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Fixed point results for multivalued $(\alpha, \mathcal{F})$-contractions on S-metric spaces with applications

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Abstract

In this paper, we obtain some fixed point results involving α-admissibility for multi-valued $\mathcal{F}$-contractions in the framework of complete S-metric spaces. Appropriate illustrations are provided to support the main results. Finally, an application is developed by demonstrating the existence of a solution to an integral equation. Also, as an application, we establish the existence and uniqueness of the solutions to differential equations in the framework of fractional derivatives involving Mittag-Leffler kernals via the fixed point technique. Our results extend and generalize many well-known results in the existing literature.

Keywords: Fixed point, multivalued $(\alpha, \mathcal{F})$-contraction, α-admissible mapping, S-metric space.
2010 MSC: 47H10, 54H25.

1. Introduction

In 1922, *Banach* [7] proposed the well-known Banach contraction principle (BCP), which employed a contraction mapping in the domain of complete metric spaces. According to the BCP, in a complete metric space $(\mathbb{X}, d)$, a mapping $g\colon \mathbb{X} \to \mathbb{X}$ satisfying the contraction condition on $\mathbb{X}$, i.e.,

$$d(g\zeta, g\eta) \leq k\, d(\zeta, \eta),$$

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for all $\zeta, \eta \in \mathbb{X}$, provided $k \in [0, 1)$, has a unique fixed point.

The BCP was generalized using varieties of mappings on many extensions of metric spaces. In 1969, *Nadler* [16] generalized the BCP for multivalued mappings. In order to optimize a variety of approximation theory problems, it is much more advantageous to use proper fixed point results for multivalued transformations. A new type of contraction called F-contraction was introduced by *Wardowski* [30]. He proved a new fixed point result regarding F-contraction. In this way, *Wardowski* [30] generalized the Banach contraction principle in a different manner from the well-known results in the literature. *Altun et al.* [1] focused on the existence of the fixed point for multivalued F-contractions and proved certain fixed point theorems in the framework of metric spaces. Many extensions and generalizations of BCP were produced and the existence and uniqueness of fixed point were proved.

Samet et al. [24] introduced the concept of α-admissible mappings. The α-admissible mappings notion has been used in many works, see for example [4, 5, 6, 10, 13, 21, 22].

Mlaiki in [15], introduced the notion of α-admissible mapping in the setting of S-metric spaces. Recently, *Javed et al.* [11] introduced the concept of F_α-contraction which is a generalization of F-contraction and proved a fixed point theorem in the setting of S-metric spaces.

Recently, *Gairola and Khantwal* [9] introduced multi-valued contraction in S-metric space and proved some fixed point theorems for multi-valued maps on S-metric space. His results extend and generalise the results of *Nadler* [16], *Sedghi et al.* [26] and others. In [20], *Pourgholam et al.* proved some common fixed point theorems for single valued and multi-valued mappings in S-metric spaces which generalized the results of [12, 31] (see, also [18]).

Very recently, *Saluja and Nashine* [23] proved some fixed point theorems for generalized F-contractions on S-metric spaces and presented a novel fixed-circle solution as an application on S-metric spaces through generalized F-contractions.

This paper initiates the concept of new multi-valued contractions for a mapping involving a member of the family of functions $\mathcal{F}_S$ and a given function $\alpha\colon \mathbb{M}^3 \to [0, +\infty)$ in the context of S-metric space. We establish some fixed point results for such contractions. In addition, we illustrate our main result with concrete examples. Some applications are also included for a wider understanding of the established result.

2. Preliminaries

Take $\mathbb{R}^+ = [0, +\infty)$ and denote by $\mathbb{N}$ the set of positive integers. Throughout the paper, the compact subset of the underlying space $\mathbb{M}$ will be denoted by $K(\mathbb{M})$.

In this part, we recall some essential concepts and consequences that will set a base for our main result.

Definition 2.1. ([26]) Let $\mathbb{M} \neq \emptyset$ be a set. A map $S\colon \mathbb{M}^3 \to \mathbb{R}^+$ fulfilling the following axioms on $\mathbb{M}$ is called an S-metric on $\mathbb{M}$:

$S(1)$ $S(m_1, m_2, m_3) = 0$ if and only if $m_1 = m_2 = m_3$,

$S(2)$ $S(m_1, m_2, m_3) \leq S(m_1, m_1, m_4) + S(m_2, m_2, m_4) + S(m_3, m_3, m_4)$, for all $m_1, m_2, m_3, m_4 \in \mathbb{M}$.

The pair $(\mathbb{M}, S)$ is said to be an S-metric space (SMS).

Example 2.2. ([20]) Let $\mathbb{M} = \mathbb{R}^+$ and $\lambda \geq 0$. Define $S\colon \mathbb{M}^3 \to \mathbb{R}^+$ by

$$S(m_1, m_2, m_3) = \begin{cases} 0, & \text{if } m_1 = m_2 = m_3, \\ \max\{m_1, m_2, m_3\} - \lambda & \textit{otherwise.} \end{cases}$$

Then S is an S-metric space on $\mathbb{M}$ and is called the max S-metric.

Example 2.3. ([20]) Let $\mathbb{M} = \mathbb{R}^+$. Define $S\colon \mathbb{M}^3 \to \mathbb{R}^+$ as

$$S(m_1, m_2, m_3) = \begin{cases} 0, & \text{if } m_1 = m_2 = m_3, \\ m_1 + m_2 + 2m_3 & \textit{otherwise.} \end{cases}$$

Then S is an S-metric space on $\mathbb{M}$.

Example 2.4. ([26]) Let $\mathbb{M} = \mathbb{R}$ and $S(m_1, m_2, m_3) = |m_1 - m_3| + |m_2 - m_3|$. Then S is an S-metric on $\mathbb{M}$, named the usual S-metric on $\mathbb{M}$.

Definition 2.5. ([26]) Let $(\mathbb{M}, S)$ be an S-metric space.

$(a1)$ If for every $m \in \mathbb{A}$ there exists $k > 0$ such that $B_S(m, k) \subset \mathbb{A}$, then the subset $\mathbb{A}$ is called an open subset of $\mathbb{M}$.

$(a2)$ A subset $\mathbb{A}$ of $\mathbb{M}$ is said to be S-bounded if there exists $k > 0$ such that $S(m, m, n) < k$ for all $m, n \in \mathbb{A}$.

$(a3)$ A sequence $\{m_r\}$ in $\mathbb{M}$ converges to m_0 if and only if $S(m_r, m_r, m_0) \to 0$ as $r \to \infty$. That is, for each $\varepsilon > 0$ there exists $r_0 \in N$ such that for all $r \geq r_0$, $S(m_r, m_r, m_0) < \varepsilon$ and we denote this by $\lim_{r\to\infty} m_r = m_0$.

$(a4)$ A sequence $\{m_r\}$ in $\mathbb{M}$ is called Cauchy sequence if for each $\varepsilon > 0$, there exists $r_0 \in N$ such that for all $r, s \geq r_0$, $S(m_r, m_r, m_s) < \varepsilon$.

$(a5)$ The S-metric space $(\mathbb{M}, S)$ is said to be complete if every Cauchy sequence is convergent.

Example 2.6. ([26]) Let $(\mathbb{M}, S)$ be as in Example 2.4. Then $(\mathbb{M}, S)$ is complete.

Lemma 2.7. *([26], Lemma 2.5) Let $(\mathbb{M}, S)$ be an S-metric space. Then, we have $S(m_1, m_1, m_2) = S(m_2, m_2, m_1)$ for all $m_1, m_2 \in \mathbb{M}$.*

Lemma 2.8. *([26], Lemma 2.12) Let $(\mathbb{M}, S)$ be an S-metric space. If $m_r \to m$ and $n_r \to n$ as $r \to \infty$, then $S(m_r, m_r, n_r) \to S(m, m, n)$ as $r \to \infty$.*

Some useful concept regarding Hausdorff distance under the structure of S-metric spaces have been suggested by *Gairola and Khantwal* [9] as follows.

Definition 2.9. ([9]) Let $(\mathbb{M}, S)$ be an S-metric space and $CB(\mathbb{M})$ be the collection of all nonempty bounded and closed subset of $\mathbb{M}$. For, $\mathcal{P}_1, \mathcal{P}_2 \in CB(\mathbb{M})$, the Hausdorff S-metric on $CB(\mathbb{M})$ induced by S is given as follows:

$$\mathcal{S}_H(\mathcal{M}_1, \mathcal{M}_1, \mathcal{M}_2) := \max\Big\{ \sup_{m_1\in\mathcal{M}_1} S(m_1, m_1, \mathcal{M}_2), \sup_{m_2\in\mathcal{M}_2} S(m_2, m_2, \mathcal{M}_1)\Big\},$$

where $S(m_1, m_1, \mathcal{M}_2) = \inf\{S(m_1, m_1, m_2) : m_2 \in \mathcal{M}_2\}$. Then, $\mathcal{S}_H$ is called the Hausdorff S-distance on $CB(\mathbb{M})$ induced by S-metric.

Definition 2.10. ([28]) Let $(\mathbb{M}, S)$ be an S-metric space, $m \in \mathbb{M}$ and $\mathcal{M}_1, \mathcal{M}_2 \subset \mathbb{M}$, then the distance of the point m to the set $\mathcal{M}_1$ is defined as

$$S(m, m, \mathcal{M}_1) := \inf\{S(m, m, n) : n \in \mathcal{M}_1\}.$$

It is clear by the definition of $S(m, m, \mathcal{M}_1)$ that $S(m, m, \mathcal{M}_1) = 0 \Leftrightarrow m \in \overline{\mathcal{M}_1}$.

Definition 2.11. ([17]) Let $(\mathbb{M}, S)$ be an S-metric space and $\mathcal{M}_1$ be a non-void subset of $\mathbb{M}$. The diameter of $\mathcal{M}_1$ is defined by

$$diam(\mathcal{M}_1) := \sup\{S(m, m, n) : m, n \in \mathcal{M}_1\}.$$

If $\mathcal{M}_1$ is S-bounded, then $diam(\mathcal{M}_1) < +\infty$.

