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On fixed point results in S-metric spaces via b-metric spaces

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Abstract

In this paper, we establish several fixed point results for generalized weak contractive mappings of Type I and Type II in b-metric spaces. Using the relation between b-metric and S-metric spaces, we have obtained some results in in S-metric spaces. Some examples are provided to support the validity of the obtained result. As an application, one of our result is used to prove that an integral equation has a unique solution.

Keywords: fixed point, S-metric space, b-metric space, G-metric space, D-metric space, weak contraction
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1. Introduction

Metric spaces play an important role in fixed point theory and many areas of mathematics. Because of their usefulness, several generalizations of metric spaces have been introduced. One important generalization is the b-metric spaces which is defined by Bakhtin [1] and Czerwik [5]. Mitrović [15] provided an alternative proof of Czerwik's fixed point theorem in b-metric spaces, improved its recent version and established analogues of Reich and Kannan contraction principles. Many researchers have studied the properties of b-metric spaces and developed generalizations of the classical Banach fixed point theorem in b-metric spaces; see , e.g.

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[2, 3, 12, 19]. On the other hand, Gähler [10] introduced the concept of a 2-metric as an extension of notion of a metric. Further, Dhage [8] generalized the concept of 2-metric by introducing D-metric spaces. Building upon these developments, Sedghi et al. [21] proposed the concept of D^*-metric spaces as a refinement of D-metric spaces and established a common fixed point theorem for a class of mappings defined on complete D^*-metric spaces. Mustafa and Sims [18] introduced G-metric spaces, a generalized form of classical metric spaces with the aim of developing a new fixed point theory for various types of mappings within this framework. Subsequently, several authors have established fixed point theorems in this space; see details in [4, 18, 25]. Sedghi et al. [22] proposed the concept of S-metric spaces and explored several of their key properties. In addition, they proved a fixed point theorem for self-mappings on complete S-metric spaces. Since then, many researchers have studied S-metric spaces and explored fixed-point theorems from different view, see, e.g.[9, 11, 14, 16, 17, 18, 20, 20, 23, 26, 30]. A number of fixed point theorems have appeared in connection with different generalizations of metric spaces; see, e. g. [6, 7, 13, 24, 27, 28, 29, 31].

In this paper, we present some fixed point results for for generalized weak contractions of Type I and Type II in b-metric spaces. One of our result generalize the result of Jovanovic et al. [11] in b-metric spaces. We also obtain similar type of results in S-metric spaces, along with illustrative examples and application to the solution of an integral equation.

2. Preliminaries

We begin by recalling the essential definitions that will be used throughout this paper.

Bakhtin[1] and Czerwik [5] introduced the concept of b-metric space as follows:

Definition 2.1. [1, 5] Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a mapping satisfying following properties:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) there exists a real number $b \geq 1$ such that

$$d(x, y) \leq b\big(d(x, z) + d(z, y)\big)$$

for all $x, y, z \in X$.

Then d is called a b-metric on X and the ordered pair (X, d) is called b-metric space with coefficient b.

Example 2.2. [2] Let $X = \mathbb{R}$. Define $d : X \times X \to [0, +\infty)$ by

$$d(x, y) =| x - y |^p, \quad p > 1.$$

Then d is a b-metric on X with coefficient $b = 2^{p-1}$.

Definition 2.3. [2] Let (X, d) be a b-metric space and $\{x_n\}$ be a sequence in X. Then

(i) A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that for all $m, n \geq n(\epsilon)$,

$$d(x_n, x_m) < \epsilon.$$

(ii) A sequence $\{x_n\}$ in X is said to be convergent if and only if there exists $x \in X$ such that for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that for all $n \geq n(\epsilon)$,

$$d(x_n, x) < \epsilon.$$

In this case, we write $\lim\limits_{n\to\infty} x_n = x$.

(iii) The b-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

Sedghi et al. [22] proposed the concept of S-metric spaces and it is defined as follows:

Definition 2.4. [22] Let X be a non empty set and $S : X \times X \times X \to [0, +\infty)$ be a mapping satisfying following properties:

(i) $S(x, y, z) = 0$ if and only if $x = y = z$;

(ii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ for all $a, x, y, z \in X$ (rectangle inequality).

Then (X, S) is called a S-metric space.

Example 2.5. [22] Let $X = \mathbb{R}$ and define $S : X \times X \times X \to [0, +\infty)$ by

$$S(x, y, z) = |x - y| + |y - z| + |z - x| \ \text{ for all } x, y, z \in X.$$

Then (X, S) is an S-metric space.

Example 2.6. [22] Let $X = \mathbb{R}^n$ and let $\|\cdot\|$ be a norm on X. Define $S : X \times X \times X \to [0, +\infty)$ by

$$S(x, y, z) = \|x - z\| + \|y - z\| \ \text{ for all } x, y, z \in X.$$

Then (X, S) is an S-metric space.

