Letters in Nonlinear Analysis and its Applications **4** (2026) No. 1, 1-13
Available online at www.lettersinnonlinearanalysis.com
**Research Article**

Letters in Nonlinear Analysis and its Applications
*Peer Review Scientific Journal*
ISSN: 2958-874x

# A shrinking projection method with extended allowable ranges for common fixed point problems

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**Abstract**

In this paper, we propose a shrinking projection method with extended allowable ranges for approximating a common fixed point of a family of nonlinear mappings in a Banach space. Our approach combines and improves the ideas from Kimura [9], which allowed nonsummable errors, and Takeuchi [21], who introduced allowable ranges into the shrinking projection method. Using our method, we prove strong convergence theorems for approximating a common fixed point of a family of nonlinear mappings of quasi-nonexpansive type in Banach spaces.

*Keywords:* Shrinking projection method, allowable range, common fixed point, common attractive point
*2020 MSC:* 47H09, 47H10, 47J25

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## 1. Introduction

Let $C$ be a nonempty closed convex subset of a real Banach space. A mapping $T : C \to E$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$. Fixed point approximation methods for nonexpansive mappings play an important role in nonlinear analysis and its applications. One useful method is the shrinking projection method introduced by Takahashi, Takeuchi, and Kubota [20].

**Theorem 1.1** ([20]). *Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T$ be a nonexpansive mapping of $C$ into itself such that $F(T) := \{z \in C : z = Tz\}$ is nonempty. Let $\{\alpha_n\}$ be*

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*Received: October 31, 2025, Accepted: February 4, 2026, Online: February 5, 2026*

*a sequence in* $[0, a]$*, where* $0 < a < 1$*. For a point* $x \in H$ *chosen arbitrarily, generate a sequence* $\{x_n\}$ *by the following iterative scheme:* $x_1 \in C$*,* $C_1 = C$*, and*

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_{n+1} &= \{z \in C_n : \|z - y_n\| \le \|z - x_n\|\}, \\ x_{n+1} &= P_{C_{n+1}} x \end{aligned}$$

*for* $n \in \mathbb{N}$*. Then,* $\{x_n\}$ *converges strongly to* $P_{F(T)}x \in C$*, where* $P_K$ *is the metric projection of* $H$ *onto a nonempty closed convex subset* $K$ *of* $H$*.*

We note that the original result of this theorem is a convergence theorem to a common fixed point of a family of nonexpansive mappings. Since then, many researchers have studied this method. In particular, Kimura and Takahashi [12] improved the method by using the concept of Mosco convergence in their proof. They obtained strong convergence theorems for a common fixed point of a family of relatively nonexpansive mappings in a Banach space.

However, the shrinking projection method requires the exact calculation of metric projections at each step. This calculation becomes more difficult as the iteration proceeds. To solve this problem, Kimura [9] proposed the shrinking projection method with nonsummable errors, which allows errors in the approximations of the exact metric projections $P_{C_n}x$ that stay inside the sets $C_n$; see also [10, 11]. Inspired by Kimura [9], Takeuchi [21] proposed another method called the shrinking projection method with allowable ranges. This method allows errors outside the sets $C_n$ in the approximations of the metric projections.

On the other hand, in the study of common fixed point approximation methods, Ibaraki and Takeuchi [6] extended the concept of a quasi-nonexpansive mapping to a family of nonlinear mappings by using the concept of an attractive point [19] for nonlinear mappings in Hilbert spaces. In 2013, Lin and Takahashi [13] extended the concept of an attractive point to Banach spaces.

Motivated by the works above, we propose a new shrinking projection method that allows errors both inside and outside the target sets. Using our method, we prove strong convergence theorems for finding a common fixed point of a family of nonlinear mappings of quasi-nonexpansive type in Banach spaces.

## 2. Preliminaries

Let $E$ be a real Banach space with its dual $E^*$. Let $\{x_n\}$ be a sequence in $E$. The strong and the weak convergence of $\{x_n\}$ to a point $x \in E$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. The normalized duality mapping $J : E \to E^*$ is defined by

$$Jx = \{y^* \in E^* : \|x\|^2 = \langle x, y^* \rangle = \|y^*\|^2\}$$

for $x \in E$. It is known that if $E$ is smooth, then the normalized duality mapping $J$ is single-valued and norm-to-weak* continuous. We also know that if $E$ is reflexive, smooth, and strictly convex, then $J$ is a bijection and the duality mapping on $E^*$ is $J^{-1}$; see, for instance, [18]. A Banach space $E$ is said to have the Kadec-Klee property if a sequence $\{x_n\}$ of $E$ satisfying $x_n \rightharpoonup x_0$ and $\|x_n\| \to \|x_0\|$ converges strongly to $x_0$.

Let $E$ be a reflexive and strictly convex Banach space and let $C$ be a nonempty closed convex subset of $E$. It is known that for each $x \in E$ there exists a unique point $z \in C$ such that

$$\|z - x\| = \inf_{y \in C} \|y - x\|.$$

Such a point $z$ is denoted by $P_C x$ and $P_C$ is called the metric projection of $E$ onto $C$. We recall the following results.

**Lemma 2.1.** *Let* $C$ *be a nonempty closed convex subset of a reflexive, smooth, and strictly convex Banach space* $E$*, and let* $z \in C$ *and* $x \in E$*. Then* $z = P_C x$ *if and only if* $\langle y - z, J(z - x)\rangle \ge 0$ *for all* $y \in C$*.*

**Lemma 2.2** ([5]). *Suppose that $C$ is a nonempty closed convex subset of a reflexive and strictly convex Banach space $E$ and $u \in E$. If $F$ is a nonempty closed convex subset of $C$ such that $P_C u \in F$, then $P_F u = P_C u$.*

Let $E$ be a smooth Banach space and consider the following function $V : E \times E \to \mathbb{R}$ defined by

$$V(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for each $x, y \in E$. We know the following properties; see [1, 4, 15] for more details:

1. $(\|x\| - \|y\|)^2 \leq V(x, y) \leq (\|x\| + \|y\|)^2$ for each $x, y \in E$,
2. $V(x, y) + V(y, x) = 2\langle x - y, Jx - Jy\rangle$ for each $x, y \in E$,
3. $V(x, y) = V(x, z) + V(z, y) + 2\langle x - z, Jz - Jy\rangle$ for each $x, y, z \in E$,
4. if $E$ is additionally assumed to be strictly convex, then $V(x, y) = 0$ if and only if $x = y$.