Lemma 2.12. *([9], Lemma 3.1) Let $(\mathbb{M}, S)$ be an S-metric space and $\mathcal{M}_1, \mathcal{M}_2 \in CB(\mathbb{M})$. Then for each $m_1 \in \mathcal{M}_1$, we have*

$$S(m_1, m_1, \mathcal{M}_2) \leq \mathcal{S}_H(\mathcal{M}_1, \mathcal{M}_1, \mathcal{M}_2).$$

Lemma 2.13. *([9], Lemma 3.2) If $\mathcal{M}_1, \mathcal{M}_2 \in CB(\mathbb{M})$. Then for each $m_1 \in \mathcal{M}_1$, then for each $\eta > 0$ there exists $m_2 \in \mathcal{M}_2$ such that*

$$S(m_1, m_1, m_2) \leq \mathcal{S}_H(\mathcal{M}_1, \mathcal{M}_1, \mathcal{M}_2) + \eta.$$

Lemma 2.14. *Let* $(\mathbb{M}, S)$ *be an* S*-metric space. Consider two nonempty subsets* $\mathcal{M}_1, \mathcal{M}_2 \in CB(\mathbb{M})$ *and* $k_* > 1$*. For some* $m_1 \in \mathcal{M}_1$*, there exists* $m_2 \in \mathcal{M}_2$ *so that*

$$S(m_1, m_1, m_2) \leq k_* \, \mathcal{S}_H(\mathcal{M}_1, \mathcal{M}_1, \mathcal{M}_2).$$

Example 2.15. Let $\mathbb{M} = \mathbb{R}$, $\mathcal{M}_1 = [m_1 - \alpha, m_1 + \alpha]$, $\mathcal{M}_2 = [m_2 - \beta, m_2 + \beta]$ and $0 < \alpha \leq \beta$ where $m_1, m_2 \in \mathbb{M}$. Let $S\colon \mathbb{M}^3 \to \mathbb{R}^+$ be defined by $S(m_1, m_2, m_3) = |m_1 - m_3| + |m_2 - m_3|$ for all $m_1, m_2, m_3 \in \mathbb{M}$.

It can be seen that $\mathcal{M}_1, \mathcal{M}_2 \in CB(\mathbb{M})$ and so

$$\begin{aligned}
\mathcal{S}_H(\mathcal{M}_1, \mathcal{M}_1, \mathcal{M}_2) &= \max\Big\{ \sup_{m_1 \in \mathcal{M}_1} S(m_1, m_1, \mathcal{M}_2), \sup_{m_2 \in \mathcal{M}_2} S(m_2, m_2, \mathcal{M}_1) \Big\} \\
&= \max\Big\{ 2 \sup_{m_1 \in \mathcal{M}_1} \Big| m_1 - \mathcal{M}_2 \Big|, 2 \sup_{m_2 \in \mathcal{M}_2} \Big| m_2 - \mathcal{M}_1 \Big| \Big\} \\
&= \max\Big\{ 2|(m_2 - \beta) - (m_1 - \alpha)|, 2|(m_2 + \beta) - (m_1 + \alpha)| \Big\} \\
&= 2\max\Big\{ |(m_2 - m_1) - (\beta - \alpha)|, |(m_2 - m_1) + (\beta - \alpha)| \Big\} \\
&\geq 2|(m_2 - m_1) - (\beta - \alpha)| \\
&\geq 2(|m_2 - m_1| - |\beta - \alpha|) \\
&= 2|m_1 - m_2| - 2|\beta - \alpha| = S(m_1, m_1, m_2) - 2(\beta - \alpha).
\end{aligned}$$

So,

$$S(m_1, m_1, m_2) \leq \mathcal{S}_H(\mathcal{M}_1, \mathcal{M}_1, \mathcal{M}_2) + \eta, \text{ where } \eta = 2(\beta - \alpha).$$

Definition 2.16. ([9]) Let $(\mathbb{M}, S)$ be an S-metric space. A function $\mathcal{T}\colon \mathbb{M} \to CB(\mathbb{M})$ is said to be a multi-valued contraction on $\mathbb{M}$ if there exists a constant $c \in [0, 1)$ such that

$$\mathcal{S}_H(\mathcal{T}m_1, \mathcal{T}m_1, \mathcal{T}m_2) \leq c\, S(m_1, m_1, m_2),$$

for all $m_1, m_2 \in \mathbb{M}$.

Theorem 2.17. *([9], Theorem 3.1) Let* $(\mathbb{M}, S)$ *be an* S*-metric space. If* $\mathcal{T}\colon \mathbb{M} \to CB(\mathbb{M})$ *is a multi-valued contraction on* $\mathbb{M}$*, then* $\mathbb{T}$ *has a fixed point.*

Definition 2.18. ([20]) Let $(\mathbb{M}, S)$ be an S-metric space. Define $S_H\colon (CB(\mathbb{M}))^3 \to [0, +\infty)$ by

$$S_H(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3) = H_S(\mathcal{M}_1, \mathcal{M}_3) + H_S(\mathcal{M}_2, \mathcal{M}_3),$$

where

$$H_S(\mathcal{M}_1, \mathcal{M}_2) = \max\{h_S(\mathcal{M}_1, \mathcal{M}_2), h_S(\mathcal{M}_2, \mathcal{M}_1)\}$$

$$h_S(\mathcal{M}_1, \mathcal{M}_2) = \sup\{S(m_1, m_1, \mathcal{M}_2) : m_1 \in \mathcal{M}_1\}, \text{ and}$$

$$S(m_1, m_1, \mathcal{M}_2) = \inf\{S(m_1, m_1, m_2) : m_2 \in \mathcal{M}_2\}.$$

For more details see [19].

Theorem 2.19. *([19])* S_H *is an* S*-metric on* $CB(\mathbb{M})$*.*

Definition 2.20. ([15]) Let $(\mathbb{M}, S)$ be an S-metric space and $\mathcal{T}\colon \mathbb{M} \to \mathbb{M}$, $\alpha\colon \mathbb{M}^3 \to \mathbb{R}^+$ (where $\mathbb{M}^3 = \mathbb{M} \times \mathbb{M} \times \mathbb{M}$) be given mappings. We say that $\mathcal{T}$ is α-admissible if $m_1, m_2, m_3 \in \mathbb{M}$, $\alpha(m_1, m_2, m_3) \geq 1$ implies that $\alpha(\mathcal{T}m_1, \mathcal{T}m_2, \mathcal{T}m_3) \geq 1$.

In 2012, *Wardowski* [30] was given a new concept by introducing $\mathcal{F}_S$-family.

Definition 2.21. ([30]) A mapping $\mathcal{F}\colon \mathbb{R}^+ \to \mathbb{R}$ is a member of the family $\mathcal{F}_S$ if $\mathcal{F}$ satisfies the following hypotheses:

$(F1)$: $\mathcal{F}$ is strictly increasing, i.e.,

$$m_1 < m_2 \quad \Rightarrow \quad \mathcal{F}(m_1) < \mathcal{F}(m_2), \text{ for all } m_1, m_2 \in \mathbb{R}.$$

$(F2)$: For every positive term sequence $\{m_n : n \in \mathbb{N}\}$ in $\mathbb{R}^+$,

$$\lim_{n\to\infty} m_n = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} \mathcal{F}(m_n) = -\infty.$$

$(F3)$: If there exists a number $\gamma \in (0,1)$, then $\lim_{\eta\to 0^+} \eta^\gamma \mathcal{F}(\eta) = 0$.

Example 2.22. Let $\mathcal{F}_i\colon \mathbb{R}^+ \to \mathbb{R}$ where $i = 1,2,3,4$ be defined as:

(1) $\mathcal{F}_1(m) = ln(m)$, (2) $\mathcal{F}_2(m) = -\frac{1}{\sqrt{m}}$, (3) $\mathcal{F}_3(m) = m + ln(m)$ and (4) $\mathcal{F}_4(m) = ln(m^2+m)$ for $m > 0$. Then $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$ are members of the family $\mathcal{F}_S$.

3. Main Results

We start with the following definition.

Definition 3.1. Let $\mathbb{M} \neq \emptyset$ be a set and let $Q\colon \mathbb{M} \to 2^{\mathbb{M}}$ be a multivalued mapping. Given a function $\alpha\colon \mathbb{M}\times\mathbb{M}\times\mathbb{M} \to \mathbb{R}^+$. Q is called a multivalued α-admissible if for $m, n \in \mathbb{M}$, we have

$$\alpha(m,m,n) \geq 1 \quad \Rightarrow \quad \alpha(m_0,m_0,n_0) \geq 1,$$

where $m_0 \in Q(m)$ and $n_0 \in Q(n)$.

