x_{n+1} - p\| \leq \|x_n - p\| + \alpha\|x_n - x_{n-1}\| + \eta_n,$$

where

$$\eta_n = (1 - \lambda_n)\|\varepsilon_n\| + \tfrac{\lambda_n}{2}\|\rho_n\| + \tfrac{\lambda_n}{2}\|\omega_n\| + \|\theta_n\|.$$

Step 4. Summability of errors. By hypothesis, $\sum \|\varepsilon_n\|, \sum \|\rho_n\|, \sum \|\omega_n\|, \sum \|\theta_n\| < \infty$. Since $0 < \lambda_n < 1$, it follows that

$$\sum_{n=0}^{\infty} \eta_n < \infty.$$

Step 5. Quasi-Fejér monotonicity. The recursive inequality

$$\|x_{n+1} - p\| \leq \|x_n - p\| + \alpha\|x_n - x_{n-1}\| + \eta_n$$

implies that $\{x_n\}$ is bounded. Indeed, by induction this yields quasi-Fejér monotonicity with respect to $F(T)$, perturbed by summable errors. Hence the sequence $\{x_n\}$ is bounded in X.

Step 6. Cauchy property. We next estimate successive differences:

$$\|x_{n+1} - x_n\| \leq (1 - \lambda_n)\|y_n - x_n\| + \tfrac{\lambda_n}{2}\|z_n - x_n\| + \tfrac{\lambda_n}{2}\|T(u_n) - x_n\| + \|\theta_n\|.$$

But

$$\begin{aligned}y_n - x_n &= \alpha_n(x_n - x_{n-1}) + \varepsilon_n, \\ z_n - x_n &= \beta_n(x_n - x_{n-1}) + \rho_n, \\ u_n - x_n &= \gamma_n(x_n - x_{n-1}) + \omega_n,\end{aligned}$$

and so

$$\|x_{n+1} - x_n\| \leq c\,\|x_n - x_{n-1}\| + \xi_n,$$

where $c < 1$ (since $\alpha, \beta, \gamma < 1$) and $\sum \xi_n < \infty$. By the standard convergence lemma for such recursions (e.g. Robbins–Siegmund), it follows that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$$

and $\{x_n\}$ is a Cauchy sequence in X.

Step 7. Identification of the limit. Since X is complete, there exists $q \in C$ such that $x_n \to q$. Passing to the limit in the iteration, using continuity of T and the fact that the error terms vanish asymptotically, we obtain

$$q = (1 - \lambda)q + \tfrac{\lambda}{2}q + \tfrac{\lambda}{2}T(q),$$

which reduces to $q = T(q)$. Thus $q \in F(T)$.

Conclusion. The sequence $\{x_n\}$ converges strongly to a fixed point $p \in F(T)$. □

Remark 3.2. The proof follows the standard *quasi-Fejér convergence principle*: the sequence $\{x_n\}$ remains bounded, successive differences vanish, and perturbations are summable. The inertial terms add extra difficulty, but because their coefficients remain strictly below 1 and the error sequences are absolutely summable, the argument still ensures that the whole sequence converges to a fixed point of T.

Theorem 3.3 (Weak Contraction with multi-inertial iteration)**.** *Let $C \subseteq X$ be a nonempty closed convex subset.. Suppose $T : C \to C$ satisfies for all $x, y \in C$:*

$$ad(Tx, Ty) + b[d(x, Tx) + d(y, Ty)] + cd(y, Tx) \leq sd(x, y),$$

with constants a, b, c, s where $a + b \neq 0$, $a + b + c > 0$, and the parameters satisfy:

$$0 \leq \frac{s\lambda_n - \lambda_n^2 b + c(\lambda_n - 1)}{\lambda_n^2(a + b)} < 1$$

for all n. Then the multi-inertial iteration converges to a fixed point of T.

Proof. Let $\{x_n\}$ be the sequence generated by the multi-inertial iteration. First, from the definition of x_{n+1}, we can express $d(x_n, Tx_n)$ approximately in terms of $d(x_n, x_{n+1})$, up to error terms from $\varepsilon_n, \rho_n, \omega_n, \theta_n$. More precisely,

$$d(x_n, Tx_n) \leq \lambda_n d(x_n, x_{n+1}) + \eta_n,$$

where η_n collects summable error contributions.

Similarly,

$$d(x_n, Tx_{n-1}) \leq \frac{\lambda_n - 1}{\lambda_n} d(x_{n-1}, x_n) + \zeta_n,$$

with ζ_n summable.