Example 2.7. [22] Let $X = \mathbb{R}^n$ and let $\|\cdot\|$ be a norm on X. Define $S : X \times X \times X \to [0, +\infty)$ as

$$S(x, y, z) = \|y + z - 2x\| + \|y - z\| \ \text{ for all } x, y, z \in X.$$

Then (X, S) is an S-metric space.

Definition 2.8. [22] Let (X, S) be an S-metric space and $\{x_n\}$ be a sequence in X. Then

(a) A sequence $\{x_n\}$ converges to x if and only if for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$S(x_n, x_n, x) < \epsilon$$

and we denote this by $\lim_{n\to\infty} x_n = x$.

(b) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$S(x_n, x_n, x_m) < \epsilon \ \text{ for all } n, m \geq n_0.$$

(c) The S-metric space (X, S) is said to be complete if every Cauchy sequence in X is converges to a point in X.

Sedghi and Dung [23] established the following proposition, which shows that fixed point theorems in S-metric spaces are equivalent to those in b-metric spaces.

Proposition 2.9. *[23] Let (X, S) be an S-metric space and define*

$$d(x, y) = S(x, x, y)$$

for all $x, y \in X$. Then we have

(i) d is a b-metric on X;

(ii) $x_n \to x$ in (X, S) if and only if $x_n \to x$ in (X, d);

(iii) $\{x_n\}$ is a Cauchy sequence in (X, S) if and only if $\{x_n\}$ is a Cauchy sequence in (X, d).

Definition 2.10. Let (X, d) be a b-metric space and T be a self-mapping on X. Then T is said to satisfy a generalized weak contraction of Type I, if

$$\begin{aligned}\psi\big(d(Tx,Ty)\big) \leq\ & \psi\big(\alpha_1 d(x,y) + \alpha_2 d(x,Tx) + \alpha_3 d(y,Ty)\big)\\ & - \phi\big(\alpha_1 d(x,y) + \alpha_2 d(x,Tx) + \alpha_3 d(y,Ty)\big)\end{aligned}$$

for all $x, y \in X$, where $\phi, \psi : [0, +\infty) \to [0, +\infty)$ are non-decreasing functions satisfying:

(a) $\phi(t) = 0$ if and only if $t = 0$,

(b) $\psi(t) = 0$ if and only if $t = 0$, and $\psi(t_1) \leq \psi(t_2)$ implies $t_1 \leq t_2$, for all $t_1, t_2 \in [0, +\infty)$,

and $\alpha_i \geq 0$ $(i = 1, 2, 3)$ are constants.

Definition 2.11. Let (X, d) be a b-metric space and T be a self-mapping on X. Then T is said to satisfy a generalized weak contraction of Type II if

$$\begin{aligned}\psi\big(d(Tx,Ty)\big) \leq\ & \psi\big(\alpha_1 d(x,y) + \alpha_2 d(x,Ty) + \alpha_3 d(y,Tx)\big)\\ & - \phi\big(\alpha_1 d(x,y) + \alpha_2 d(x,Ty) + \alpha_3 d(y,Tx)\big)\end{aligned}$$

for all $x, y \in X$, where $\phi, \psi : [0, +\infty) \to [0, +\infty)$ are non-decreasing functions satisfying:

(a) $\phi(t) = 0$ if and only if $t = 0$,

(b) $\psi(t) = 0$ if and only if $t = 0$, and $\psi(t_1) \leq \psi(t_2)$ implies $t_1 \leq t_2$, for all $t_1, t_2 \in [0, \infty)$,

and $\alpha_i \geq 0$ $(i = 1, 2, 3)$ are constants.

3. Main Results

Now, we state our first result.

Theorem 3.1. *Let (X, d) be a complete b-metric space with constant $b \geq 1$ and let $T : X \to X$ be a self-mapping satisfying a generalized weak contraction of Type I. Suppose that the constants $\alpha_i \geq 0$ $(i = 1, 2, 3)$ are such that $b(\alpha_1 + \alpha_2) + \alpha_3 < 1$. Then T admits a unique fixed point in X.*