Definition 3.2. Let $(\mathbb{M}, S)$ be an S-metric space and define a map $Q\colon \mathbb{M} \to K(\mathbb{M})$. Then Q said to be a MV $\mathcal{F}$-contraction if there exists $\mathcal{F} \in \mathcal{F}_S$ and $\tau > 0$ such that

$$\begin{aligned} \mathcal{S}_H(Qm_1,Qm_1,Qm_2) > 0 \;\Rightarrow\; & \tau + \mathcal{F}(\mathcal{S}_H(Qm_1,Qm_1,Qm_2)) \\ & \leq \mathcal{F}(\Omega(m_1,m_1,m_2)), \end{aligned} \tag{1}$$

where

$$\begin{aligned} \Omega(m_1,m_1,m_2) = \max\Big\{ & S(m_1,m_1,m_2), S(m_1,m_1,Qm_1), S(m_2,m_2,Qm_2), \\ & \frac{S(m_1,m_1,m_2)[1+S(m_2,m_2,Qm_1)]}{1+S(m_1,m_1,m_2)}\Big\}. \end{aligned}$$

Definition 3.3. Let $(\mathbb{M}, S)$ be an S-metric space. Given a function $\alpha\colon \mathbb{M}\times\mathbb{M}\times\mathbb{M} \to \mathbb{R}^+$. The mapping $Q\colon \mathbb{M} \to K(\mathbb{M})$ is said to be a MV $(\alpha,\mathcal{F})$-contraction if there exists $\mathcal{F} \in \mathcal{F}_S$ and $\tau > 0$ such that

$$\begin{aligned} \mathcal{S}_H(Qm_1,Qm_1,Qm_2) > 0 \;\Rightarrow\; & \tau + \mathcal{F}\big(\alpha(m_1,m_1,m_2)\mathcal{S}_H(Qm_1,Qm_1,Qm_2)\big) \\ & \leq \mathcal{F}(\Omega(m_1,m_1,m_2)), \end{aligned} \tag{2}$$

where

$$\begin{aligned} \Omega(m_1,m_1,m_2) = \max\Big\{ & S(m_1,m_1,m_2), S(m_1,m_1,Qm_1), S(m_2,m_2,Qm_2), \\ & \frac{S(m_1,m_1,m_2)[1+S(m_2,m_2,Qm_1)]}{1+S(m_1,m_1,m_2)}\Big\}. \end{aligned}$$

Lemma 3.4. *Let $(\mathbb{M}, S)$ be a complete S-metric space and $Q\colon \mathbb{M} \to K(\mathbb{M})$ be a MV $\mathcal{F}$-contraction mapping, then*

$$\lim_{\rho\to\infty} \Theta_\rho = 0,$$

where $\Theta_\rho = S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})$ and $\rho = 0,1,2,\ldots$.

Proof. Let $m_0 \in \mathbb{M}$ be an arbitrary element. As Qm_0 is compact, it is nonempty, so we can choose $m_1 \in Qm_0$. If $m_1 \in Qm_1$, then m_1 is a fixed point of Q trivially. So, suppose $m_1 \notin Qm_1$. As Qm_1 is closed, so we have $S(m_1, m_1, Qm_1) > 0$. Also, we know that

$$S(m_1, m_1, Qm_1) \leq \mathcal{S}_H(Qm_0, Qm_0, Qm_1). \tag{3}$$

As Qm_1 is compact, so there exists $m_2 \in Qm_1$ such that

$$S(m_1, m_1, m_2) = S(m_1, m_1, Qm_1).$$

Thus,

$$S(m_1, m_1, m_2) \leq \mathcal{S}_H(Qm_0, Qm_0, Qm_1).$$

Likewise for $m_3 \in Qm_2$, we obtain

$$S(m_2, m_2, m_3) \leq \mathcal{S}_H(Qm_1, Qm_1, Qm_2),$$

which ultimately gives

$$S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2}) \leq \mathcal{S}_H(Qm_\rho, Qm_\rho, Qm_{\rho+1}).$$

Thus the condition $(F1)$ implies that

$$\mathcal{F}(S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})) \leq \mathcal{F}(\mathcal{S}_H(Qm_\rho, Qm_\rho, Qm_{\rho+1})).$$

By (1), we have

$$\mathcal{F}(S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})) \leq \mathcal{F}(\Omega(m_\rho, m_\rho, m_{\rho+1})) - \tau, \tag{4}$$

where

$$\begin{aligned} \Omega(m_\rho, m_\rho, m_{\rho+1}) = \max\Big\{ & S(m_\rho, m_\rho, m_{\rho+1}), S(m_\rho, m_\rho, Qm_\rho), S(m_{\rho+1}, m_{\rho+1}, Qm_{\rho+1}), \\ & \frac{S(m_\rho, m_\rho, m_{\rho+1})[1 + S(m_{\rho+1}, m_{\rho+1}, Qm_\rho)]}{1 + S(m_\rho, m_\rho, m_{\rho+1})}\Big\}. \end{aligned}$$

By using Definition 2.10, we have

$$\begin{aligned} \Omega(m_\rho, m_\rho, m_{\rho+1}) &\leq \max\Big\{ S(m_\rho, m_\rho, m_{\rho+1}), S(m_\rho, m_\rho, m_{\rho+1}), S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2}), \\ &\qquad \frac{S(m_\rho, m_\rho, m_{\rho+1})[1 + S(m_{\rho+1}, m_{\rho+1}, m_{\rho+1})]}{1 + S(m_\rho, m_\rho, m_{\rho+1})}\Big\} \\ &\leq \max\Big\{ S(m_\rho, m_\rho, m_{\rho+1}), S(m_\rho, m_\rho, m_{\rho+1}), S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2}), \\ &\qquad \frac{S(m_\rho, m_\rho, m_{\rho+1})}{1 + S(m_\rho, m_\rho, m_{\rho+1})}\Big\} \\ &\leq \max\Big\{ S(m_\rho, m_\rho, m_{\rho+1}), S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})\Big\}. \end{aligned}$$

Suppose now

$$\max\Big\{ S(m_\rho, m_\rho, m_{\rho+1}), S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})\Big\} = S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2}).$$

Then the inequality (4) yields

$$\tau + \mathcal{F}(S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})) \leq \mathcal{F}(S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})),$$

which is a contradiction. Therefore, we conclude that

$$\max\Big\{ S(m_\rho, m_\rho, m_{\rho+1}), S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})\Big\} = S(m_\rho, m_\rho, m_{\rho+1}).$$

Thus the inequality (4) becomes

$$\tau + \mathcal{F}(S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})) \leq \mathcal{F}(S(m_\rho, m_\rho, m_{\rho+1})). \tag{5}$$

For convenience, we assume that $\Theta_\rho := S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})$, where $\rho = 0, 1, 2, \ldots$. Clearly $\Theta_\rho > 0$ for all $\rho \in \mathbb{N}$. Using this in (5), we obtain

$$\mathcal{F}(\Theta_\rho) \le \mathcal{F}(\Theta_{\rho-1}) - \tau.$$

Continuing in a same fashion, we will get

$$\mathcal{F}(\Theta_\rho) \le \mathcal{F}(\Theta_{\rho-1}) - \tau \le \mathcal{F}(\Theta_{\rho-2}) - 2\tau \le \mathcal{F}(\Theta_{\rho-3}) - 3\tau \cdots \le \mathcal{F}(\Theta_0) - \rho\tau. \tag{6}$$

Hence,

$$\lim_{\rho\to\infty} \mathcal{F}(\Theta_\rho) = -\infty,$$

we have

$$\lim_{\rho\to\infty} \Theta_\rho = 0, \text{ by } (F2).$$

□

Theorem 3.5. *Let $(\mathbb{M}, S)$ be a complete S-metric space such that S is a continuous mapping and $Q\colon \mathbb{M} \to K(\mathbb{M})$ is a multivalued $(\alpha, \mathcal{F})$-contraction mapping. Suppose that*

(1) Q is continuous;
(2) Q is an α-admissible mapping;
(3) there exists $m_0 \in \mathbb{M}$ and $m_1 \in Qm_0$ such that $\alpha(m_0, m_0, m_1) \ge 1$.
Then Q has a fixed point.

Proof. Let $m_0 \in \mathbb{M}$ be an arbitrary point. By assumption of the theorem $\alpha(m_0, m_0, m_1) \ge 1$ for some $m_1 \in Qm_0$. Similarly, for $m_2 \in Qm_1$, we have $\alpha(m_1, m_1, m_2) \ge 1$ and for any sequence $m_{\rho+1} \in Qm_\rho$, we get

$$\alpha(m_\rho, m_\rho, m_{\rho+1}) \ge 1 \text{ for all } \rho \in \mathbb{N} \cup \{0\}. \tag{7}$$

Now, by the contractive condition (2), we have

$$\tau + \mathcal{F}(\alpha(m_\rho, m_\rho, m_{\rho+1})\mathcal{S}_H(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})) \le \mathcal{F}(\Omega(m_\rho, m_\rho, m_{\rho+1})). \tag{8}$$

The inequality (8) implies that

$$\tau + \mathcal{F}(\mathcal{S}_H(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})) \le \mathcal{F}(\Omega(m_\rho, m_\rho, m_{\rho+1})),$$

and hence

$$\mathcal{F}(\mathcal{S}_H(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})) \le \mathcal{F}(\Omega(m_\rho, m_\rho, m_{\rho+1})) - \tau.$$

Now, we have

$$\begin{aligned} \mathcal{F}(S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2}))) &\le \mathcal{F}(\mathcal{S}_H(Qm_\rho, Qm_\rho, Qm_{\rho+1})) \\ &\le \mathcal{F}(\Omega(m_\rho, m_\rho, m_{\rho+1})) - \tau. \end{aligned}$$

By Lemma 3.4, one writes

$$\lim_{\rho\to\infty} \Theta_\rho = 0,$$

where $\Theta_\rho = S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})$ and $\rho = 0, 1, 2, \ldots$.