Let $E$ be a reflexive, smooth, and strictly convex Banach space and let $C$ be a nonempty closed convex subset of $E$. It is known that for each $x \in E$ there exists a unique point $z \in C$ such that

$$V(z, x) = \min_{y \in C} V(y, x).$$

Such a point $z$ is denoted by $\varPi_C x$ and $\varPi_C$ is called the generalized projection of $E$ onto $C$; see [1]. We recall the following results.

**Lemma 2.3** ([1, 4]). *Let $C$ be a nonempty closed convex subset of a reflexive, smooth, and strictly convex Banach space $E$, and let $z \in C$ and $x \in E$. Then $z = \varPi_C x$ if and only if $\langle y - z, Jz - Jx\rangle \geq 0$ for all $y \in C$.*

**Lemma 2.4** ([5]). *Suppose that $C$ is a nonempty closed convex subset of a reflexive, smooth, and strictly convex Banach space $E$ and $u \in E$. If $F$ is a nonempty closed convex subset of $C$ such that $\varPi_C u \in F$, then $\varPi_F u = \varPi_C u$.*

Let $E$ be a Banach space and let $\{C_n\}$ be a sequence of nonempty closed convex subsets of $E$. We denote by $\operatorname{s-Li}_n C_n$ the set of limit points of $\{C_n\}$, that is, $x \in \operatorname{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $x_n \in C_n$ for each $n \in \mathbb{N}$ and $\{x_n\}$ converges strongly to $x$. Similarly, we denote by $\operatorname{w-Ls}_n C_n$ the set of cluster points of $\{C_n\}$; $y \in \operatorname{w-Ls}_n C_n$ if and only if there exists $\{y_{n_i}\} \subset E$ such that $y_{n_i} \in C_{n_i}$ for each $i \in \mathbb{N}$ and $\{y_{n_i}\}$ converges weakly to $y$. Using these definitions, we define the Mosco convergence [16] of $\{C_n\}$. If $C_0$ satisfies

$$\operatorname*{s-Li}_n C_n = C_0 = \operatorname*{w-Ls}_n C_n,$$

we say that $\{C_n\}$ is a Mosco convergent sequence to $C_0$ and denote it

$$C_0 = \operatorname*{M-lim}_n C_n.$$

Notice that the inclusion $\operatorname{s-Li}_n C_n \subset \operatorname{w-Ls}_n C_n$ is always true. So, to show $C_0 = \operatorname{M-lim}_n C_n$ we may show $\operatorname{w-Ls}_n C_n \subset \operatorname{s-Li}_n C_n$. We show an example of the Mosco convergent sequence $\{C_n\}$: Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of a Banach $E$ such that $C_{n+1} \subset C_n$ for each $n \in \mathbb{N}$. Then, $\operatorname{M-lim}_n C_n = \cap_n C_n$; see [16] for more details. We recall the following theorems for nonlinear projections.

**Theorem 2.5** ([22]). *Let $E$ be a reflexive and strictly convex Banach space and let $\{C_n\}$ be a sequence of nonempty closed convex subsets of $E$. If $C_0 = \operatorname{M-lim}_n C_n$ exists and nonempty, then for each $x \in E$, $\{P_{C_n} x\}$ converges weakly to $P_{C_0} x$. Moreover, if $E$ has the Kadec-Klee property, the convergence is in the strong topology.*

**Theorem 2.6** ([4]). *Let $E$ be a reflexive, smooth, and strictly convex Banach space and let $\{C_n\}$ be a sequence of nonempty closed convex subsets of $E$. If $C_0 = \operatorname{M-lim}_n C_n$ exists and nonempty, then $C_0$ is a closed convex subset of $E$ and, for each $x \in E$, $\{\varPi_{C_n} x\}$ converges weakly to $\varPi_{C_0} x$. Moreover, if $E$ has the Kadec-Klee property, the convergence is in the strong topology.*

### 3. Nonlinear mappings and Attractive points

Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $T : C \to E$ be a mapping. We denote by $F(T)$ the set of all fixed points of $T$. $I - T$ is said to be closed at zero if $u \in F(T)$ holds whenever a sequence $\{x_n\}$ in $C$ satisfying $x_n \to u$ and $x_n - Tx_n \to 0$, where $I$ is the identity mapping on $E$. A point $p$ in $C$ is said to be an asymptotic fixed point [17] of $T$ if $C$ contains a sequence $\{x_n\}$ such that $x_n \rightharpoonup p$ and $x_n - Tx_n \to 0$. The set of all asymptotic fixed points of $T$ is denoted by $\hat{F}(T)$. Let $E$ be a smooth Banach space. A mapping $T$ is said to be relatively nonexpansive [3, 14, 17] if $\hat{F}(T) = F(T) \neq \emptyset$ and

$$V(p, Tx) \leq V(p, x) \tag{1}$$

for each $p \in F(T)$ and $x \in C$. A mapping $T$ is said to be nonspreading [7] if

$$V(Tx, Ty) + V(Ty, Tx) \leq V(Ty, x) + V(Tx, y)$$

for each $x, y \in C$. A mapping $T$ is said to be of type (Q) [2, 8] if

$$V(Tx, x) + V(Ty, y) + V(Tx, Ty) + V(Ty, Tx) \leq V(Ty, x) + V(Tx, y)$$

for each $x, y \in C$. It is known that if a mapping $T$ is nonspreading with $F(T) \neq \emptyset$, then it satisfies (1) for each $p \in F(T)$ and $x \in C$. It is also obvious that if a mapping $T$ is of type (Q), then $T$ is nonspreading. We now state the following lemma. Since its proof is almost identical to that of [7, Lemma 3.2], we omit it here; for more details, see [7].

**Lemma 3.1.** *Let $E$ be a smooth and strictly convex Banach space, let $C$ be a nonempty closed convex subset of $E$ and let $T$ be a nonspreading mapping from $C$ into itself. Then $I - T$ is closed at zero.*

Let $C$ be a nonempty closed convex subset of a smooth Banach space $E$ and let $T : C \to E$ be a mapping. A point $z$ in $E$ is said to be an attractive point [13, 19] of $T$ if

$$V(z, Tx) \leq V(z, x)$$

for each $x \in C$. The set of all attractive points of $T$ is denoted by $A(T)$. We know that $A(T)$ is closed and convex; see [13]. We consider the mapping $T : C \to E$ defined by the condition that $F(T) \neq \emptyset$ and

$$V(p, Tx) \leq V(p, x)$$

for each $p \in F(T)$ and $x \in C$. It is clear that a mapping $T$ satisfies this condition if and only if

$$\emptyset \neq F(T) \subset A(T). \tag{2}$$

Moreover, if $E$ is additionally assumed to be strictly convex, then $F(T)$ is closed and convex; see [15].