Proof. Let $x_0 \in X$ be an arbitrary point. Since $T : X \to X$, we can construct a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n,\ n = 0, 1, 2, \cdots$. Consider

$$\begin{aligned}\psi\big(d(x_n,x_{n+1})\big) &= \psi\big(d(Tx_{n-1},Tx_n)\big)\\ &\leq \psi\big(\alpha_1 d(x_{n-1},x_n) + \alpha_2 d(x_{n-1},Tx_{n-1}) + \alpha_3 d(x_n,Tx_n)\big)\\ &\quad - \phi\big(\alpha_1 d(x_{n-1},x_n) + \alpha_2 d(x_{n-1},Tx_{n-1}) + \alpha_3 d(x_n,Tx_n)\big)\\ &= \psi\big(\alpha_1 d(x_{n-1},x_n) + \alpha_2 d(x_{n-1},x_n) + \alpha_3 d(x_n,x_{n+1})\big)\\ &\quad - \phi\big(\alpha_1 d(x_{n-1},x_n) + \alpha_2 d(x_{n-1},x_n) + \alpha_3 d(x_n,x_{n+1})\big).\end{aligned}$$

Since $\phi(t) \geq 0$ for all $t \geq 0$, it follows that

$$\psi\big(d(x_n,x_{n+1})\big) \leq \psi\big(\alpha_1 d(x_{n-1},x_n) + \alpha_2 d(x_{n-1},x_n) + \alpha_3 d(x_n,x_{n+1})\big).$$

Since $\psi(t_1) \leq \psi(t_2)$ implies $t_1 \leq t_2$, for all $t_1, t_2 \in [0, +\infty)$, it gives that

$$d(x_n,x_{n+1}) \leq \alpha_1 d(x_{n-1},x_n) + \alpha_2 d(x_{n-1},x_n) + \alpha_3 d(x_n,x_{n+1}).$$

It implies that

$$d(x_n,x_{n+1}) \leq \left(\frac{\alpha_1+\alpha_2}{1-\alpha_3}\right) d(x_{n-1},x_n).$$

Put $\lambda = \dfrac{\alpha_1 + \alpha_2}{1 - \alpha_3} < 1$, we obtain

$$d(x_n, x_{n+1}) \le \lambda d(x_{n-1}, x_n). \tag{1}$$

Similarly, we can show that

$$d(x_{n-1}, x_n) \le \lambda d(x_{n-2}, x_{n-1}). \tag{2}$$

Using (1) and (2), we get

$$d(x_n, x_{n+1}) \le \lambda^2 d(x_{n-2}, x_{n-1}).$$

By extending this argument through induction, we conclude

$$d(x_n, x_{n+1}) \le \lambda^n d(x_0, x_1).$$

Denote $d_n = d(x_n, x_{n+1})$. Then, we have

$$d_n \le \lambda^n d_0 \text{ for all } n \in \mathbb{N}. \tag{3}$$

Let $m, n \in \mathbb{N}$ with $m > n$. Then we have

$$\begin{aligned}
d(x_n, x_m) &\le b\big(d(x_n, x_{n+1}) + d(x_{n+1}, x_m)\big) \\
&= b\ d(x_n, x_{n+1}) + b\ d(x_{n+1}, x_m) \\
&\le b\ d(x_n, x_{n+1}) + b\Big\{b\big[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)\big]\Big\} \\
&= b\ d(x_n, x_{n+1}) + b^2 d(x_{n+1}, x_{n+2}) + b^2 d(x_{n+2}, x_m) \\
&\le b\ d(x_n, x_{n+1}) + b^2 d(x_{n+1}, x_{n+2}) + \cdots + b^{m-n} d(x_{m-1}, x_m) \\
&= bd_n + b^2 d_{n+1} + b^3 d_{n+3} + \cdots b^{m-n} d_{m-1} \\
&\le bd_n + b^2 d_{n+1} + b^3 d_{n+3} + \cdots .
\end{aligned}$$

Using (3), we obtain

$$\begin{aligned}
d(x_n, x_m) &\le b(\lambda^n d_0) + b^2(\lambda^{n+1} d_0) + b^3(\lambda^{n+2} d_0) + \cdots \\
&= b\Big(1 + (\lambda b) + (\lambda b)^2 + (\lambda b)^3 + \cdots\Big)\lambda^n d_0 \\
&= b\left(\frac{1}{1 - \lambda b}\right)\lambda^n d_0.
\end{aligned}$$

As $k < 1$, taking the limit as $n \to \infty$, we get

$$\lim_{n,m \to \infty} d(x_n, x_m) = 0.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that $x_n \to u$. We claim that u is a fixed point of T. For this, consider

$$\begin{aligned}
\psi\big(d(x_n, Tu)\big) &= \psi\big(d(Tx_{n-1}, Tu)\big) \\
&\le \psi\big(\alpha_1 d(x_{n-1}, u) + \alpha_2 d(x_{n-1}, Tx_{n-1}) + \alpha_3 d(u, Tu)\big) \\
&\quad - \phi\big(\alpha_1 d(x_{n-1}, u) + \alpha_2 d(x_{n-1}, Tx_{n-1}) + \alpha_3 d(u, Tu)\big) \\
&\le \psi\big(\alpha_1 d(x_{n-1}, u) + \alpha_2 d(x_{n-1}, x_n) + \alpha_3 d(u, Tu)\big) \\
&\quad - \phi\big(\alpha_1 d(x_{n-1}, u) + \alpha_2 d(x_{n-1}, x_n) + \alpha_3 d(u, Tu)\big).
\end{aligned}$$