Now, by $\mathcal{F} \in \mathcal{F}_S$ and $(F3)$, there exists $\gamma \in (0, 1)$ such that

$$\lim_{\rho\to\infty} (\Theta_\rho)^\gamma \mathcal{F}(\Theta_\rho) = 0, \quad \text{for all } \rho \in \mathbb{N}. \tag{9}$$

Using (6), one writes

$$(\Theta_\rho)^\gamma \Big(\mathcal{F}(\Theta_\rho) - \mathcal{F}(\Theta_0)\Big) \le -\rho(\Theta_\rho)^\gamma \tau \le 0. \tag{10}$$

As $\tau > 0$, using (9), we have

$$\lim_{\rho\to\infty} \rho(\Theta_\rho)^\gamma = 0. \tag{11}$$

So, there exists $\rho_1 \in \mathbb{N}$, such that

$$\rho(\Theta_\rho)^\gamma \le 1, \quad \forall\, \rho \ge \rho_1.$$

It implies that

$$\Theta_\rho \le \frac{1}{\rho^{\frac{1}{\gamma}}}. \tag{12}$$

Now, we will prove that $\{m_\rho\}$ is a Cauchy sequence in $\mathbb{M}$. For this, let $\rho, \sigma \in \mathbb{N}$ such that $\rho > \sigma \ge \rho_1$. Using condition $S(2)$ of an S-MS, we have

$$\begin{aligned} S(m_\rho, m_\rho, m_\sigma) &\le 2S(m_\rho, m_\rho, m_{\rho+1}) + S(m_{\rho+1}, m_{\rho+1}, m_\sigma) \\ &\le 2S(m_\rho, m_\rho, m_{\rho+1}) + 2S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2}) \\ &\quad + S(m_{\rho+2}, m_{\rho+2}, m_\sigma) \\ &\;\vdots \\ &\le 2[S(m_\rho, m_\rho, m_{\rho+1}) + S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2}) \\ &\quad + \cdots + S(m_{\sigma-2}, m_{\sigma-2}, m_{\sigma-1})] + S(m_{\sigma-1}, m_{\sigma-1}, m_\sigma) \\ &= 2\sum_{t=\rho}^{\sigma-2} S(m_t, m_t, m_{t+1}) + S(m_{\sigma-1}, m_{\sigma-1}, m_\sigma) \\ &\le 2\sum_{t=\rho}^{\infty} S(m_t, m_t, m_{t+1}) \le 2\sum_{t=\rho}^{\infty} S(m_{t+1}, m_{t+1}, m_{t+2}) \\ &= 2\sum_{t=\rho}^{\infty} \Theta_t \le 2\sum_{t=\rho}^{\infty} \frac{1}{t^{\frac{1}{\gamma}}}. \end{aligned} \tag{13}$$

The convergence of the series $\sum_{t=1}^{\infty} \frac{1}{t^{\frac{1}{\gamma}}}$ implies that $\lim_{\rho\to\infty} S(m_\rho, m_\rho, m_\sigma) = 0$, which shows that $\{m_\rho\}$ is a Cauchy sequence in $\mathbb{M}$. Since $\mathbb{M}$ is complete, there exists $m^\star \in \mathbb{M}$ such that

$$\lim_{\rho\to\infty} S(m_\rho, m_\rho, m^\star) = S(m^\star, m^\star, m^\star) = 0. \tag{14}$$

We claim that $m^\star$ is a fixed point of Q, that is,

$$S(m^\star, m^\star, Qm^\star) = S(m^\star, m^\star, m^\star).$$

Assume that $S(m^\star, m^\star, Qm^\star) > 0$. So, there exists $r_0 \in \mathbb{N}$ such that $S(m_\rho, m_\rho, Qm^\star) > 0$ for all $\rho > r_0$. We have

$$S(m_\rho, m_\rho, Qm^\star) \le \mathcal{S}_H(Qm_{\rho+1}, Qm_{\rho+1}, Qm^\star).$$

Now, using contractive condition (2) and taking the limit as $\rho \to \infty$, we have

$$\begin{aligned} \tau + \mathcal{F}(S(m^\star, m^\star, Qm^\star)) &\le \tau + \mathcal{F}(\alpha(m^\star, m^\star, m^\star)\mathcal{S}_H(Qm^\star, Qm^\star, Qm^\star)) \\ &\le \mathcal{F}(\Omega(m^\star, m^\star, m^\star)) \le \mathcal{F}(S(m^\star, m^\star, Qm^\star)), \end{aligned} \tag{15}$$

where

$$\begin{aligned}\Omega(m^\star, m^\star, m^\star) &= \max\Big\{S(m^\star, m^\star, m^\star), S(m^\star, m^\star, Qm^\star), S(m^\star, m^\star, Qm^\star), \\ &\qquad \frac{S(m^\star, m^\star, m^\star)[1 + S(m^\star, m^\star, Qm^\star)]}{1 + S(m^\star, m^\star, m^\star)}\Big\} \\ &= \max\Big\{0, S(m^\star, m^\star, Qm^\star), S(m^\star, m^\star, Qm^\star), 0\Big\} \\ &= S(m^\star, m^\star, Qm^\star).\end{aligned}$$

Hence (15) yields

$$\tau + \mathcal{F}(S(m^\star, m^\star, Qm^\star)) \le \mathcal{F}(S(m^\star, m^\star, Qm^\star)).$$

Since $\tau > 0$, the above inequality yields a contradiction. Hence, $S(m^\star, m^\star, Qm^\star) = 0$. Also, $S(m^\star, m^\star, m^\star) = 0$. This gives that $m^\star \in Qm^\star$. This proves that $m^\star$ is a fixed point of Q. The proof is completed. □

Example 3.6. Let $\mathbb{M} = \{0, 1, 2, \dots\}$. Define $S\colon \mathbb{M}^3 \to \mathbb{R}^+$ by $S(m_1, m_2, m_3) = |m_1 - m_3| + |m_2 - m_3|$ for all $m_1, m_2, m_3 \in \mathbb{M}$. Then $(\mathbb{M}, S)$ is a complete S-metric space.

We also define a multivalued map $Q\colon \mathbb{M} \to 2^{\mathbb{M}}$ by

$$Q(m) = \begin{cases} \{0, 1\}, & \text{if } m = 0, 1, \\ \{m-1, m\}, & \text{otherwise.}\end{cases}$$

Consider a map $\alpha\colon \mathbb{M}^3 \to \mathbb{R}^+$ as

$$\alpha(m, m, n) = \begin{cases} 2, & \text{if } m, n \in \{0, 1\}, \\ \frac{1}{2}, & \text{otherwise.}\end{cases}$$

Let $m_0 = 0$, $m_1 = 1$, then $Qm_0 = \{0, 1\}$ and $m_1 = \{0, 1\}$. Giving $\alpha(m_0, m_0, m_1) = \alpha(0, 0, 1) = 2 > 1$, for some $m_2 = 0 \in Qm_1$, we get $\alpha(m_1, m_1, m_2) = \alpha(1, 1, 0) = 2 > 1$. Thus Q is an α-admissible map.

Define $\mathcal{F}\colon \mathbb{R}^+ \to \mathbb{R}$ as $\mathcal{F}(m) = ln(m) + m$. It can be easily seen that $\mathcal{F}$ is a member of the family $\mathcal{F}_S$. Now, applying $\mathcal{F}$ on our contractive condition, we get

$$\tau + \mathcal{F}(\alpha(m, m, n)\mathcal{S}_H(Qm, Qm, Qn)) \le \mathcal{F}(\Omega(m, m, n)).$$

That is,

$$\tau + ln\{\alpha(m, m, n)\mathcal{S}_H(Qm, Qm, Qn)\} + \alpha(m, m, n)\mathcal{S}_H(Qm, Qm, Qn)$$

$$\le ln\big(\Omega(m, m, n)\big) + \Omega(m, m, n).$$

Hence,

$$\tau + \alpha(m, m, n)\mathcal{S}_H(Qm, Qm, Qn) - \Omega(m, m, n)$$

$$\le ln\big(\Omega(m, m, n)\big) - ln\{\alpha(m, m, n)\mathcal{S}_H(Qm, Qm, Qn)\}.$$

Therefore,

$$e^{\tau + \alpha(m,m,n)\mathcal{S}_H(Qm,Qm,Qn) - \Omega(m,m,n)} \le \frac{\Omega(m, m, n)}{\alpha(m, m, n)\mathcal{S}_H(Qm, Qm, Qn)}.$$

That is,

$$\frac{\alpha(m, m, n)\mathcal{S}_H(Qm, Qm, Qn)}{\Omega(m, m, n)} e^{\alpha(m,m,n)\mathcal{S}_H(Qm,Qm,Qn) - \Omega(m,m,n)} \le e^{-\tau}. \tag{16}$$

Now,

$$\begin{aligned}\mathcal{S}_H(Qm,Qm,Qn) &= \max\Big\{\sup_{a\in Qm} S(a,a,Qn), \sup_{b\in Qn} S(b,b,Qm)\Big\}\\ &= \max\big\{S(m,m,Qn), S(m-1,m-1,Qn)\big\}\\ &= \max\Big\{\inf\big\{S(m,m,n), S(m,m,n-1)\big\},\\ &\qquad \inf\big\{S(m-1,m-1,n), S(m-1,m-1,n-1)\big\}\Big\}\\ &= \max\{2|m-n|, 2|m-n-1|\} = 2|m-n|.\end{aligned}$$

Hence,

$$\mathcal{S}_H(Qm,Qm,Qn) = 2|m-n|. \tag{17}$$

Also,

$$\Omega(m,m,n) \geq S(m,m,n) = 2|m-n|. \tag{18}$$

Putting the values of (17) and (18) in the L.H.S. of (16), we have

$$\frac{\alpha(m,m,n)\mathcal{S}_H(Qm,Qm,Qn)}{\Omega(m,m,n)} e^{\alpha(m,m,n)\mathcal{S}_H(Qm,Qm,Qn)-\Omega(m,m,n)}$$

$$\begin{aligned}&= \frac{2|m-n|}{2\Omega(m,m,n)} e^{\frac{1}{2}.2|m-n|-\Omega(m,m,n)} \quad (\text{using } (17))\\ &= \frac{2|m-n|}{4|m-n|} e^{\frac{1}{2}.2|m-n|-2|m-n|} \quad (\text{using } (18))\\ &= \frac{1}{2}e^{-|m-n|} = \frac{1}{2}e^{-\tau} < e^{-\tau}.\end{aligned}$$

This implies that (16) is satisfied with $\tau = |m-n|$, which is a positive number for $m \neq n$. Thus all conditions of Theorem 3.5 are true, and 0 and 1 are two fixed points of Q.

Theorem 3.7. *Let $(\mathbb{M}, S)$ be a complete S-metric space such that S is a continuous mapping. Let $Q\colon \mathbb{M} \to CB(\mathbb{M})$ be a multivalued $(\alpha, \mathcal{F})$-contraction mapping and $D \subset (0,\infty)$ with $\inf D > 0$. Suppose that*

(1) *Q is continuous;*
(2) *Q is an α-admissible mapping;*
(3) *there exists $m_0 \in \mathbb{M}$ and $m_1 \in Qm_0$ such that $\alpha(m_0,m_0,m_1) \geq 1$;*
(4) *$\mathcal{F}(\inf D) = \inf \mathcal{F}(D)$, where $\mathcal{F} \in \mathcal{F}_S$.*

Then Q has a fixed point.