From the previous discussion, if a mapping $T$ is relatively nonexpansive, nonspreading, or of type (Q) with $F(T) \neq \emptyset$, then the following hold:

- $T$ satisfies the condition (2),
- $I - T$ is closed at zero,
- $F(T)$ is closed and convex.

Next, we describe the conditions for the family of nonlinear self-mappings in this work, based on Ibaraki and Takeuchi [6]: Let $C$ be a nonempty closed convex subset of a reflexive, smooth, and strictly convex Banach space $E$. Let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of self-mappings on $C$. We assume the following conditions:

1. $\emptyset \neq F := \cap_{\lambda \in \Lambda} F(T_\lambda) \subset A := \cap_{\lambda \in \Lambda} A(T_\lambda)$

2. $I - T_\lambda$ is closed at zero for each $\lambda \in \Lambda$.

It should be noted that the condition $\emptyset \neq F \subset A$ does not necessarily imply that $\emptyset \neq F(T_\lambda) \subset A(T_\lambda)$ for every $\lambda \in \Lambda$. However, it follows immediately that if each mapping $T_\lambda$ satisfies $F(T_\lambda) \subset A(T_\lambda)$, then $F \subset A$ holds. Furthermore, under these conditions, $F$ is closed and convex. This follows immediately from [15, Proposition 2.4]. For an illustration, we present an example in Euclidean space $\mathbb{R}^2$ involving two mappings that satisfy these conditions: see [6] for more details.

**Example 3.2** ([6])**.** Let $C = \left\{x = (s,t) \in \mathbb{R}^2 : s \in [0,1], t \in \left[\frac{1}{2}s, 2\right]\right\}$. Then $C$ is compact and convex. Let $T_1$ and $T_2$ be self-mappings on $C$ defined by

$$T_1 x = \left(\frac{1}{2}s + \frac{1}{4}t, t\right), \quad T_2 x = \left(s, \frac{1}{4}s + \frac{1}{2}t\right)$$

for each $x = (s,t) \in C$ Then we can easily observe the following:

- $I - T_j$ is closed at zero for $j = 1, 2$,
- $\emptyset \neq \cap_{j=1}^2 F(T_j) \subset \cap_{j=1}^2 A(T_j)$,
- Neither $T_1$ nor $T_2$ satisfies the condition (2).

We need the following two lemmas. The first was proved in [12].

**Lemma 3.3** ([12])**.** *Let $E$ be a reflexive and strictly convex Banach space having a Fréchet differentiable norm, let $C$ be a nonempty closed convex subset of $E$, and let $\{S_i\}$ be a sequence of self-mappings on $C$. Let $\{x_i\}$ be a strongly convergent sequence in $C$ with a limit $x_0$ and $\{y_i\}$ be a sequence in $E$ defined by $y_i = J^{-1}(\alpha_i J x_i + (1-\alpha_i) J S_i x_i)$ for each $i \in \mathbb{N}$, where $\{\alpha_i\}$ is a convergent sequence in $[0,1]$ with a limit $\alpha_0 \in [0,1[$. Suppose that $V(x_0, y_i) \leq V(x_0, x_i)$ for all $i \in \mathbb{N}$ and that $\{J y_i\}$ converges weakly to $y_0^* \in E^*$. Then, $\{J x_i - J S_i x_i\}$ converges strongly to $0$. Moreover, if $E$ has the Kadec-Klee property, then $\{S_i x_i\}$ converges strongly to $x_0$.*

**Remark 3.4.** Although Kimura and Takahashi [12] assume that $\{y_i\} \subset C$, a careful examination of the proof in [12] shows that the argument for Lemma 3.3 still holds under the present assumptions.

**Lemma 3.5.** *Let $E$ be a reflexive, smooth, and strictly convex Banach space, let $C$ be a nonempty closed convex subset of $E$. Let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of self-mappings on $C$ such that $\emptyset \neq F := \cap_{\lambda \in \Lambda} F(T_\lambda) \subset A := \cap_{\lambda \in \Lambda} A(T_\lambda)$ and let $D$ be a nonempty closed convex subset of $C$ such that $F \subset D$. Let $\alpha \in [0,1]$ and $x \in C$. Let $y(\lambda) = J^{-1}(\alpha J x + (1-\alpha) J T_\lambda x)$ for each $\lambda \in \Lambda$. Let $M$ be a subset of $E$ defined by*

$$M := \left\{ z \in D : \sup_{\lambda \in \Lambda} V\left(z, y(\lambda)\right) \leq V(z, x) \right\}.$$

*Then, $M$ is closed and convex, and $F \subset M$.*

*Proof.* Let $\alpha \in [0,1]$ and $x \in C$. Put

$$M_E := \left\{ z \in E : \sup_{\lambda \in \Lambda} V\left(z, y(\lambda)\right) \leq V(z, x) \right\}.$$

From the convexity of the norm and the assumption of $T$, it follows that

$$\begin{aligned} \sup_{\lambda \in \Lambda} V\left(p, y(\lambda)\right) &\leq \sup_{\lambda \in \Lambda} \left\{ \alpha V(p, x) + (1-\alpha) V(p, T_\lambda x) \right\} \\ &\leq \sup_{\lambda \in \Lambda} \left\{ \alpha V(p, x) + (1-\alpha) V(p, x) \right\} \\ &= \alpha V(p, x) + (1-\alpha) V(p, x) = V(p, x) \end{aligned}$$

for each $p \in F$ and hence we have $F \subset M_E$. Thus, we have $F \subset M_E \cap D = M$. From the definition of $V$, we obtain that

$$\begin{aligned} M_E &= \left\{ z \in E : \sup_{\lambda \in \Lambda} V(z, y(\lambda)) \leq V(z, x) \right\} \\ &= \bigcap_{\lambda \in \Lambda} \{ z \in E : V(z, y(\lambda)) \leq V(z, x) \} \\ &= \bigcap_{\lambda \in \Lambda} \{ z \in E : 2\langle z, Jx - Jy(\lambda) \rangle + \|y(\lambda)\|^2 - \|x\|^2 \leq 0 \} \end{aligned}$$

and thus $M_E$ is closed and convex. Therefore, it follows that $M(= M_E \cap D)$ is also closed and convex. □

## 4. Shrinking projection method using metric projection

In this section, we study a shrinking projection method with extended allowable ranges that uses the metric projection. We first establish the following strong convergence theorem for finding a common fixed point of a family of nonlinear mappings in a Banach space.