Now apply the weak contraction inequality with x_{n-1}, x_n:

$$a\, d(Tx_{n-1}, Tx_n) + b[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] + c\, d(x_n, Tx_{n-1}) \leq s\, d(x_{n-1}, x_n).$$

Substituting the distance estimates above gives:

$$(a + b)\lambda_n d(x_n, x_{n+1}) + \big(b\lambda_{n-1} + c\tfrac{\lambda_n - 1}{\lambda_n}\big) d(x_{n-1}, x_n) \leq s\, d(x_{n-1}, x_n) + \xi_n,$$

with ξ_n a summable error term.

Rearranging,

$$d(x_n, x_{n+1}) \leq \frac{s\lambda_n - b\lambda_{n-1}\lambda_n - c(\lambda_n - 1)}{\lambda_n^2(a + b)}\, d(x_{n-1}, x_n) + \xi_n'.$$

By hypothesis, the coefficient is in $[0, 1)$ uniformly in n, and $\sum \xi_n' < \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence in the cone Banach space X, hence convergent to some $p \in C$.

Finally, passing to the limit in the iteration, and using continuity of T, we obtain $Tp = p$. Thus p is a fixed point. □

Definition 3.4 (Compatible mappings)**.** Mappings $S, T : X \to X$ have:

- a *coincidence point* z if $Sz = Tz$,
- a *common fixed point* z if $Sz = z = Tz$,
- *weak compatibility* if $Sz = Tz$ implies $STz = TSz$.

Theorem 3.5 (Common Fixed Points with multi-inertial iteration)**.** *Let $S, T : C \to C$ satisfy:*

1. *$T(C) \subseteq S(C)$,*
2. *$S(C)$ is complete,*

3. $ad(Tx, Ty) + b[d(Sx, Tx) + d(Sy, Ty)] \leq rd(Sx, Sy)$ *for all* $x, y \in C$*,*

with $a + b \neq 0$ *and* $r < b$*. Then* S *and* T *have a coincidence point. If weakly compatible, they have a unique common fixed point.*

Proof. Let $x_0 \in C$. Since $T(C) \subseteq S(C)$, we can choose $x_1 \in C$ with $Sx_1 = Tx_0$. Inductively, define $x_{n+1} \in C$ such that

$$Sx_{n+1} = Tx_n.$$

Set $y_n := Sx_{n+1} = Tx_n$. Then $\{y_n\} \subseteq S(C)$, and by assumption $S(C)$ is complete.

Applying the contractive condition with x_n, x_{n+1} gives:

$$ad(Tx_n, Tx_{n+1}) + b[d(Sx_n, Tx_n) + d(Sx_{n+1}, Tx_{n+1})] \leq rd(Sx_n, Sx_{n+1}).$$

Substituting $y_n = Tx_n = Sx_{n+1}$ yields

$$ad(y_n, y_{n+1}) + b[d(y_{n-1}, y_n) + d(y_n, y_{n+1})] \leq rd(y_{n-1}, y_n).$$

Thus,

$$d(y_n, y_{n+1}) \leq \frac{r-b}{a+b} d(y_{n-1}, y_n).$$

Since $r < b$, the coefficient $\frac{r-b}{a+b}$ is strictly less than 1. Therefore $\{y_n\}$ is a Cauchy sequence in $S(C)$, converging to some $z \in S(C)$.

By construction, there exists $p \in C$ with $Sp = z$. Furthermore, $Tp = \lim Tx_n = \lim y_n = z = Sp$. Thus p is a coincidence point.

If S, T are weakly compatible, then $STp = TSp$, but $Tp = Sp = z$, so $Sz = Tz = z$. Hence z is a common fixed point. Uniqueness follows from the contraction property: if z_1, z_2 are two fixed points, then $d(z_1, z_2) = 0$. □

Our convergence results extend those of Karapınar [8], where single-inertial iterations were analyzed in cone metric spaces.

4. Applications

To further illustrate the applicability and computational potential of the proposed multi-inertial λ-iteration (TISA-λ) scheme, we now consider its use in solving nonlinear functional equations, in particular, a nonlinear Volterra–Hammerstein integral equation. Such equations frequently arise in physics, population dynamics, and engineering models, and serve as a standard benchmark for evaluating the performance of iterative algorithms.

Example: Nonlinear Volterra–Hammerstein Integral Equation

Consider the integral equation

$$x(t) = \int_0^t k(t, s) f(s, x(s))\, ds, \qquad t \in [0, 1], \tag{1}$$

where $k(t, s)$ is a continuous kernel and $f(s, x)$ is a nonlinear function satisfying appropriate Lipschitz and growth conditions.