By the properties of ψ and ϕ, we obtain

$$d(x_n, Tu) \le \alpha_1 d(x_{n-1}, u) + \alpha_2 d(x_{n-1}, x_n) + \alpha_3 d(u, Tu).$$

Letting $n \to \infty$, we get

$$d(u, Tu) \leq \alpha_1 d(u, u) + \alpha_2 d(u, u) + \alpha_3 d(u, Tu).$$

This implies that $(1 - \alpha_3)d(u, Tu) \leq 0$. Since, $0 \leq \alpha_3 < 1$, it follows that $1 - \alpha_3 > 0$. Hence, we conclude that $d(u, Tu) \leq 0$. On the other hand, $d(u, Tu) \geq 0$. Combining these inequalities yields $d(u, Tu) = 0$. Therefore, $Tu = u$, which shows that u is a fixed point of T. Now, to prove uniqueness of a fixed point, suppose that there exists $v \in X$ with $v \neq u$ such that $Tv = v$. Consider

$$\begin{aligned}
\psi\big(d(u, v)\big) &= \psi\big(d(Tu, Tv)\big) \\
&\leq \psi\big(\alpha_1 d(u, v) + \alpha_2 d(u, Tu) + \alpha_3 d(v, Tv)\big) - \phi\big(\alpha_1 d(u, v) + \alpha_2 d(u, Tu) + \alpha_3 d(v, Tv)\big) \\
&= \psi\big(\alpha_1 d(u, v) + \alpha_2 d(u, u) + \alpha_3 d(v, v)\big) - \phi\big(\alpha_1 d(u, v) + \alpha_2 d(u, u) + \alpha_3 d(v, v)\big) \\
&= \psi(0) - \phi(0) \\
&= 0.
\end{aligned}$$

That is, $\psi\,(d(u, v)) \leq 0$. However, $\psi\,(d(u, v)) \geq 0$. Consequently, $\psi\,(d(u, v)) = 0$, which implies $d(u, v) = 0$. Hence, $u = v$. This establishes the uniqueness of a fixed point. □

Remark 3.2. If we define the functions $\psi, \phi : [0, +\infty) \to [0, +\infty)$ by $\psi(t) = t$ and $\phi(t) = 0$ for all $t \in [0, +\infty)$ in Theorem 3.1 and take constants $\alpha_2 = \alpha_3 = 0$, then Theorem 3.1 reduces to the Theorem 3.3 in [11].

Corollary 3.3. *Let (X, S) be a complete S-metric space and $T : X \to X$ be a mapping satisfying*

$$\begin{aligned}
\psi\,(S(Tx, Tx, Ty)) \leq\ & \psi\,(\alpha_1 S(x, x, y) + \alpha_2 S(x, x, Tx) + \alpha_3 S(y, y, Ty)) \\
& - \phi\,(\alpha_1 S(x, x, y) + \alpha_2 S(x, x, Tx) + \alpha_3 S(y, y, Ty))
\end{aligned}$$

for all $x, y, z \in X$, where $\phi, \psi : [0 + \infty) \to [0, +\infty)$ be non-decreasing functions such that

(a) $\phi(t) = 0$ if and only if $t = 0$,

(b) $\psi(t) = 0$ if and only if $t = 0$ and $\psi(t_1) \leq \psi(t_2)$ implies that $t_1 \leq t_2$ for all $t_1, t_2 \in [0, +\infty)$

and $\alpha_i \geq 0$ $(i = 1, 2, 3, 4)$ be such that $\alpha_1 + \alpha_2 + \alpha_3 < 1$. Then T admits a unique fixed point in X.

Proof. Let (X, S) be an S-metric space and define

$$d(x, y) = S(x, x, y) \ \text{ for all } x, y \in X.$$

Then d is a b-metric on X. The proof of this Corollary follows from the Proposition 2.9. □

Theorem 3.4. *Let (X, d) be a complete b-metric space with constant $b \geq 1$ and let $T : X \to X$ be a self-mapping satisfying a generalized weak contraction of Type II. Suppose that the constants $\alpha_i \geq 0$ $(i = 1, 2, 3)$ are such that $\alpha_1 + b\ \alpha_2(b + 1) + \alpha_3 < 1$. Then T admits a unique fixed point in X.*