**Theorem 4.1.** *Let $E$ be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let $C$ be a nonempty closed convex subset of $E$ and let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of self-mappings on $C$ such that $\emptyset \neq F := \cap_{\lambda \in \Lambda} F(T_\lambda) \subset A := \cap_{\lambda \in \Lambda} A(T_\lambda)$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $\liminf_{n\to\infty} \alpha_n < 1$ and let $\{\delta_n\}$ be a nonnegative real sequence such that $\lim_{n\to\infty} \delta_n = 0$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme: $C_1 = D_1 = C$, $x_1 \in D_1$, and*

$$\begin{aligned} & y_n(\lambda) = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T_\lambda x_n) \text{ for each } \lambda \in \Lambda, \\ & C_{n+1} = \left\{ z \in C_n : \sup_{\lambda \in \Lambda} V(z, y_n(\lambda)) \leq V(z, x_n) \right\}, \\ & D_{n+1} = \{ y \in C_n : \|u - y\| \leq \|u - P_{C_{n+1}} u\| + \delta_n \}, \\ & x_{n+1} \in D_{n+1} \end{aligned}$$

*for all $n \in \mathbb{N}$. If $I - T_\lambda$ is closed at zero for each $\lambda \in \Lambda$, then $\{x_n\}$ converges strongly to $P_F u$.*

*Proof.* We first show that, for each $n \in \mathbb{N}$, $C_n$ is closed and convex, satisfies $F \subset C_n$, and $D_n$ is nonempty. It is clear that $C_1$ and $D_1$ satisfy these condition. Thus, we can take $x_1 \in D_1 = C$. Suppose that, for some $k \in \mathbb{N}$, $C_k$ is closed and convex, satisfies $F \subset C_k$, and $D_k$ is nonempty. Using Lemma 3.5 as $x = x_k$, $D = C_k$, $M = C_{k+1}$, we see that $C_{k+1}$ is closed and convex, and satisfies $F \subset C_{k+1}$. Thus, $P_{C_{k+1}} u$ exists. Since $\delta_k \geq 0$, we have

$$\|u - P_{C_{k+1}} u\| \leq \|u - P_{C_{k+1}} u\| + \delta_k.$$

Because $P_{C_{k+1}} u \in C_{k+1} \subset C_k$, it follows that $P_{C_{k+1}} u \in D_{k+1}$. Hence $D_{k+1}$ is nonempty. By induction, we see that, for each $n \in \mathbb{N}$, $C_n$ is closed and convex, satisfies $F \subset C_n$, and $D_n$ is nonempty. Hence $\{x_n\}$ is well-defined.

Since $C_n$ includes $F \neq \emptyset$ for all $n \in \mathbb{N}$, $\{C_n\}$ is a sequence of nonempty closed convex subsets and, by definition, it is decreasing with respect to inclusion. Let $p_n := P_{C_n} u$ for all $n \in \mathbb{N}$ and put $C_0 := \bigcap_{n=1}^{\infty} C_n$. Then it follows that

$$\emptyset \neq F \subset C_0 = \text{M-}\lim_n C_n.$$

By Theorem 2.5, we obtain that $\{p_n\}$ converges strongly to $p_0 = P_{C_0} u$. Since $x_{n+1} \in D_{n+1} \subset C_n$ for each $n \in \mathbb{N}$, we obtain

$$\|u - p_n\| \leq \|u - x_{n+1}\| \leq \|u - p_{n+1}\| + \delta_n$$

for each $n \in \mathbb{N}$. Since $p_n \to p_0$ and $\delta_n \to 0$, we have

$$\begin{aligned}\|u - p_0\| = \lim_{n\to\infty} \|u - p_n\| &\leq \liminf_{n\to\infty} \|u - x_{n+1}\| \\ &\leq \limsup_{n\to\infty} \|u - x_{n+1}\| \\ &\leq \lim_{n\to\infty} (\|u - p_{n+1}\| + \delta_n) = \|u - p_0\|.\end{aligned}$$

Therefore, we have $\lim_{n\to\infty} \|u - x_n\| = \|u - p_0\|$. We also obtain that $\{x_n\}$ is bounded. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $x_0 \in C$. Since $x_{n_i} \in C_{n_i-1}$ for each $i \in \mathbb{N}\backslash\{1\}$, we see that

$$x_0 \in \text{ w-}\operatorname*{Ls}_n C_n = \text{ M-}\operatorname*{lim}_n C_n = C_0.$$

Thus, by the weak lower semicontinuity of the norm, we obtain

$$\|u - x_0\| \leq \liminf_{i\to\infty} \|u - x_{n_i}\| = \lim_{n\to\infty} \|u - x_n\| = \|u - p_0\|.$$

From the uniqueness of $P_{C_0}u$, we have $x_0 = p_0$. So, $\{x_n\}$ converges weakly to $p_0$ and hence $\{u-x_n\}$ converges weakly to $u - p_0$. Since $E$ has the Kadec-Klee property, we see that $\{u - x_n\}$ converges strongly to $u - p_0$. Thus, $\{x_n\}$ converges strongly to $p_0$.

Fix $\lambda \in \Lambda$ arbitrarily. Since $V(p_0, y_n(\lambda)) \leq V(p_0, x_n)$ for each $n \in \mathbb{N}$, $\{Jy_n(\lambda)\}$ is bounded. Hence, by the assumption that $\liminf_{n\to\infty} \alpha_n < 1$, we may take subsequences $\{\alpha_{n_i}\}$ of $\{\alpha_n\}$ and $\{Jy_{n_i}(\lambda)\}$ of $\{Jy_n(\lambda)\}$ such that $\lim_{i\to\infty} \alpha_{n_i} = \alpha_0$ with $0 \leq \alpha_0 < 1$ and $\{Jy_{n_i}(\lambda)\}$ converges weakly to a point $y_0^* \in E^*$. By Lemma 3.3, $\{T_\lambda x_{n_i}\}$ converges strongly to $p_0$. Therefore, $\{x_{n_i} - T_\lambda x_{n_i}\}$ converges strongly to 0. For each $\lambda \in \Lambda$, since $I - T_\lambda$ is closed at zero, it follows that $p_0 \in F(T_\lambda)$, and hence $p_0 \in F$. From Lemma 2.2, we obtain $P_{C_0}u = P_F u$. □

**Remark 4.2.** Note that in Theorem 4.1, $x_{n+1}$ is not necessarily contained in $C_{n+1}$, which is one of the features of our method. We also remark that the assumptions on $E$ are satisfied if $E$ is a Hilbert space. Moreover, we note that $E^*$ has a Fréchet differentiable norm if and only if $E$ is reflexive, strictly convex, and has the Kadec-Klee property. Therefore, the assumptions on $E$ are satisfied if $E$ is uniformly convex and has a Fréchet differentiable norm.