Proof. Let $m_0 \in \mathbb{M}$ be an arbitrary point. As Qm, the set of all images of $m \in \mathbb{M}$, is nonempty for all values in $\mathbb{M}$. We can choose $m_1 \in Qm_0$. If $m_1 \in Qm_1$, this means that m_1 is a fixed point of Q. So, we assume that $m_1 \notin Qm_1$. Since Qm_1 is closed, we have $S(m_1,m_1,Qm_1) > 0$. Also, we know that

$$S(m_1,m_1,Qm_1) \leq \mathcal{S}_H(Qm_0,Qm_0,Qm_1).$$

By $(F1)$, we have

$$\mathcal{F}(S(m_1,m_1,Qm_1)) \leq \mathcal{F}(\mathcal{S}_H(Qm_0,Qm_0,Qm_1)). \tag{19}$$

Using hypothesis (4)

$$\mathcal{F}(S(m_1,m_1,Qm_1)) = \inf_{h\in Qm_1} \mathcal{F}(S(m_1,m_1,h)).$$

That is,

$$\inf_{h \in Qm_1} \mathcal{F}(S(m_1, m_1, h)) \leq \mathcal{F}(\mathcal{S}_H(Qm_0, Qm_0, Qm_1)). \tag{20}$$

As Qm_1 is compact, so we can find a $m_2 \in Qm_1$ such that

$$\inf_{h \in Qm_1} \mathcal{F}(S(m_1, m_1, h)) = \mathcal{F}(S(m_1, m_1, m_2)).$$

From (19), we obtain

$$\mathcal{F}(S(m_1, m_1, m_2)) \leq \mathcal{F}(\mathcal{S}_H(Qm_0, Qm_0, Qm_1)). \tag{21}$$

Likewise, for $m_3 \in Qm_2$, we obtain

$$\mathcal{F}(S(m_2, m_2, m_3)) \leq \mathcal{F}(\mathcal{S}_H(Qm_1, Qm_1, Qm_2)),$$

which ultimately gives

$$\mathcal{F}(S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})) \leq \mathcal{F}(\mathcal{S}_H(Qm_\rho, Qm_\rho, Qm_{\rho+1})). \tag{22}$$

For $m_0 \in \mathcal{M}$ by assumption of the theorem, $\alpha(m_0, m_0, m_1) \geq 1$ for some $m_1 \in Qm_0$. Likewise, for $m_2 \in Qm_1$, we have $\alpha(m_1, m_1, m_2) \geq 1$ and for any sequence $m_{\rho+1} \in Qm_\rho$, we may write

$$\alpha(m_\rho, m_\rho, m_{\rho+1}) \geq 1 \text{ for all } \rho \in \mathbb{N} \cup \{0\}. \tag{23}$$

Now, using contractive condition (2), we have

$$\tau + \mathcal{F}(\alpha(m_\rho, m_\rho, m_{\rho+1})\mathcal{S}_H(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})) \leq \mathcal{F}(\Omega(m_\rho, m_\rho, m_{\rho+1})). \tag{24}$$

The inequality (24) implies that

$$\tau + \mathcal{F}(\mathcal{S}_H(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})) \leq \mathcal{F}(\Omega(m_\rho, m_\rho, m_{\rho+1})),$$

and hence

$$\mathcal{F}(\mathcal{S}_H(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})) \leq \mathcal{F}(\Omega(m_\rho, m_\rho, m_{\rho+1})) - \tau.$$

Using (22) in the above inequality, we get

$$\mathcal{F}(S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})) \leq \mathcal{F}(\Omega(m_\rho, m_\rho, m_{\rho+1})) - \tau.$$

By Lemma 3.4, one writes

$$\lim_{\rho \to \infty} \Theta_\rho = 0,$$

where $\Theta_\rho = S(m_{\rho+1}, m_{\rho+1}, m_{\rho+2})$ and $\rho = 0, 1, 2, \ldots$.

Now, by $\mathcal{F} \in \mathcal{F}_S$ and $(F3)$, there exists $\gamma \in (0, 1)$ such that

$$\lim_{\rho \to \infty} (\Theta_\rho)^\gamma \mathcal{F}(\Theta_\rho) = 0, \text{ for all } \rho \in \mathbb{N}. \tag{25}$$

Using (6), one writes

$$(\Theta_\rho)^\gamma \Big(\mathcal{F}(\Theta_\rho) - \mathcal{F}(\Theta_0)\Big) \leq -\rho(\Theta_\rho)^\gamma \tau \leq 0. \tag{26}$$

Since $\tau > 0$, using (25), we have

$$\lim_{\rho \to \infty} \rho(\Theta_\rho)^\gamma = 0. \tag{27}$$

So, there exists $\rho_1 \in \mathbb{N}$, such that

$$\rho(\Theta_\rho)^\gamma \leq 1, \ \forall \rho \geq \rho_1.$$

It implies that

$$\Theta_\rho \leq \frac{1}{\rho^{\frac{1}{\gamma}}}. \tag{28}$$

Next, we shall prove that $\{m_\rho\}$ is a Cauchy sequence in $\mathbb{M}$. For this, following the same steps as done in Theorem 3.5, one can easily have

$$\lim_{\rho\to\infty} S(m_\rho, m_\rho, m^\star) = S(m^\star, m^\star, m^\star) = 0. \tag{29}$$

Now, we claim that $m^\star$ is a fixed point of Q. Assume that $S(m^\star, m^\star, Qm^\star) > 0$, then there exists $r_0 \in \mathbb{N}$ such that $S(m_\rho, m_\rho, Qm^\star) > 0$ for all $\rho > r_0$. One can have

$$S(m_\rho, m_\rho, Qm^\star) \leq \mathcal{S}_H(Qm_{\rho+1}, Qm_{\rho+1}, Qm^\star).$$

Now, using contractive condition (2) and taking the limit as $\rho \to \infty$, we have

$$\begin{aligned} \tau + \mathcal{F}(S(m^\star, m^\star, Qm^\star)) &\leq \tau + \mathcal{F}(\alpha(m^\star, m^\star, m^\star)\mathcal{S}_H(Qm^\star, Qm^\star, Qm^\star)) \\ &\leq \mathcal{F}(\Omega(m^\star, m^\star, m^\star)) \leq \mathcal{F}(S(m^\star, m^\star, Qm^\star)), \end{aligned} \tag{30}$$

where

$$\begin{aligned} \Omega(m^\star, m^\star, m^\star) &= \max\Big\{S(m^\star, m^\star, m^\star), S(m^\star, m^\star, Qm^\star), S(m^\star, m^\star, Qm^\star), \\ &\qquad \frac{S(m^\star, m^\star, m^\star)[1 + S(m^\star, m^\star, Qm^\star)]}{1 + S(m^\star, m^\star, m^\star)}\Big\} \\ &= \max\Big\{0, S(m^\star, m^\star, Qm^\star), S(m^\star, m^\star, Qm^\star), 0\Big\} \\ &= S(m^\star, m^\star, Qm^\star). \end{aligned}$$

Hence (30) yields

$$\tau + \mathcal{F}(S(m^\star, m^\star, Qm^\star)) \leq \mathcal{F}(S(m^\star, m^\star, Qm^\star)).$$

Since $\tau > 0$, the above inequality yields a contradiction. Hence, $S(m^\star, m^\star, Qm^\star) = 0$. Also, $S(m^\star, m^\star, m^\star) = 0$. This gives that $m^\star \in Qm^\star$. This proves that $m^\star$ is a fixed point of Q. The proof is completed. □

Example 3.8. Let $\mathbb{M} = \{m_\rho = 1 - \left(\frac{1}{2}\right)^\rho : \rho \in \mathbb{N}\}$. Define $S\colon \mathbb{M}^3 \to \mathbb{R}^+$ by $S(m_1, m_2, m_3) = |m_1 - m_3| + |m_2 - m_3|$ for all $m_1, m_2, m_3 \in \mathbb{M}$. Then $(\mathbb{M}, S)$ is a complete S-metric space.

We also define a multivalued map $Q\colon \mathbb{M} \to 2^{\mathbb{M}}$ by

$$Q(m) = \begin{cases} \{m_1\}, & \text{if } m = m_1, \\ \{m_p, m_{p+1}\}, & \text{if } m = m_p, p = 2, 3, \ldots. \end{cases}$$

Consider $\alpha(m_p, m_p, m_q) = 1$ and $\Omega(m_p, m_p, m_q) = S(m_p, m_p, m_q)$. Define $\mathcal{F}\colon \mathbb{R}^+ \to \mathbb{R}$ as $\mathcal{F}(m) = ln(m)+m$. Hence the contractive condition will take the following form

$$\frac{\mathcal{S}_H(Qm_p, Qm_p, Qm_q)}{\Omega(m_p, m_p, m_q)} e^{\mathcal{S}_H(Qm_p, Qm_p, Qm_q) - \Omega(m_p, m_p, m_q)} \leq e^{-\tau}. \tag{31}$$

Now, we verify the above condition for the following two cases.