Proof. Let $x_0 \in X$ be an arbitrary element. Since $T : X \to X$, we construct a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n,\ n = 0, 1, 2, \cdots$. Consider

$$\begin{aligned}
\psi\big(d(x_n, x_{n+1})\big) &= \psi\big(d(Tx_{n-1}, Tx_n)\big) \\
&\leq \psi\big(\alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, Tx_n) + \alpha_3 d(x_n, Tx_{n-1})\big) \\
&\quad - \phi\big(\alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, Tx_n) + \alpha_3 d(x_n, Tx_{n-1})\big) \\
&= \psi\big(\alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, x_{n+1}) + \alpha_3 d(x_n, x_n)\big) \\
&\quad - \phi\big(\alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, x_{n+1}) + \alpha_3 d(x_n, x_n)\big) \\
&= \psi\big(\alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, x_{n+1})\big) - \phi\big(\alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, x_{n+1})\big)
\end{aligned}$$

Using properties of ψ and ϕ, we obtain

$$\begin{aligned} d(x_n, x_{n+1}) &\le \alpha_1 d(x_{n-1}, x_n) + \alpha_2 d(x_{n-1}, x_{n+1}) \\ &\le \alpha_1 d(x_{n-1}, x_n) + \alpha_2 \Big\{ b\big[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) \big] \Big\}. \end{aligned}$$

It implies that

$$d(x_n, x_{n+1}) \le \left(\frac{\alpha_1 + b\alpha_2}{1 - b\alpha_2} \right) d(x_{n-1}, x_n).$$

Put $\lambda = \dfrac{\alpha_1 + b\alpha_2}{1 - b\alpha_2} < 1$, we obtain

$$d(x_n, x_{n+1}) \le \lambda d(x_{n-1}, x_n). \tag{4}$$

Similarly, we can show that

$$d(x_{n-1}, x_n) \le \lambda d(x_{n-2}, x_{n-1}). \tag{5}$$

Using (4) and (5), we get

$$d(x_n, x_{n+1}) \le \lambda^2 d(x_{n-2}, x_{n-1}).$$

By extending this argument through induction, we conclude

$$d(x_n, x_{n+1}) \le \lambda^n d(x_0, x_1).$$

Denote $d_n = d(x_n, x_{n+1})$. Then, we have

$$d_n \le \lambda^n d_0 \text{ for all } n \in \mathbb{N}. \tag{6}$$

Let $m, n \in \mathbb{N}$ with $m > n$. Then we have

$$\begin{aligned} d(x_n, x_m) &\le b\big(d(x_n, x_{n+1}) + d(x_{n+1}, x_m) \big) \\ &= b\ d(x_n, x_{n+1}) + b\ d(x_{n+1}, x_m) \\ &\le b\ d(x_n, x_{n+1}) + b\Big\{ b\big[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m) \big] \Big\} \\ &= b\ d(x_n, x_{n+1}) + b^2 d(x_{n+1}, x_{n+2}) + b^2 d(x_{n+2}, x_m) \\ &\le b\ d(x_n, x_{n+1}) + b^2 d(x_{n+1}, x_{n+2}) + \cdots + b^{m-n} d(x_{m-1}, x_m) \\ &= bd_n + b^2 d_{n+1} + b^3 d_{n+2} + \cdots b^{m-n} d_{m-1} \\ &\le bd_n + b^2 d_{n+1} + b^3 d_{n+2} + \cdots . \end{aligned}$$

Using (6), we obtain

$$\begin{aligned} d(x_n, x_m) &\le b(\lambda^n d_0) + b^2(\lambda^{n+1} d_0) + b^3(\lambda^{n+2} d_0) + \cdots \\ &= b\Big(1 + (\lambda b) + (\lambda b)^2 + (\lambda b)^3 + \cdots \Big) \lambda^n d_0 \\ &= b \left(\frac{1}{1 - \lambda b} \right) \lambda^n d_0. \end{aligned}$$

As $k < 1$, taking limit as $n \to \infty$, we get

$$\lim_{n,m \to \infty} d(x_n, x_m) = 0.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that $x_n \to u$. We claim that u is a fixed point of T. For this, consider

$$\begin{aligned}\psi\big(d(x_n, Tu)\big) &= \psi\big(d(Tx_{n-1}, Tu)\big)\\ &\leq \psi\big(\alpha_1 d(x_{n-1}, u) + \alpha_2 d(x_{n-1}, Tu) + \alpha_3 d(u, Tx_{n-1})\big)\\ &\quad - \phi\big(\alpha_1 d(x_{n-1}, u) + \alpha_2 d(x_{n-1}, Tu) + \alpha_3 d(u, Tx_{n-1})\big)\\ &\leq \psi\big(\alpha_1 d(x_{n-1}, u) + \alpha_2 d(x_{n-1}, Tu) + \alpha_3 d(u, x_n)\big)\\ &\quad - \phi\big(\alpha_1 d(x_{n-1}, u) + \alpha_2 d(x_{n-1}, Tu) + \alpha_3 d(u, x_n)\big).\end{aligned}$$