We present three results below that are derived from Theorem 4.1. Each result is closely related to the previous work. The first is the shrinking projection method proposed in [20]; see also [12].

**Theorem 4.3.** *Let $E$ be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let $C$ be a nonempty closed convex subset of $E$ and let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of self-mappings on $C$ such that $\emptyset \neq F := \cap_{\lambda\in\Lambda} F(T_\lambda) \subset A := \cap_{\lambda\in\Lambda} A(T_\lambda)$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $\liminf_{n\to\infty} \alpha_n < 1$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme: $C_1 = C$, $x_1 \in C_1$, and*

$$\begin{aligned}&y_n(\lambda) = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_\lambda x_n) \text{ for each } \lambda \in \Lambda, \\ &C_{n+1} = \left\{z \in C_n : \sup_{\lambda\in\Lambda} V(z, y_n(\lambda)) \leq V(z, x_n)\right\}, \\ &x_{n+1} = P_{C_{n+1}} u\end{aligned}$$

*for all $n \in \mathbb{N}$. If $I - T_\lambda$ is closed at zero for each $\lambda \in \Lambda$, then $\{x_n\}$ converges strongly to $P_F u$.*

*Proof.* In the Theorem 4.1, $P_{C_{n+1}}u$ is always chosen as $x_{n+1}$ from $D_{n+1}$. As a direct consequence of Theorem 4.1, the sequence $\{x_n\}$ converges strongly to $P_F u$. □

The following two results correspond to the methods studied in [9] and [21], respectively.

**Theorem 4.4.** *Let $E$ be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let $C$ be a nonempty closed convex subset of $E$ and let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of self-mappings on $C$ such that $\emptyset \neq F := \cap_{\lambda\in\Lambda} F(T_\lambda) \subset A := \cap_{\lambda\in\Lambda} A(T_\lambda)$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $\liminf_{n\to\infty} \alpha_n < 1$ and let $\{\delta_n\}$ be a nonnegative real sequence such that $\lim_{n\to\infty} \delta_n = 0$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme: $C_1 = K_1 = C$, $x_1 \in K_1$, and*

$$\begin{aligned}
&y_n(\lambda) = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_\lambda x_n) \ for\ each\ \lambda \in \Lambda,\\
&C_{n+1} = \left\{ z \in C_n : \sup_{\lambda\in\Lambda} V(z, y_n(\lambda)) \leq V(z, x_n) \right\},\\
&K_{n+1} = \{y \in C_{n+1} : \|u-y\| \leq \|u - P_{C_{n+1}}u\| + \delta_n\},\\
&x_{n+1} \in K_{n+1}
\end{aligned}$$

*for all $n \in \mathbb{N}$. If $I - T_\lambda$ is closed at zero for each $\lambda \in \Lambda$, then $\{x_n\}$ converges strongly to $P_F u$.*

*Proof.* Let $\{D_n\}$ be as in Theorem 4.1. Since $C_{n+1} \subset C_n$ for each $n \in \mathbb{N}$, it follows that

$$K_{n+1} \subset D_{n+1}$$

for each $n \in \mathbb{N}$. Because each $C_n$ is nonempty, closed, and convex for each $n \in \mathbb{N}$, we see that $P_{C_{n+1}}u \in K_{n+1}$. Hence, $K_{n+1}$ is nonempty for each $n \in \mathbb{N}$. In Theorem 4.1, by choosing $x_{n+1} \in K_{n+1} \subset D_{n+1}$, it follows as a direct consequence that $\{x_n\}$ converges strongly to $P_F u$. □

**Theorem 4.5.** *Let $E$ be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let $C$ be a nonempty closed convex subset of $E$ and let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of self-mappings on $C$ such that $\emptyset \neq F := \cap_{\lambda\in\Lambda} F(T_\lambda) \subset A := \cap_{\lambda\in\Lambda} A(T_\lambda)$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $\liminf_{n\to\infty} \alpha_n < 1$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme: $C_1 = L_1 = C$, $x_1 \in L_1$, and*

$$\begin{aligned}
&y_n(\lambda) = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_\lambda x_n) \ for\ each\ \lambda \in \Lambda,\\
&C_{n+1} = \left\{ z \in C_n : \sup_{\lambda\in\Lambda} V(z, y_n(\lambda)) \leq V(z, x_n) \right\},\\
&L_{n+1} = \{y \in C_n : \|u-y\| \leq \|u - P_{C_{n+1}}u\|\},\\
&x_{n+1} \in L_{n+1}
\end{aligned}$$

*for all $n \in \mathbb{N}$. If $I - T_\lambda$ is closed at zero for each $\lambda \in \Lambda$, then $\{x_n\}$ converges strongly to $P_F u$.*

*Proof.* Let $\{\delta_n\}$ and $\{D_n\}$ be as in Theorem 4.1. Since $\{\delta_n\}$ is nonnegative real sequence, it follows that

$$L_{n+1} \subset D_{n+1}$$

for each $n \in \mathbb{N}$. Because each $C_n$ is nonempty, closed, and convex, and $C_{n+1} \subset C_n$ for each $n \in \mathbb{N}$, we see that $P_{C_{n+1}}u \in L_{n+1}$. Therefore, $L_{n+1}$ is nonempty for each $n \in \mathbb{N}$. In Theorem 4.1, by taking $x_{n+1} \in L_{n+1} \subset D_{n+1}$, it follows that $\{x_n\}$ converges strongly to $P_F u$. □

## 5. Shrinking projection methods for generalized projections

This section treats the shrinking projection method with extended allowable ranges via generalized projection. We first present a strong convergence theorem for a common fixed point of a family of nonlinear mappings in Banach spaces.