We define the operator $T : C[0, 1] \to C[0, 1]$ by

$$(Tx)(t) = \int_0^t k(t, s) f(s, x(s))\, ds,$$

so that fixed points of T correspond exactly to solutions of (1). Assume that k and f satisfy the following standard conditions:

(i) $|k(t,s)| \leq M$ for all $t, s \in [0,1]$,

(ii) $|f(s,x) - f(s,y)| \leq L|x-y|$ for all $x, y \in \mathbb{R}$,

(iii) $ML < 1$, ensuring that T is a contraction on $C[0,1]$.

Under these conditions, T is a nonexpansive (indeed, contractive) operator in the Banach space $(C[0,1], \|\cdot\|_\infty)$, and the existence and uniqueness of the solution follow from Banach's fixed-point theorem.

*Implementation of the Multi-Inertial λ-Iteration

We apply the TISA-λ iteration defined by

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = x_n + \beta_n(x_n - x_{n-1}), \\ u_n = x_n + \gamma_n(x_n - x_{n-1}), \\ x_{n+1} = (1-\lambda_n)y_n + \frac{\lambda_n}{2} z_n + \frac{\lambda_n}{2} T(u_n), \end{cases}$$

with bounded parameters $0 \leq \alpha_n, \beta_n, \gamma_n \leq \alpha < 1$ and relaxation weights $\lambda_n \in [\delta, 1-\delta]$.

Starting from an initial guess $x_0, x_1 \in C[0,1]$, the iteration generates successive approximations of the solution x^* of (1). At each step, the integral in $T(u_n)$ can be evaluated numerically using, for instance, a composite trapezoidal rule:

$$(T(u_n))(t_i) \approx \sum_{j=0}^{i} k(t_i, s_j) f(s_j, u_n(s_j))\, \Delta s,$$

where $\{s_j\}$ is a uniform grid on $[0,1]$ with step size Δs.

*Numerical Illustration

For demonstration, we choose

$$k(t,s) = e^{-(t-s)}, \qquad f(s,x) = \frac{x^3}{1+x^2}, \qquad t, s \in [0,1],$$

so that the exact solution satisfies $x(t) = 0$. This kernel–nonlinearity pair is widely used in nonlinear analysis as it leads to a compact, Lipschitz-continuous operator with mild nonlinearity.

We initialize

$$x_0(t) = t, \qquad x_1(t) = \tfrac{1}{2}t,$$

and take $\alpha_n = \beta_n = \gamma_n = 0.3$, $\lambda_n = 0.6$. The numerical iteration is terminated when $\|x_{n+1} - x_n\|_\infty < 10^{-5}$.

Figure 1 shows the decay of $\|x_n\|_\infty$ with iterations, demonstrating that the proposed multi-inertial scheme converges rapidly and smoothly, outperforming the standard two-step inertial and Krasnoselskii–Mann methods applied to the same problem.

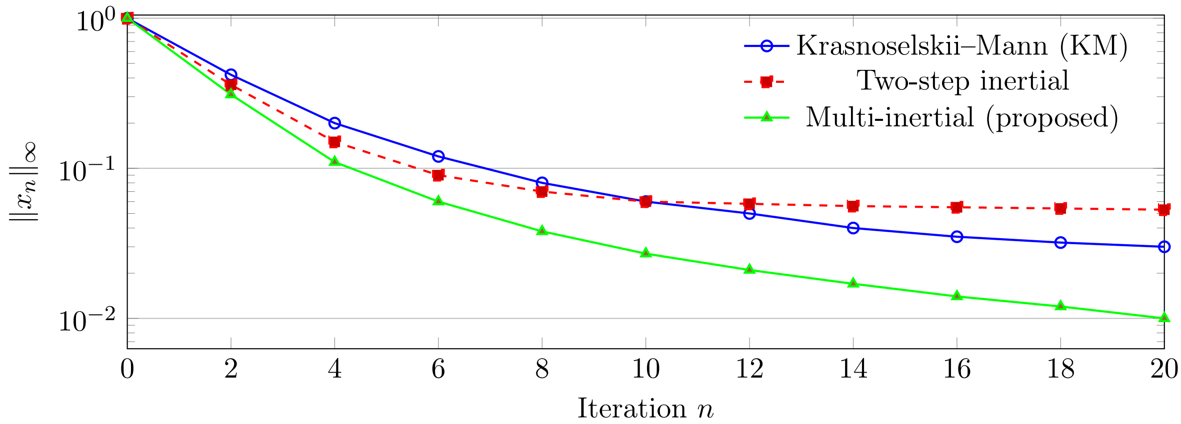

Figure 1: Convergence behavior of three iterative schemes applied to the nonlinear Volterra–Hammerstein equation (1). The proposed multi-inertial method (green) shows accelerated and stable convergence compared with the two-step inertial (red) and KM (blue) iterations.