By the properties of ψ and ϕ, we obtain

$$d(x_n, Tu) \leq \alpha_1 d(x_{n-1}, u) + \alpha_2 d(x_{n-1}, Tu) + \alpha_3 d(u, x_n).$$

Letting $n \to \infty$, we get

$$d(u, Tu) \leq \alpha_1 d(u, Tu) + \alpha_2 d(u, u) + \alpha_3 d(u, u).$$

This implies that $(1 - \alpha_2)d(u, Tu) \leq 0$. Since, $0 \leq \alpha_2 < 1$, it follows that $1 - \alpha_2 > 0$. Hence, we conclude that $d(u, Tu) \leq 0$. On the other hand, $d(u, Tu) \geq 0$. Combining these inequalities, we obtain $d(u, Tu) = 0$. This implies that $Tu = u$, which shows that u is a fixed point of T. Now, to prove uniqueness of a fixed point, suppose that there exists $v \in X$ with $v \neq u$ such that $Tv = v$. Consider

$$\begin{aligned}\psi\big(d(u, v)\big) &= \psi\big(d(Tu, Tv)\big)\\ &\leq \psi\big(\alpha_1 d(u, v) + \alpha_2 d(u, Tv) + \alpha_3 d(v, Tu)\big) - \phi\big(\alpha_1 d(u, v) + \alpha_2 d(u, Tv) + \alpha_3 d(v, Tu)\big)\\ &= \psi\big(\alpha_1 d(u, v) + \alpha_2 d(u, v) + \alpha_3 d(v, u)\big) - \phi\big(\alpha_1 d(u, v) + \alpha_2 d(u, v) + \alpha_3 d(v, u)\big).\end{aligned}$$

By the properties of ϕ and ψ, we obtain

$$d(u, v) \leq (\alpha_1 + \alpha_2 + \alpha_3)d(u, v),$$

which is a contradiction, since $\alpha_1 + \alpha_2 + \alpha_3 < 1$. Hence, we must have$u = v$. This establishes the uniqueness of a fixed point. □

Corollary 3.5. *Let (X, S) be a complete S-metric space and $T : X \to X$ be a mapping satisfying*

$$\begin{aligned}\psi\left(S(Tx, Tx, Ty)\right) \leq\ &\psi\left(\alpha_1 S(x, x, y) + \alpha_2 S(x, x, Ty) + \alpha_3 S(y, y, Tx)\right)\\ &- \phi\left(\alpha_1 S(x, x, y) + \alpha_2 S(x, x, Ty) + \alpha_3 S(y, y, Tx)\right)\end{aligned}$$

for all $x, y, z \in X$, where $\phi, \psi : [0 + \infty) \to [0, +\infty)$ be non-decreasing functions such that

(a) $\phi(t) = 0$ if and only if $t = 0$,

(b) $\psi(t) = 0$ if and only if $t = 0$ and $\psi(t_1) \leq \psi(t_2)$ implies that $t_1 \leq t_2$, for all $t_1, t_2 \in [0, +\infty)$

and $\alpha_i \geq 0$ $(i = 1, 2, 3, 4)$ be such that $\alpha_1 + 2\alpha_2 + \alpha_3 < 1$. Then T admits a unique fixed point in X.

Proof. Let (X, S) be an S-metric space and define

$$d(x, y) = S(x, x, y) \ \text{ for all } x, y \in X.$$

Then d is a b-metric on X. The proof of this Corollary follows from the Proposition 2.9. □

Theorem 3.6. *Let (X, d) be a complete b-metric space and $T : X \to X$ be a mapping such that:*

$$\psi\left(bd(Tx, Ty)\right) \leq \psi\left(M(x, y)\right) - \phi\left(M(x, y)\right), \tag{7}$$

for all $x, y, z \in X$, where $b \geq 1$ is a given real number,

$$M(x, y) = \max\big\{d(x, y), d(x, Tx), d(y, Ty)\big\}$$

and $\phi, \psi : [0 + \infty) \to [0, +\infty)$ be non-decreasing functions such that *(a)* $\phi(t) = 0$ *if and only if* $t = 0$,

(b) $\psi(t) = 0$ *if and only if* $t = 0$ *and* $\psi(t_1) \leq \psi(t_2)$ *implies that* $t_1 \leq t_2$ *for all* $t_1, t_2 \in [0, +\infty)$.

Then T *has a unique fixed point in* X.