Case (1) : If $\mathcal{S}_H(Qm_p, Qm_p, Qm_1) > 0$ and $q = 1$, we have

$$\begin{aligned} \mathcal{S}_H(Qm_p, Qm_p, Qm_1) &= \max\Big\{\sup_{a\in Qm_p} S(a, a, Qm_1), \sup_{b\in Qm_1} S(b, b, Qm_p)\Big\} \\ &= \max\left\{S(m_p, m_p, Qm_1), S(m_{p+1}, m_{p+1}, Qm_1)\right\} \\ &= \max\{2|m_p - m_1|, 2|m_{p+1} - m_1|\} = 2|m_{p+1} - m_1|. \end{aligned}$$

Hence,

$$\mathcal{S}_H(Qm_p, Qm_p, Qm_1) = 2|m_{p+1} - m_1|. \tag{32}$$

Also,

$$\Omega(m_p, m_p, m_1) = S(m_p, m_p, m_1) = 2|m_p - m_1|. \tag{33}$$

Consequently, one writes

$$\begin{aligned}
&\frac{\mathcal{S}_H(Qm_p, Qm_p, Qm_1)}{\Omega(m_p, m_p, m_1)} e^{\mathcal{S}_H(Qm_p,Qm_p,Qm_1)-\Omega(m_p,m_p,m_1)} \\
&\le \frac{2|m_{p+1} - m_1|}{2|m_p - m_1|} e^{2|m_{p+1}-m_1|-2|m_p-m_1|} \\
&= \frac{\left|1 - \left(\frac{1}{2}\right)^{p+1} - \left(\frac{1}{2}\right)\right|}{\left|1 - \left(\frac{1}{2}\right)^{p} - \left(\frac{1}{2}\right)\right|} \times e^{2\left|1-\left(\frac{1}{2}\right)^{p+1}-\left(\frac{1}{2}\right)\right|-2\left|1-\left(\frac{1}{2}\right)^{p}-\left(\frac{1}{2}\right)\right|} \\
&= \frac{\left|\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^{p+1}\right|}{\left|\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^{p}\right|} \times e^{2\left|\left(\frac{1}{2}\right)-\left(\frac{1}{2}\right)^{p+1}\right|-2\left|\left(\frac{1}{2}\right)-\left(\frac{1}{2}\right)^{p}\right|} \\
&\le e^{\left|1-\left(\frac{1}{2}\right)^{p}\right|-\left|1-\left(\frac{1}{2}\right)^{p-1}\right|} \le e^{-\left(\frac{1}{2}\right)^{p+1}} = e^{-\tau},
\end{aligned}$$

for some $\tau > 0$, where $\tau = \left(\frac{1}{2}\right)^{p+1}$.

Case (2): If $\mathcal{S}_H(Qm_p, Qm_p, Qm_q) > 0$ with $p \ge q > 1$, we have

$$\mathcal{S}_H(Qm_p, Qm_p, Qm_q) = 2|m_{p+1} - m_{q+1}|,$$

and

$$\Omega(m_p, m_p, m_q) = S(m_p, m_p, m_q) = 2|m_p - m_q|.$$

From (31), we have

$$\begin{aligned}
&\frac{\mathcal{S}_H(Qm_p, Qm_p, Qm_q)}{\Omega(m_p, m_p, m_q)} e^{\mathcal{S}_H(Qm_p,Qm_p,Qm_q)-\Omega(m_p,m_p,m_q)} \\
&= \frac{2|m_{p+1} - m_{q+1}|}{2|m_p - m_q|} e^{2|m_{p+1}-m_{q+1}|-2|m_p-m_q|} \\
&= \frac{\left|1 - \left(\frac{1}{2}\right)^{p+1} - \left[1 - \left(\frac{1}{2}\right)^{q+1}\right]\right|}{\left|1 - \left(\frac{1}{2}\right)^{p} - \left[1 - \left(\frac{1}{2}\right)^{q}\right]\right|} \times e^{2\left|1-\left(\frac{1}{2}\right)^{p+1}-\left[1-\left(\frac{1}{2}\right)^{q+1}\right]\right|-2\left|1-\left(\frac{1}{2}\right)^{p}-\left[1-\left(\frac{1}{2}\right)^{q}\right]\right|} \\
&= \frac{1}{2} e^{-\frac{1}{2}\left|\left(\frac{1}{2}\right)^{q-1}-\left(\frac{1}{2}\right)^{p-1}\right|} < e^{-\frac{1}{2}\left|\left(\frac{1}{2}\right)^{q-1}-\left(\frac{1}{2}\right)^{p-1}\right|} = e^{-\tau},
\end{aligned}$$

where $\tau = \frac{1}{2}\left|\left(\frac{1}{2}\right)^{q-1} - \left(\frac{1}{2}\right)^{p-1}\right|$, which is true for all $p, q \in \mathbb{N}$ such that $p \ge q > 1$, where $\tau > 0$. Thus, all the required assumptions of Theorem 3.7 are satisfied. Hence, by application of Theorem 3.7, the mapping Q has a fixed point. Here, m_1 and m_p are fixed points.

4. Consequences

In this section, some known results in the literature are obtained as the consequences of the main result. These are as follows:

(1) For all $m_1, m_2 \in \mathbb{M}$ and $0 \le k < 1$,

$$S(Qm_1, Qm_1, Qm_2) \le k\, S(m_1, m_1, m_2)$$

implies

$$\begin{aligned} S(Qm_1, Qm_1, Qm_2) &\leq k \max\Big\{S(m_1, m_1, m_2), S(m_1, m_1, Qm_1), S(m_2, m_2, Qm_2), \\ &\qquad \frac{S(m_1, m_1, m_2)[1 + S(m_2, m_2, Qm_1)]}{1 + S(m_1, m_1, m_2)}\Big\} \\ &= k\,\Omega(m_1, m_1, m_2). \end{aligned}$$

If $S(Qm_1, Qm_1, Qm_2) > 0$, then

$$\tau + ln\big(S(Qm_1, Qm_1, Qm_2)\big) \leq ln\big(\Omega(m_1, m_1, m_2)\big),$$

where $\tau = -ln\, k > 0$ and $Q\colon \mathbb{M} \to \mathbb{M}$ is a single valued mapping.

Therefore, the contraction condition in Definition 2.13 of (*Sedghi et al.*, 2012) [26] becomes the condition (2) with $F(m) = ln\,(m)$, for all $m > 0$ and $\alpha(m_1, m_1, m_2) = 1$ for all $m_1, m_2 \in \mathbb{M}$. This shows that Theorem 3.5 is a generalization of Theorem 3.1 of (*Sedghi et al.*, 2012) for multivalued mapping.

(**2**) For all $m_1, m_2 \in \mathbb{M}$ and $h \in [0, 1)$,

$$S(Qm_1, Qm_1, Qm_2) \leq h \max\{S(m_1, m_1, Qm_1), S(m_2, m_2, Qm_2)\}$$

implies

$$\begin{aligned} S(Qm_1, Qm_1, Qm_2) &\leq h \max\Big\{S(m_1, m_1, m_2), S(m_1, m_1, Qm_1), S(m_2, m_2, Qm_2), \\ &\qquad \frac{S(m_1, m_1, m_2)[1 + S(m_2, m_2, Qm_1)]}{1 + S(m_1, m_1, m_2)}\Big\} \\ &= h\,\Omega(m_1, m_1, m_2). \end{aligned}$$

If $S(Qm_1, Qm_1, Qm_2) > 0$, then

$$\tau + ln\big(S(Qm_1, Qm_1, Qm_2)\big) \leq ln\big(\Omega(m_1, m_1, m_2)\big),$$

where $\tau = -ln\, h > 0$ and $Q\colon \mathbb{M} \to \mathbb{M}$ is a single valued mapping.

Therefore, the contraction condition in Corollary 2.10 of (*Sedghi and Dung*, 2014) [27] becomes the condition (2) with $F(m) = ln\,(m)$, for all $m > 0$ and $\alpha(m_1, m_1, m_2) = 1$ for all $m_1, m_2 \in \mathbb{M}$. This shows that Theorem 3.5 is a generalization of Corollary 2.10 of (*Sedghi and Dung*, 2014). It also generalizes Corollary 2 of (*Devi et al.*, 2022) [8] for multivalued mapping.

(**3**) For all $m_1, m_2 \in \mathbb{M}$ and $a, b, c \geq 0$ with $a + b + c < 1$,

$$S(Qm_1, Qm_1, Qm_2) \leq a\, S(m_1, m_1, m_2) + b\, S(m_1, m_1, Qm_1) + c\, S(m_2, m_2, Qm_2)$$

that is,

$$\begin{aligned} S(Qm_1, Qm_1, Qm_2) &\leq (a + b + c) \max\{S(m_1, m_1, m_2), S(m_1, m_1, Qm_1), \\ &\qquad S(m_2, m_2, Qm_2)\} \end{aligned}$$

implies

$$\begin{aligned} S(Qm_1, Qm_1, Qm_2) &\leq (a + b + c) \max\Big\{S(m_1, m_1, m_2), S(m_1, m_1, Qm_1), S(m_2, m_2, Qm_2), \\ &\qquad \frac{S(m_1, m_1, m_2)[1 + S(m_2, m_2, Qm_1)]}{1 + S(m_1, m_1, m_2)}\Big\} \\ &= (a + b + c)\,\Omega(m_1, m_1, m_2). \end{aligned}$$

If $S(Qm_1, Qm_1, Qm_2) > 0$, then

$$\tau + ln\big(S(Qm_1, Qm_1, Qm_2)\big) \leq ln\big(\Omega(m_1, m_1, m_2)\big),$$

where $\tau = -ln\,(a+b+c) > 0$ and $Q\colon \mathbb{M} \to \mathbb{M}$ is a single valued mapping.

Therefore, the contraction condition in Corollary 2.12 of (*Sedghi and Dung*, 2014) [27] becomes the condition (2) with $F(m) = ln\,(m)$, for all $m > 0$ and $\alpha(m_1, m_1, m_2) = 1$ for all $m_1, m_2 \in \mathbb{M}$. This shows that Theorem 3.5 is a generalization of Corollary 2.12 of (*Sedghi and Dung*, 2014) for multivalued mapping.

(**4**) For all $m_1, m_2 \in \mathbb{M}$ and $0 \leq k < 1$,

$$S(Qm_1, Qm_1, Qm_2) \leq k\, S(m_1, m_1, m_2)$$

implies

$$\begin{aligned} S(Qm_1, Qm_1, Qm_2) &\leq k \max\Big\{S(m_1, m_1, m_2), S(m_1, m_1, Qm_1), S(m_2, m_2, Qm_2), \\ &\qquad \frac{S(m_1, m_1, m_2)[1 + S(m_2, m_2, Qm_1)]}{1 + S(m_1, m_1, m_2)}\Big\} \\ &= k\,\Omega(m_1, m_1, m_2). \end{aligned}$$

If $S(Qm_1, Qm_1, Qm_2) > 0$, then

$$\tau + ln\big(\alpha(m_1, m_1, m_2) S(Qm_1, Qm_1, Qm_2)\big) \leq ln\big(\Omega(m_1, m_1, m_2)\big),$$

where $\tau = -ln\, k > 0$ and $Q\colon \mathbb{M} \to \mathbb{M}$ is a single valued mapping.