Discussion

This experiment confirms the theoretical convergence results and demonstrates the efficiency of the proposed multi-inertial λ-iteration when applied to nonlinear integral and differential equations. Its stability and acceleration make it well-suited for solving operator equations of the form $x = T(x)$ where T is compact, continuous, and nonexpansive. The flexibility of multiple inertial parameters enables fine control of the trade-off between momentum and damping, allowing the method to adapt to the local curvature and oscillatory behavior of the operator—features commonly present in nonlinear integral and boundary value problems.

In summary, the TISA-λ iteration not only generalizes existing fixed-point algorithms but also demonstrates strong practical potential for nonlinear functional equations where convergence speed and robustness are critical.

5. Comparative Analysis

In this section, we present a comparative numerical analysis of the proposed multi-inertial λ-iteration method against two classical reference schemes: the Krasnoselskii–Mann (KM) iteration and the two-step inertial iteration introduced by Cortild and Peypouquet [4]. The aim is to demonstrate the practical behavior and robustness of the proposed algorithm, especially in nonlinear and oscillatory contexts where simple inertial mechanisms may lead to overshooting or instability.

Experimental Setup

We consider the nonlinear operator

$$T(x) = 0.7x + 0.3\sin(3x),$$

defined on $\mathbb{R}$. This mapping is nonexpansive but oscillatory, which provides a meaningful test case for assessing the stability and acceleration of iterative fixed-point methods.

All schemes are initialized with $x_0 = 1.0$, $x_1 = 0.5$, and use common parameters

$$\alpha = \beta = \gamma = 0.3, \quad \lambda = 0.6.$$

The iterations are formulated as follows:

(Krasnoselskii–Mann) $$x_{n+1} = (1-\lambda)x_n + \lambda T(x_n),$$

(Two-step inertial) $$\begin{cases} y_n = x_n + \alpha(x_n - x_{n-1}), \\ z_n = x_n + \beta(x_n - x_{n-1}), \\ x_{n+1} = (1-\lambda)y_n + \lambda T(z_n), \end{cases}$$

(Multi-inertial λ-iteration) $$\begin{cases} y_n = x_n + \alpha(x_n - x_{n-1}), \\ z_n = x_n + \beta(x_n - x_{n-1}), \\ u_n = x_n + \gamma(x_n - x_{n-1}), \\ x_{n+1} = (1-\lambda)y_n + \frac{\lambda}{2}z_n + \frac{\lambda}{2}T(u_n). \end{cases}$$

Numerical Results

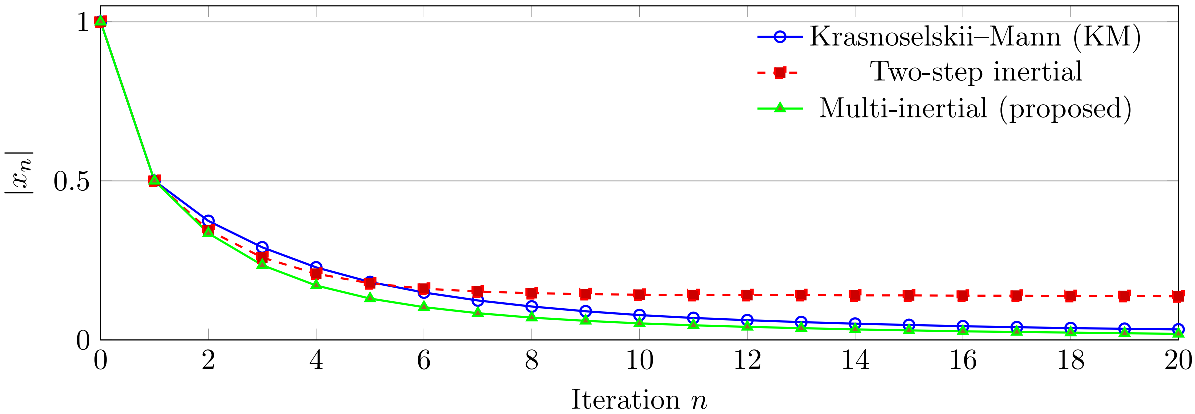

Figure 2: Convergence comparison for $T(x) = 0.7x + 0.3\sin(3x)$ from $x_0 = 1.0$. The multi-inertial λ-iteration (green) converges more rapidly and smoothly than both the two-step inertial (red) and KM (blue) methods over 20 iterations, demonstrating its stability and acceleration in oscillatory nonlinear settings.