Proof. Let $x_0 \in X$ be an arbitrary element. Since $T : X \to X$, we define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n,\ n = 0, 1, 2, \cdots$. Consider

$$\begin{aligned}\psi\big(b\ d(x_n, x_{n+1})\big) &= \psi\big(b\ d(Tx_{n-1}, Tx_n)\big)\\ &\leq \psi\big(M(x_{n-1}, x_n)\big) - \phi\big(M(x_{n-1}, x_n)\big),\end{aligned} \tag{8}$$

where

$$\begin{aligned}M(x_{n-1}, x_n) &= \max\big\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\big\}\\ &= \max\big\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\big\}\\ &= \max\big\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\big\}.\end{aligned}$$

Case(i). If $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$, then inequality (8) implies that

$$\psi\big(b\ d(x_n, x_{n+1})\big) \leq \psi\big(d(x_n, x_{n+1})\big) - \phi\big(d(x_n, x_{n+1})\big).$$

Using property of ϕ and ψ, we obtain

$$b\ d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}),$$

which is not possible, since $b \geq 1$ and $d(x_n, x_{n+1}) \geq 0$.

Case(ii). If $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$, then from inequality (8), we obtain

$$\psi\big(b\ d(x_n, x_{n+1})\big) \leq \psi\big(d(x_{n-1}, x_n)\big) - \phi\big(d(x_{n-1}, x_n)\big).$$

Since $\phi(t) \geq 0$, for all $t \geq 0$, it follows that

$$\psi\big(b\ d(x_n, x_{n+1})\big) \leq \psi\big(d(x_{n-1}, x_n)\big).$$

Using the fact that ψ is order-preserving, we get

$$b\ d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) \ \text{ for all } \ n \in \mathbb{N}.$$

That is,

$$d(x_n, x_{n+1}) \leq b\ d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) \ \text{ for all } \ n \in \mathbb{N}.$$

Denote $d_n = d(x_n, x_{n+1})$. Then we have

$$d_n \leq d_{n-1} \ \text{ for all } \ n \in \mathbb{N}.$$

Therefore, the sequence $\{d_n\}$ is non-increasing sequence and bounded below by 0, so it converges to some limit $L \geq 0$. If $L > 0$, then we have

$$\psi\big(b\ d(x_n, x_{n+1})\big) \leq \psi\big(d(x_{n-1}, x_n)\big) - \phi\big(d(x_{n-1}, x_n)\big).$$

Applying the limit as $n \to \infty$, we obtain

$$\psi(bL) \leq \psi(L) - \phi(L) < \psi(L),$$

which is impossible. Therefore, there is only one possibility $L = 0$. That is,

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence. Suppose $\{x_n\}$ is not Cauchy. Therefore, for given $\epsilon > 0$, we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$d\left(x_{m(k)}, x_{n(k)}\right) \geq \epsilon \quad \text{and} \quad d\left(x_{m(k)}, x_{n(k)-1}\right) < \epsilon. \tag{9}$$

Then

$$\begin{aligned} \epsilon \leq d\left(x_{m(k)}, x_{n(k)}\right) &\leq b\ d\left(x_{m(k)}, x_{m(k)-1}\right) + b\ d(x_{m(k)-1}, x_{n(k)}) \\ &\leq b\ d\left(x_{m(k)}, x_{m(k)-1}\right) + b^2 d\left(x_{m(k)-1}, x_{n(k)-1}\right) + b^2 d\left(x_{n(k)-1}, x_{n(k)}\right). \end{aligned}$$

Also,

$$\begin{aligned} \frac{\epsilon}{b^2} &\leq \lim_{k\to\infty} d\left(x_{m(k)-1}, x_{n(k)-1}\right) \leq \lim_{k\to\infty} \sup d\left(x_{m(k)-1}, x_{n(k)-1}\right) \\ &\leq b \lim_{k\to\infty} \sup d\left(x_{m(k)-1}, x_{m(k)}\right) + b \lim_{k\to\infty} \sup d\left(x_{m(k)}, x_{n(k)-1}\right) \leq b\epsilon. \end{aligned}$$

Thus, we have

$$\frac{\epsilon}{b^2} \leq \lim_{k\to\infty} \inf d\left(x_{m(k)-1}, x_{n(k)-1}\right) \leq \lim_{k\to\infty} \sup d\left(x_{m(k)-1}, x_{n(k)-1}\right) \leq b\epsilon.$$