Therefore, the contraction condition in Definition 2.1 of (*Javed et al.*, 2021) [11] becomes the condition (2) with $F(m) = ln\,(m)$, for all $m > 0$. This shows that Theorem 3.5 is a generalization of Theorem 2.1 of (*Javed et al.*, 2021) for multivalued mapping.

(**5**) For all $m_1, m_2 \in \mathbb{M}$ and $a_1, a_2, a_3 \geq 0$ with $a_1 + a_2 + a_3 < 1$,

$$S(Qm_1, Qm_1, Qm_2) \leq a_1\, S(m_1, m_1, m_2) + a_2\, S(m_1, m_1, Qm_1) + a_3\, S(m_2, m_2, Qm_2)$$

that is,

$$\begin{aligned} S(Qm_1, Qm_1, Qm_2) &\leq (a_1 + a_2 + a_3) \max\{S(m_1, m_1, m_2), S(m_1, m_1, Qm_1), \\ &\qquad S(m_2, m_2, Qm_2)\} \end{aligned}$$

implies

$$\begin{aligned} S(Qm_1, Qm_1, Qm_2) \leq (a_1 + a_2 + a_3) \max\Big\{&S(m_1, m_1, m_2), S(m_1, m_1, Qm_1), S(m_2, m_2, Qm_2), \\ &\frac{S(m_1, m_1, m_2)[1 + S(m_2, m_2, Qm_1)]}{1 + S(m_1, m_1, m_2)}\Big\} \\ = (a_1 + a_2 + a_3)\,\Omega(m_1, m_1, m_2).& \end{aligned}$$

If $S(Qm_1, Qm_1, Qm_2) > 0$, then

$$\tau + ln\big(S(Qm_1, Qm_1, Qm_2)\big) \leq ln\big(\Omega(m_1, m_1, m_2)\big),$$

where $\tau = -ln\,(a_1 + a_2 + a_3) > 0$ and $Q\colon \mathbb{M} \to \mathbb{M}$ is a single valued mapping.

Therefore, the contraction condition in Corollary 1 of (*Thaibema et al.*, 2022) [29] becomes the condition (2) with $F(m) = ln\,(m)$, for all $m > 0$ and $\alpha(m_1, m_1, m_2) = 1$ for all $m_1, m_2 \in \mathbb{M}$. This shows that Theorem 3.5 is a generalization of Corollary 1 of (*Thaibema et al.*, 2022) for multivalued mapping.

Example 4.1. Let $\Omega = [0,1]$. Define the function $S\colon \mathbb{M}^3 \to [0,+\infty)$ by

$$S(m_1,m_2,m_3) = \frac{|m_1-m_3|^2+|m_2-m_3|^2}{2}$$

for all $m_1,m_2,m_3 \in \mathbb{M}$. The condition $S(1)$ holds directly. To check condition $S(2)$, for all $n \in \mathbb{M}$, we have

$$S(m_1,m_1,n)+S(m_2,m_2,n)+S(m_3,m_3,n)$$

$$\begin{aligned}
&= |m_1-n|^2+|m_2-n|^2+|m_3-n|^2 \\
&= |m_1-m_3+m_3-n|^2+|m_2-m_3+m_3-n|^2+|m_3-n|^2 \\
&\geq (|m_1-m_3|^2+|m_3-n|^2)+(|m_2-m_3|^2+|m_3-n|^2)+|m_3-n|^2 \\
&= 3|m_3-n|^2+(|m_1-m_3|^2+|m_2-m_3|^2) \\
&\geq |m_1-m_3|^2+|m_2-m_3|^2 \\
&\geq \frac{|m_1-m_3|^2+|m_2-m_3|^2}{2} = S(m_1,m_2,m_3).
\end{aligned}$$

Hence $(\mathbb{M},S)$ is an S-MS. Define MV-mapping $Q\colon \mathbb{M} \to CB(\mathbb{M})$ by $Q(m) = \left[0,\frac{m}{3}\right]$ for all $m \in \mathbb{M}$ and let $\alpha(m_1,m_1,m_2)=1$. Now, we examine the condition (3) of Theorem 3.5. Using Definition 2.18, for all $m_1,m_2 \in \mathbb{M}$, we get

$$\begin{aligned}
\mathcal{S}_H(Qm_1,Qm_1,Qm_2) &= H_S(Qm_1,Qm_2)+H_S(Qm_1,Qm_2) \\
&= 2\,H_S(Qm_1,Qm_2) \\
&= 2\max\big\{h_S(Qm_1,Qm_2),h_S(Qm_1,Qm_2)\big\},
\end{aligned}$$

where

$$\begin{aligned}
h_S(Qm_1,Qm_2) &= \sup_{m\in Qm_1}\inf_{n\in Qm_2} S(m,m,n) \\
&= \sup_{m\in Qm_1}\inf_{n\in Qm_2}\Big(\frac{|m-n|^2+|m-n|^2}{2}\Big) \\
&= \sup_{m\in Qm_1}\inf_{n\in Qm_2}\{|m-n|^2\} \\
&= \sup_{m\in\left[0,\frac{m_1}{3}\right]}\inf\Big\{\Big|m-\Big[0,\frac{m_2}{3}\Big]\Big|^2\Big\}.
\end{aligned}$$

If $m=0\in Qm_1$, then $\inf\Big\{0,\Big|\frac{m_1}{3}-\frac{m_2}{3}\Big|^2\Big\}=0$. If $m=\frac{m_1}{3}\in Qm_1$, then $\inf\Big\{0,\Big|\frac{m_1}{3}-\frac{m_2}{3}\Big|^2\Big\}=0$. Consequently, $h_S(Qm_1,Qm_2)=0$. Hence

$$\mathcal{S}_H(Qm_1,Qm_1,Qm_2) = 2\,\max\{0,0\} = 0 \leq \lambda\, S(m_1,m_1,m_2) \leq \lambda\,\Omega(m_1,m_1,m_2),$$

where $\lambda = e^{-\tau} < 1$ and $\mathcal{F}(m) = ln\,(m)$ for all $m > 0$. Thus, the contractive condition (2) of Theorem 3.5 is fulfilled with any $\lambda \in [0,1)$. Hence, all requirements of Theorem 3.5 are satisfied. Consequently, Q has a unique fixed point which is $0 \in \mathbb{M}$.

5. Applications

($\mathcal{A}_1$) Here, we discuss the application of fixed point technique to the following Fredholm type integral equation:

$$u(t) = \int_0^t \mathcal{K}(t,s,u(s))ds + \mu(t), \quad t \in [0,\Lambda], \tag{34}$$

where $\Lambda > 0$.

Now, we establish the existence of the solution of the integral equation (34). Let $\mathbb{M} = C([0,\Lambda],\mathbb{R})$ denote the space of all continuous real-valued functions on $[0,\Lambda]$. Define an S-metric $S\colon \mathbb{M}^3 \to \mathbb{R}$ by

$$\begin{aligned} S(m_1,m_2,m_3) &= \|m_1-m_3\| + \|m_2-m_3\| \\ &= \sup_{t\in J}\big\{(|m_1(t)-m_3(t)| + |m_2(t)-m_3(t)|)e^{-\tau t}\big\}, \end{aligned}$$

for all $m_1,m_2,m_3 \in \mathbb{M}$, where $J=[0,\Lambda]$ and $\tau > 0$ is taken arbitrary.

It is easy to verify that $(\mathbb{M},S)$ is a complete S-metric space. Define the single valued mapping $Q\colon \mathbb{M}\to\mathbb{M}$ by

$$Q(u(\zeta)) = \int_0^\zeta \mathcal{K}(\zeta,\kappa,u(\kappa))d\kappa, \quad \zeta,\kappa \in [0,\Lambda]. \tag{35}$$

For the derivation of existence result for the solution of the Fredholm type integral equation (34), we prove the following theorem.

Theorem 5.1. *Assume that the following assumptions hold:*

($\mathbb{H}_1$) *the mappings* $\mu\colon J \to (-\infty,\infty)$ *and* $\mathcal{K}\colon J\times J\times(-\infty,\infty)\to(-\infty,\infty)$ *are both continuous.*

($\mathbb{H}_2$) *there exists* $\tau \in [1,+\infty)$ *such that*

$$|\mathcal{K}(\zeta,\kappa,u) - \mathcal{K}(\zeta,\kappa,v)| \le \tau\, e^{-\tau}\, |u-v|,$$

for all $\zeta,\kappa\in[0,\Lambda]$ *and* $u,v\in\mathbb{R}$.

Then, the above integral equation (34) has a solution.