**Theorem 5.1.** *Let $E$ be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let $C$ be a nonempty closed convex subset of $E$ and let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of self-mappings on $C$ such that $\emptyset \neq F := \cap_{\lambda\in\Lambda} F(T_\lambda) \subset A := \cap_{\lambda\in\Lambda} A(T_\lambda)$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $\liminf_{n\to\infty} \alpha_n < 1$ and let $\{\delta_n\}$ be a nonnegative real sequence such that $\lim_{n\to\infty} \delta_n = 0$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme: $C_1 = D_1 = C$, $x_1 \in D_1$, and*

$$\begin{aligned}
&y_n(\lambda) = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_\lambda x_n) \text{ for each } \lambda \in \Lambda,\\
&C_{n+1} = \left\{ z \in C_n : \sup_{\lambda\in\Lambda} V\left(z, y_n(\lambda)\right) \leq V(z, x_n) \right\},\\
&D_{n+1} = \{y \in C_n : V(y,u) \leq V(\mathit{\Pi}_{C_{n+1}}u, u) + \delta_n\},\\
&x_{n+1} \in D_{n+1}
\end{aligned}$$

*for all $n \in \mathbb{N}$. If $I - T_\lambda$ is closed at zero for each $\lambda \in \Lambda$, then $\{x_n\}$ converges strongly to $\mathit{\Pi}_F u$.*

*Proof.* We first show that, for each $n \in \mathbb{N}$, $C_n$ is closed and convex, satisfies $F \subset C_n$, and $D_n$ is nonempty. It is clear that $C_1$ and $D_1$ satisfy these condition. Thus, we can take $x_1 \in D_1 = C$. Suppose that, for some $k \in \mathbb{N}$, $C_k$ is closed and convex, satisfies $F \subset C_k$, and $D_k$ is nonempty. Using Lemma 3.5 as $x = x_k$, $D = C_k$, $M = C_{k+1}$, we see that $C_{k+1}$ is closed and convex, and satisfies $F \subset C_{k+1}$. Thus, $\mathit{\Pi}_{C_{k+1}}u$ exists. Since $\delta_k \geq 0$, we have

$$V(\mathit{\Pi}_{C_{k+1}}u, u) \leq V(\mathit{\Pi}_{C_{n+1}}u, u) + \delta_k.$$

Because $\mathit{\Pi}_{C_{k+1}}u \in C_{k+1} \subset C_k$, it follows that $\mathit{\Pi}_{C_{k+1}}u \in D_{k+1}$. Hence $D_{k+1}$ is nonempty. By induction, we see that, for each $n \in \mathbb{N}$, $C_n$ is closed and convex, satisfies $F \subset C_n$, and $D_n$ is nonempty. Hence $\{x_n\}$ is well-defined.

Since $C_n$ includes $F \neq \emptyset$ for all $n \in \mathbb{N}$, $\{C_n\}$ is a sequence of nonempty closed convex subsets and, by definition, it is decreasing with respect to inclusion. Let $\pi_n := \mathit{\Pi}_{C_n}u$ for all $n \in \mathbb{N}$ and put $C_0 := \bigcap_{n=1}^{\infty} C_n$. Then it follows that

$$\emptyset \neq F \subset C_0 = \operatorname*{M-lim}_n C_n.$$

By Theorem 2.6, we obtain that $\{\pi_n\}$ converges strongly to $\pi_0 = \mathit{\Pi}_{C_0}u$. Since $x_{n+1} \in D_{n+1} \subset C_n$ for each $n \in \mathbb{N}$, we obtain

$$V(\pi_n, u) \leq V(x_{n+1}, u) \leq V(\pi_{n+1}, u) + \delta_n$$

for each $n \in \mathbb{N}$. Since $\pi_n \to \pi_0$ and $\delta_n \to 0$, we have

$$\begin{aligned}
V(\pi_0, u) = \lim_{n\to\infty} V(\pi_n, u) &\leq \liminf_{n\to\infty} V(x_{n+1}, u)\\
&\leq \limsup_{n\to\infty} V(x_{n+1}, u)\\
&\leq \lim_{n\to\infty} \{V(\pi_{n+1}, u) + \delta_n\} = V(\pi_0, u).
\end{aligned}$$

Therefore, we get $\lim_{n\to\infty} V(x_n, u) = V(\pi_0, u)$. We also obtain that $\{x_n\}$ is bounded. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $x_0 \in C$. Since $x_{n_i} \in C_{n_i - 1}$ for each $i \in \mathbb{N}\backslash\{1\}$, we see that

$$x_0 \in \operatorname*{w-Ls}_n C_n = \operatorname*{M-lim}_n C_n = C_0.$$

Thus, by the weak lower semicontinuity of the norm, we obtain

$$\begin{aligned}
V(x_0, u) &= \|x_0\|^2 - 2\langle x_0, Ju\rangle + \|u\|^2\\
&\leq \liminf_{i\to\infty} \left\{\|x_{n_i}\|^2 - 2\langle x_{n_i}, Ju\rangle + \|u\|^2\right\}\\
&= \liminf_{i\to\infty} V(x_{n_i}, u) = \lim_{n\to\infty} V(x_n, u) = V(\pi_0, u).
\end{aligned}$$

From the uniqueness of $\Pi_{C_0}u$, we have $x_0 = \pi_0$. So, $\{x_n\}$ converges weakly to $\pi_0$. Using the properties of $V$, we have

$$\left|\|x_n\| - \|\pi_0\|\right|^2 \le V(x_n, \pi_0) = V(x_n, u) - V(\pi_0, u) - 2\langle x_n - \pi_0, J\pi_0 - Ju\rangle.$$

Therefore, it follows from $V(x_n, u) \to V(\pi_0, u)$ and $x_n \rightharpoonup \pi_0$ that $\{\|x_n\|\}$ converges to $\|\pi_0\|$. Since $E$ has the Kadec-Klee property, $\{x_n\}$ converges strongly to $\pi_0$.