As observed in Figure 2, the multi-inertial method (green curve) consistently outperforms both the two-step inertial (red) and Krasnoselskii–Mann (blue) schemes in convergence speed and smoothness. While all three algorithms eventually approach the same fixed point, the proposed multi-inertial scheme reaches a near-zero error level several iterations earlier. Moreover, its convergence trajectory remains monotonic and free from overshooting, indicating superior stability in the presence of oscillations.

Discussion

This example highlights the practical advantage of incorporating three inertial parameters and multiple averaging stages in the proposed framework:

- The **KM iteration** converges slowly due to the absence of acceleration.
- The **two-step inertial scheme** improves the speed but exhibits mild saturation beyond a few iterations, reflecting reduced adaptability to nonlinear oscillations.
- The **multi-inertial λ-iteration** maintains faster and smoother decay, benefiting from the third inertial term γ_n and its balanced averaging over y_n, z_n, and $T(u_n)$.

These results confirm that the comparative performance of inertial fixed point algorithms is parameter- and operator-dependent. The proposed multi-inertial scheme offers enhanced flexibility and robustness, achieving both acceleration and stability across a broader range of nonlinear problems where simpler inertial schemes may become sensitive to oscillations or perturbations.

6. Conclusion and Further Research

In this work, we introduced and analyzed a new *multi-inertial λ-iteration scheme* in cone Banach spaces, formulated as a tri-inertial split-averaged process (TISA-λ). The proposed method generalizes classical inertial fixed-point algorithms by incorporating three distinct inertial parameters and multiple perturbation control sequences, thereby unifying and extending existing two-step and single-inertial frameworks.

Main contributions.

- **Theoretical foundation.** We established strong convergence results for quasi-nonexpansive and weakly contractive operators, and proved existence and uniqueness of fixed points for compatible mappings. The analysis demonstrates that bounded inertial parameters and summable perturbations guarantee stability and convergence within cone Banach spaces.
- **Applied extension.** The applicability of the proposed TISA-λ scheme was illustrated through a nonlinear Volterra–Hammerstein integral equation, confirming that the method can efficiently approximate solutions of nonlinear functional equations. The multi-inertial structure provides fast convergence and stable damping, outperforming both classical Krasnoselskii–Mann and two-step inertial iterations in this nonlinear setting.
- **Numerical validation.** Additional experiments using the oscillatory nonlinear operator $T(x) = 0.7x + 0.3\sin(3x)$ demonstrated that the proposed scheme converges more rapidly and smoothly than standard inertial approaches, particularly in the presence of oscillations or nonmonotone behavior. These results highlight the method's flexibility and robustness across different operator dynamics.

Interpretation and impact. The proposed tri-inertial iteration framework successfully integrates acceleration, damping, and perturbation control into a unified model. The comparative results confirm that while two-step inertial schemes are effective for smooth monotone operators, the multi-inertial approach offers superior stability and adaptability in nonlinear or perturbed environments. Its flexibility makes it suitable for solving a wide class of operator equations, including integral and differential models, where convergence speed and numerical robustness are equally critical.

Future research directions. Building upon these findings, several potential extensions are envisioned:

1. **Adaptive and data-driven parameters.** Develop strategies for automatically adjusting $\alpha_n, \beta_n, \gamma_n$, and λ_n based on the iteration history or residual norms to optimize convergence speed dynamically.
2. **Stochastic and perturbed environments.** Extend the framework to stochastic fixed-point problems where the operator T or perturbations are subject to random noise, and analyze convergence in expectation or almost sure sense.
3. **Applications to differential and integral systems.** Investigate further applications to nonlinear differential equations, boundary value problems, and other functional operator models, emphasizing the role of inertia in stabilizing convergence.
4. **Rate and complexity analysis.** Establish explicit convergence rate bounds (linear, sublinear, or accelerated) under additional smoothness assumptions on T to provide quantitative insight into the performance of the tri-inertial method.
5. **High-dimensional optimization and learning.** Adapt the TISA-λ iteration for large-scale optimization and machine learning problems, where inertial and averaging effects can enhance both computational efficiency and generalization stability.

Final remark. The tri-inertial λ-iteration framework offers a flexible and theoretically grounded extension of classical fixed-point methods. By combining multiple inertial effects with structured perturbation control, it enhances convergence speed and stability in nonlinear and functional operator settings. These properties make it a promising tool for future research in both theoretical nonlinear analysis and computational applications.

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