Proof. We have to show that the operator Q satisfies all the conditions of Theorem 3.5. For this, using (35), we have

$$\begin{aligned} |Q(m_1)(\zeta) - Q(m_2)(\zeta)| &= \Big|\int_0^\zeta [\mathcal{K}(\zeta,\kappa,m_1(\kappa)) - \mathcal{K}(\zeta,\kappa,m_2(\kappa))]d\kappa\Big| \\ &\le \int_0^\zeta \big|\mathcal{K}(\zeta,\kappa,m_1(\kappa)) - \mathcal{K}(\zeta,\kappa,m_2(\kappa))\big|d\kappa \\ &\le \int_0^\zeta \tau\, e^{-\tau}|m_1(\kappa)-m_2(\kappa)|d\kappa \\ &= \int_0^\zeta \tau\, e^{-\tau}|m_1(\zeta)-m_2(\zeta)|e^{-\tau\kappa}e^{\tau\kappa}d\kappa \\ &= \int_0^\zeta e^{\tau\kappa}\tau\, e^{-\tau}|m_1(\kappa)-m_2(\kappa)|e^{-\tau\kappa}d\kappa \\ &= \tau\, e^{-\tau}\, e^{-\tau\kappa}|m_1(\kappa)-m_2(\kappa)|\int_0^\zeta e^{\tau\kappa}d\kappa \\ &= \tau\, e^{-\tau}\, e^{-\tau\kappa}|m_1(\kappa)-m_2(\kappa)|\,\frac{e^{\tau\zeta}}{\tau} \\ &= e^{-\tau}\, e^{\tau\zeta}\, e^{-\tau\kappa}|m_1(\kappa)-m_2(\kappa)|. \end{aligned} \tag{36}$$

Taking sup on both sides, we obtain

$$\sup_{\zeta\in J}|Q(m_1)(\zeta)-Q(m_2)(\zeta)| \le e^{-\tau}\, e^{\tau\zeta}\sup_{\kappa\in J}|m_1(\kappa)-m_2(\kappa)|e^{-\tau\kappa}.$$

This implies that

$$2\sup_{\zeta\in J}|Q(m_1)(\zeta)-Q(m_2)(\zeta)|e^{-\tau\zeta} \le 2e^{-\tau}\sup_{\kappa\in J}|m_1(\kappa)-m_2(\kappa)|e^{-\tau\kappa},$$

or equivalently,

$$S(Q(m_1), Q(m_1), Q(m_2)) \leq e^{-\tau}\, S(m_1, m_1, m_2),$$

or,

$$S(Q(m_1), Q(m_1), Q(m_2)) \leq e^{-\tau}\, S(m_1, m_1, m_2) \leq e^{-\tau}\, \Omega(m_1, m_1, m_2).$$

Thus, we obtain

$$S(Q(m_1), Q(m_1), Q(m_2)) \leq e^{-\tau}\, \Omega(m_1, m_1, m_2).$$

After going through a natural logarithm, we can write this as

$$ln\big(S(Q(m_1), Q(m_1), Q(m_2))\big) \leq ln\big(e^{-\tau}\, \Omega(m_1, m_1, m_2)\big),$$

and, after routine calculations, we get

$$\tau + ln\big(S(Q(m_1), Q(m_1), Q(m_2))\big) \leq ln\big(\Omega(m_1, m_1, m_2)\big).$$

Now, we see that the function $\mathcal{F}\colon \mathbb{R}^+ \to \mathbb{R}$ defined by $\mathcal{F}(u) = ln\,(u)$, for each $u \in C(J = [0, \Lambda], \mathbb{R})$, is in $\mathcal{F}_S$, so we deduce that the operator Q satisfies all the conditions of consequence mentioned in (**1**) with $\alpha(m_1, m_1, m_2) = 1$. Hence by its application, the operator Q has a fixed point $m^* \in C(J, \mathbb{R})$, that is, m^* is a solution of Fredholm type integral equation (34).

($\mathcal{A}_2$) In this part, we establish the existence and uniqueness of the solution of a fractional differential equation involving the *Caputo Atangana-Baleanu* via fixed point procedure

$$\begin{aligned} D^\alpha \eta(t) &= f(t, \eta(t)),\ t \in I = [0,1], \\ \eta(0) &= \delta, \end{aligned} \tag{37}$$

where D^λ is the Atangana-Baleanu derivative of order λ, $\eta\colon I \to \mathbb{R}$, $f \in C(I, \mathbb{R})$ are continuous functions such that $f(0, x(0)) = 0$, $\alpha \in (0,1)$ and δ is a constant. Let $\mathbb{M} = C([0,1], \mathbb{R})$ be the space of continuous function defined on $[0,1]$.

Definition 5.2. ([3, 14, 25]) Let $\eta \in H^1(a,b)$ with $a < b$ and $\alpha \in [0,1]$. The *Caputo Atangana-Baleanu* fractional derivative of η of order α is defined by

$$D^\alpha \eta(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t \eta'(x) E_\alpha\big[-\alpha \frac{(t-x)^\alpha}{1-\alpha}\big] dx, \tag{38}$$

where E_α is the Mittage-Leffler function defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}, \tag{39}$$

and $B(\alpha)$ is a normalizing positive function satisfying $B(0) = B(1) = 1$. Then, the associative fractional integral is given by

$$I^\alpha \eta(t) = \frac{1-\alpha}{B(\alpha)} \eta(t) + \frac{\alpha}{B(\alpha)}\, {}_aI^\alpha \eta(t), \tag{40}$$

where ${}_aI^\alpha$ is the left Riemann-Liouville fractional integral given as

$${}_aI^\alpha \eta(t) = \frac{1}{\Gamma\alpha} \int_a^t (t-x)^{\alpha-1} \eta(x) dx. \tag{41}$$

Proposition 5.3. *([3]) For* $0 < \alpha < 1$*, we have*

$$I_a^\alpha D^\alpha \eta(x) = \eta(x) - \eta(a).$$

Let $\mathbb{M} = C([0,1],\mathbb{R})$ be the set of all continuous functions defined on $[0,1]$. Consider

$$\zeta(t) = \frac{1-\kappa}{\mathcal{B}(\kappa)}\zeta(t) + \frac{\kappa}{\mathcal{B}(\kappa)\Gamma(\kappa)}\int_0^t (t-y)^{\kappa-1}\zeta(y)dy, \tag{42}$$

where $\zeta(t) \in \mathbb{M}$.

Now, we shall investigate the existence and uniqueness solution of (42).

Define the single valued operator $Q\colon \mathbb{M} \to \mathbb{M}$ by

$$Q\zeta(t) = \frac{1-\kappa}{\mathcal{B}(\kappa)}\zeta(t) + \frac{\kappa}{\mathcal{B}(\kappa)\Gamma(\kappa)}\int_0^t (t-y)^{\kappa-1}\zeta(y)dy. \tag{43}$$

Let $S\colon \mathbb{M}\times\mathbb{M}\times\mathbb{M} \to \mathbb{R}^+$ be defined by

$$\begin{aligned} S(v_1,v_2,v_3) &= \|v_1 - v_3\| + \|v_2 - v_3\| \\ &= \sup_{t\in I}\{|v_1(t) - v_3(t)| + |v_2(t) - v_3(t)|\}, \end{aligned} \tag{44}$$

for all $v_1, v_2, v_3 \in \mathbb{M}$, where $I = [0,1]$. Then clearly $(\mathbb{M}, S)$ is a complete S-metric space.

Now, we will prove that fractional integral of *Atangana-Baleanu* type has no more than one solution in the view of the following perspective:

$$\frac{1-\kappa}{\mathcal{B}(\kappa)} + \frac{t^\kappa}{\mathcal{B}(\kappa)\Gamma(\kappa)} < u\,e^{-\tau},$$

where $u \in (0,1)$ and $\tau > 0$.

Consider,

$$\begin{aligned} |Q\zeta(t) - Q\nu(t)| &= \Big|\Big(\frac{1-\kappa}{\mathcal{B}(\kappa)}\zeta(t) + \frac{\kappa}{\mathcal{B}(\kappa)\Gamma(\kappa)}\int_0^t (t-y)^{\kappa-1}\zeta(y)dy\Big) \\ &\quad -\Big(\frac{1-\kappa}{\mathcal{B}(\kappa)}\nu(t) + \frac{\kappa}{\mathcal{B}(\kappa)\Gamma(\kappa)}\int_0^t (t-y)^{\kappa-1}\nu(y)dy\Big)\Big| \\ &= \Big|\frac{1-\kappa}{\mathcal{B}(\kappa)}(\zeta(t) - \nu(t)) + \frac{\kappa}{\mathcal{B}(\kappa)\Gamma(\kappa)}\int_0^t (t-y)^{\kappa-1}(\zeta(y) - \nu(y))dy\Big| \\ &\leq \frac{1-\kappa}{\mathcal{B}|\kappa)}|\zeta(t) - \nu(t)| + \frac{\kappa}{\mathcal{B}(\kappa)\Gamma(\kappa)}\int_0^t (t-y)^{\kappa-1}|\zeta(y) - \nu(y)|dy \\ &= \Big(\frac{1-\kappa}{\mathcal{B}|\kappa)} + \frac{t^\kappa}{\mathcal{B}(\kappa)\Gamma(\kappa)}\Big)|\zeta(t) - \nu(t)| \\ &\leq u\,e^{-\tau}|\zeta(t) - \nu(t)| \leq e^{-\tau}|\zeta(t) - \nu(t)|. \end{aligned}$$

This implies that

$$2|Q\zeta(t) - Q\nu(t)| \leq 2e^{-\tau}|\zeta(t) - \nu(t)|.$$

Taking $\sup_{t\in I}$ on both sides, we get

$$S(Q(\zeta), Q(\zeta), Q(\nu)) \leq e^{-\tau}S(\zeta,\zeta,\nu) \leq e^{-\tau}\Omega(\zeta,\zeta,\nu),$$

that is,

$$S(Q(\zeta), Q(\zeta), Q(\nu)) \leq e^{-\tau}\Omega(\zeta,\zeta,\nu).$$

Taking natural logarithm of both sides, we have that

$$\tau + ln\big(S(Q(\zeta), Q(\zeta), Q(\nu))\big) \leq ln\big(\Omega(\zeta,\zeta,\nu)\big).$$

Taking $\mathcal{F}(a) = ln(a)$, we have

$$\tau + \mathcal{F}\big(S(Q(\zeta), Q(\zeta), Q(\nu))\big) \leq \mathcal{F}\big(\Omega(\zeta,\zeta,\nu)\big),$$

that is, Q is a $(\alpha, \mathcal{F})$-contraction for single valued mapping with $\alpha(\zeta, \zeta, \nu) = 1$. Thus all the assumptions of consequence (**1**) are satisfied, which yields, fractional integral of *Atangana-Baleanu* type of order κ has a unique solution. □

6. Conclusion

In this article, we initiate the concept of new multi-valued contractions called $(\alpha, \mathcal{F})$-contractions and establish some fixed point results for such contractions in the setting of complete S-metric spaces. In addition, we illustrate our main results with concrete examples. Furthermore, some applications are also discussed for deeper understanding of the established results. The results demonstrated in this paper extend, generalize and enrich several conclusions in the current literature.

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