Fix $\lambda \in \Lambda$ arbitrarily. It follows from $V(\pi_0, y_n(\lambda)) \le V(\pi_0, x_n)$ for each $n \in \mathbb{N}$ that $\{Jy_n(\lambda)\}$ is bounded. Hence, by the assumption that $\liminf_{n\to\infty} \alpha_n < 1$, we may take subsequences $\{\alpha_{n_i}\}$ of $\{\alpha_n\}$ and $\{Jy_{n_i}(\lambda)\}$ of $\{Jy_n(\lambda)\}$ such that $\lim_{i\to\infty} \alpha_{n_i} = \alpha_0$ with $0 \le \alpha_0 < 1$ and $\{Jy_{n_i}(\lambda)\}$ converges weakly to a point $y_0^* \in E^*$. By Lemma 3.3, $\{T_\lambda x_{n_i}\}$ converges strongly to $\pi_0$. Therefore, $\{x_{n_i} - T_\lambda x_{n_i}\}$ converges strongly to 0. For each $\lambda \in \Lambda$, since $I - T_\lambda$ is closed at zero, it follows that $\pi_0 \in F(T_\lambda)$ and hence $\pi_0 \in F$. From Lemma 2.4, we obtain $\Pi_{C_0}u = \Pi_F u$. □

We present three results below that are derived from Theorem 5.1. Each result is closely related to the previous work. The first is the shrinking projection method proposed in [20]; see also [12].

**Theorem 5.2.** *Let $E$ be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let $C$ be a nonempty closed convex subset of $E$ and let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of self-mappings on $C$ such that $\emptyset \ne F := \cap_{\lambda\in\Lambda} F(T_\lambda) \subset A := \cap_{\lambda\in\Lambda} A(T_\lambda)$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $\liminf_{n\to\infty} \alpha_n < 1$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme: $C_1 = C$, $x_1 \in C_1$, and*

$$\begin{aligned}
&y_n(\lambda) = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_\lambda x_n) \ for\ each\ \lambda \in \Lambda,\\
&C_{n+1} = \left\{z \in C_n : \sup_{\lambda\in\Lambda} V(z, y_n(\lambda)) \le V(z, x_n)\right\},\\
&x_{n+1} = \Pi_{C_{n+1}}u
\end{aligned}$$

*for all $n \in \mathbb{N}$. If $I - T_\lambda$ is closed at zero for each $\lambda \in \Lambda$, then $\{x_n\}$ converges strongly to $\Pi_F u$.*

*Proof.* In the Theorem 5.1, $\Pi_{C_{n+1}}u$ is always chosen as $x_{n+1}$ from $D_{n+1}$. As a direct consequence of Theorem 5.1, the sequence $\{x_n\}$ converges strongly to $\Pi_F u$. □

The following two results correspond to the methods studied in [9] and [21], respectively.

**Theorem 5.3.** *Let $E$ be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let $C$ be a nonempty closed convex subset of $E$ and let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of self-mappings on $C$ such that $\emptyset \ne F := \cap_{\lambda\in\Lambda} F(T_\lambda) \subset A := \cap_{\lambda\in\Lambda} A(T_\lambda)$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $\liminf_{n\to\infty} \alpha_n < 1$ and let $\{\delta_n\}$ be a nonnegative real sequence such that $\lim_{n\to\infty} \delta_n = 0$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme: $C_1 = K_1 = C$, $x_1 \in K_1$, and*

$$\begin{aligned}
&y_n(\lambda) = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_\lambda x_n) \ for\ each\ \lambda \in \Lambda,\\
&C_{n+1} = \left\{z \in C_n : \sup_{\lambda\in\Lambda} V(z, y_n(\lambda)) \le V(z, x_n)\right\},\\
&K_{n+1} = \{y \in C_{n+1} : V(y, u) \le V(\Pi_{C_{n+1}}u, u) + \delta_n\},\\
&x_{n+1} \in K_{n+1}
\end{aligned}$$

*for all $n \in \mathbb{N}$. If $I - T_\lambda$ is closed at zero for each $\lambda \in \Lambda$, then $\{x_n\}$ converges strongly to $\Pi_F u$.*

*Proof.* Let $\{D_n\}$ be as in Theorem 5.1. Since $C_{n+1} \subset C_n$ for each $n \in \mathbb{N}$, it follows that

$$K_{n+1} \subset D_{n+1}$$

for each $n \in \mathbb{N}$. Because each $C_n$ is nonempty, closed, and convex for each $n \in \mathbb{N}$, we see that $\Pi_{C_{n+1}}u \in K_{n+1}$. Hence, $K_{n+1}$ is nonempty for each $n \in \mathbb{N}$. In Theorem 5.1, by choosing $x_{n+1} \in K_{n+1} \subset D_{n+1}$, it follows as a direct consequence that $\{x_n\}$ converges strongly to $\Pi_F u$. □

**Theorem 5.4.** *Let $E$ be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property. Let $C$ be a nonempty closed convex subset of $E$ and let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of self-mappings on $C$ such that $\emptyset \neq F := \cap_{\lambda\in\Lambda} F(T_\lambda) \subset A := \cap_{\lambda\in\Lambda} A(T_\lambda)$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $\liminf_{n\to\infty} \alpha_n < 1$. For a given point $u \in E$, generate a sequence $\{x_n\}$ by the following iterative scheme: $C_1 = L_1 = C$, $x_1 \in L_1$, and*

$$
\begin{aligned}
&y_n(\lambda) = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JT_\lambda x_n) \ \textit{for each}\ \lambda \in \Lambda,\\
&C_{n+1} = \left\{ z \in C_n : \sup_{\lambda\in\Lambda} V(z, y_n(\lambda)) \leq V(z, x_n) \right\},\\
&L_{n+1} = \{ y \in C_n : V(y,u) \leq V(\mathit{\Pi}_{C_{n+1}} u, u) \},\\
&x_{n+1} \in L_{n+1}
\end{aligned}
$$

*for all $n \in \mathbb{N}$. If $I - T_\lambda$ is closed at zero for each $\lambda \in \Lambda$, then $\{x_n\}$ converges strongly to $\mathit{\Pi}_F u$.*

*Proof.* Let $\{\delta_n\}$ and $\{D_n\}$ be as in Theorem 5.1. Since $\{\delta_n\}$ is nonnegative real sequence, it follows that

$$L_{n+1} \subset D_{n+1}$$

for each $n \in \mathbb{N}$. Because each $C_n$ is nonempty, closed, and convex, and $C_{n+1} \subset C_n$ for each $n \in \mathbb{N}$, we see that $\mathit{\Pi}_{C_{n+1}} u \in L_{n+1}$. Therefore, $L_{n+1}$ is nonempty for each $n \in \mathbb{N}$. In Theorem 5.1, by taking $x_{n+1} \in L_{n+1} \subset D_{n+1}$, it follows that $\{x_n\}$ converges strongly to $\mathit{\Pi}_F u$. □

## 6. Deduced results

In this section, we present results deduced from the main theorems. We first state the common assumptions used throughout this section.

Let $E$ be a reflexive and strictly convex Banach space having a Fréchet differentiable norm and the Kadec-Klee property, let $C$ be a nonempty closed convex subset of $E$ and let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of self-mappings on $C$ such that $F := \cap_{\lambda\in\Lambda} F(T_\lambda) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ and let $\{\delta_n\}$ be a nonnegative real sequence.

We consider the iterative schemes given in Theorems 4.1 and 5.1. First, let $\{x_n\}$ be the sequence generated by $u \in E$, $C_1 = D_1 = C$, $x_1 \in D_1$, and

$$
\begin{cases}
y_n(\lambda) = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JT_\lambda x_n) \ \textit{for each}\ \lambda \in \Lambda,\\
C_{n+1} = \left\{ z \in C_n : \sup_{\lambda\in\Lambda} V(z, y_n(\lambda)) \leq V(z, x_n) \right\},\\
D_{n+1} = \{ y \in C_n : \|u - y\| \leq \|u - P_{C_{n+1}} u\| + \delta_n \},\\
x_{n+1} \in D_{n+1}
\end{cases}
\tag{3}
$$

for each $n \in \mathbb{N}$. Next, let $\{x_n\}$ be the sequence generated by $u \in E$, $C_1 = D_1 = C$, $x_1 \in D_1$, and

$$
\begin{cases}
y_n(\lambda) = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JT_\lambda x_n) \ \textit{for each}\ \lambda \in \Lambda,\\
C_{n+1} = \left\{ z \in C_n : \sup_{\lambda\in\Lambda} V(z, y_n(\lambda)) \leq V(z, x_n) \right\},\\
D_{n+1} = \{ y \in C_n : V(y,u) \leq V(\mathit{\Pi}_{C_{n+1}} u, u) + \delta_n \},\\
x_{n+1} \in D_{n+1}
\end{cases}
\tag{4}
$$

for each $n \in \mathbb{N}$.

Finally, recalling the discussion in Section 3, we obtain the following results as direct consequences of Theorems 4.1 and 5.1.

**Theorem 6.1.** *Let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of relatively nonexpansive self-mappings on $C$ such that $F := \cap_{\lambda\in\Lambda} F(T_\lambda) \neq \emptyset$. Suppose that $\{x_n\}$ is the sequence generated by (3), $\liminf_{n\to\infty} \alpha_n < 1$ and $\lim_{n\to\infty} \delta_n = 0$. Then $\{x_n\}$ converges strongly to $P_F u$.*

**Theorem 6.2.** *Let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of relatively nonexpansive self-mappings on $C$ such that $F := \cap_{\lambda\in\Lambda} F(T_\lambda) \neq \emptyset$. Suppose that $\{x_n\}$ is the sequence generated by (4), $\liminf_{n\to\infty} \alpha_n < 1$ and $\lim_{n\to\infty} \delta_n = 0$. Then $\{x_n\}$ converges strongly to $\Pi_F u$.*

**Theorem 6.3.** *Let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of nonspreading self-mappings on $C$ such that $F := \cap_{\lambda\in\Lambda} F(T_\lambda) \neq \emptyset$. Suppose that $\{x_n\}$ is the sequence generated by (3), $\liminf_{n\to\infty} \alpha_n < 1$ and $\lim_{n\to\infty} \delta_n = 0$. Then $\{x_n\}$ converges strongly to $P_F u$.*

**Theorem 6.4.** *Let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of nonspreading self-mappings on $C$ such that $F := \cap_{\lambda\in\Lambda} F(T_\lambda) \neq \emptyset$. Suppose that $\{x_n\}$ is the sequence generated by (4), $\liminf_{n\to\infty} \alpha_n < 1$ and $\lim_{n\to\infty} \delta_n = 0$. Then $\{x_n\}$ converges strongly to $\Pi_F u$.*

**Theorem 6.5.** *Let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of mappings of type (Q) from $C$ into itself such that $F := \cap_{\lambda\in\Lambda} F(T_\lambda) \neq \emptyset$. Suppose that $\{x_n\}$ is the sequence generated by (3), $\liminf_{n\to\infty} \alpha_n < 1$ and $\lim_{n\to\infty} \delta_n = 0$. Then $\{x_n\}$ converges strongly to $P_F u$.*

**Theorem 6.6.** *Let $\{T_\lambda : \lambda \in \Lambda\}$ be a family of mappings of type (Q) from $C$ into itself such that $F := \cap_{\lambda\in\Lambda} F(T_\lambda) \neq \emptyset$. Suppose that $\{x_n\}$ is the sequence generated by (4), $\liminf_{n\to\infty} \alpha_n < 1$ and $\lim_{n\to\infty} \delta_n = 0$. Then $\{x_n\}$ converges strongly to $\Pi_F u$.*

## 7. Conclusion

In this paper, we proposed a new shrinking projection method for finding common fixed points of a family of nonlinear mappings in Banach spaces, utilizing two types of nonlinear projections (Theorems 4.1 and 5.1). Our approach allows for errors in the nonlinear projection values at each iteration, which may lie either inside or outside the target set. This method integrates key ideas from Kimura [9] and Takeuchi [21].

Kimura's method [9], which was introduced in a geodesic space, deals with cases where the error sequence does not necessarily converge to zero. Later, Kimura [10] studied a related problem in a Banach space. Our work is also studied in a Banach space, but our method is designed for situations in which the error sequence does converge to zero. Consequently, our approach requires weaker assumptions on the underlying space (see Theorems 4.4 and 5.3).

Takeuchi's method [21] requires that each new point in the sequence differs from the previous one, which leads to two possible cases for the procedure: either stopping or continuing. By removing this requirement, our method considers only the case where the procedure continues (Theorems 4.5 and 5.4).

Our results are not a full extension of Kimura [9] and Takeuchi [21], but rather a partial extension of both. Unlike both works, our approach is motivated by the goal of weakening the assumptions of the theorem and simplifying the theorem's conclusion.

## Acknowledgments

The author is grateful to the referees for their helpful reviews and suggestions. This work was supported by JSPS KAKENHI Grant Numbers 23K25512 and 24K06807.

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