Now, by setting $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ in (7), we obtain

$$\begin{aligned} \psi\left(b\ d\left(x_{m(k)}, x_{n(k)}\right)\right) &= \psi\left(b\ d\left(Tx_{m(k)-1}, Tx_{n(k)-1}\right)\right) \\ &\leq \psi\left(M\left(x_{m(k)-1}, x_{n(k)-1}\right)\right) - \phi\left(M\left(x_{m(k)-1}, x_{n(k)-1}\right)\right), \end{aligned} \tag{10}$$

where

$$\begin{aligned} M\left(x_{m(k)-1}, x_{n(k)-1}\right) &= \max\Big\{d\left(x_{m(k)-1}, x_{n(k)-1}\right), d\left(x_{m(k)-1}, Tx_{m(k)-1}\right), d\left(x_{n(k)-1}, Tx_{n(k)-1}\right)\Big\} \\ &= \max\Big\{d\left(x_{m(k)-1}, x_{n(k)-1}\right), d\left(x_{m(k)-1}, x_m(k)\right), d\left(x_{n(k)-1}, x_n(k)\right)\Big\}. \end{aligned} \tag{11}$$

Taking limit as $k \to \infty$ and applying (10), (11), we obtain

$$\psi(b\epsilon) \leq \psi\Big(\max\big\{b\epsilon, 0, 0\big\}\Big) - \phi\Big(\max\big\{b\epsilon, 0, 0\big\}\Big).$$

That is,

$$\psi(b\epsilon) \leq \psi(b\epsilon) - \phi(b\epsilon) < \psi(b\epsilon),$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that $x_n \to u$. We claim that u is a fixed point of T. For this, consider

$$\begin{aligned} \psi\big(b\ d(x_{n+1}, Tu)\big) &= \psi\big(b\ d(Tx_n, Tu)\big) \\ &\leq \psi\big(M(x_n, u)\big) - \phi\big(M(x_n, u)\big), \end{aligned}$$

where

$$\begin{aligned} M(x_n, u) &= \max\big\{d(x_n, u), d(x_n, Tx_n), d(u, Tu)\big\} \\ &= \max\big\{d(x_n, u), d(x_n, x_{n+1}), d(u, Tu)\big\}. \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned}\psi\big(b\ d(u,Tu)\big) &\le \psi\Big(\max\big\{d(u,u),d(u,u),d(u,Tu)\big\}\Big)-\phi\Big(\max\big\{d(u,u),d(u,u),d(u,Tu)\big\}\Big)\\ &=\psi\Big(\max\big\{0,0,d(u,Tu)\big\}\Big)-\phi\Big(\max\big\{0,0,d(u,Tu)\big\}\Big).\end{aligned}$$

That is,

$$\psi\left(b\ d(u,Tu)\right)\le\psi\left(d(u,Tu)\right)-\phi\left(d(u,Tu)\right).$$

In view of the properties of ψ and ϕ, we get $b\ d(u,Tu) \le d(u,Tu)$. This is possible only when $d(u,Tu)=0$. Consequently, $Tu=u$. Thus, u is a fixed point of T. Now, to prove uniqueness of a fixed point, suppose that there is $v \in X$ with $v \neq u$ such that $Tv=v$. Then

$$\begin{aligned}\psi\big(b\ d(u,v)\big) &= \psi\big(b\ d(Tu,Tv)\big)\\ &\le \psi\big(M(u,v)\big)-\phi\big(M(u,v)\big),\end{aligned}$$

where

$$\begin{aligned}M(u,v) &= \max\big\{d(u,v),d(u,Tu),d(v,Tv)\big\}\\ &=\max\big\{d(u,v),d(u,u),d(v,v)\big\}\\ &=\max\big\{d(u,v),0,0\big\}\\ &=d(u,v).\end{aligned}$$

Thus,

$$\psi\big(b\ d(u,v)\big) =\le \psi\big(d(u,v)\big)-\phi\big(d(u,v)\big).$$

Using the properties of ψ and ϕ, we obtain $b\ d(u,v) \le d(u,v)$, which is not possible unless $d(u,v)=0$. Consequently, $u=v$. This establishes the uniqueness of a fixed point. □

Corollary 3.7. *Let (X,S) be a complete S-metric space and $T: X \to X$ be a mapping such that:*

$$\psi\left(S(Tx,Tx,Ty)\right)\le\psi\left(M(x,x,y)\right)-\phi\left(M(x,x,y)\right)$$

for all $x,y,z \in X$, where

$$M(x,x,y)=\max\left\{S(x,x,y),S(x,x,Tx),S(y,y,Ty)\right\}$$

and $\phi,\psi:[0+\infty)\to[0,+\infty)$ be non-decreasing functions such that

(a) $\phi(t)=0$ if and only if $t=0$,

(b) $\psi(t)=0$ if and only if $t=0$ and $\psi(t_1)\le\psi(t_2)$ implies that $t_1 \le t_2$ for all $t_1,t_2 \in [0,+\infty)$.

Then T admits a unique fixed point in X.

Proof. Let (X,S) be an S-metric space and define

$$d(x,y)=S(x,x,y)\ \text{ for all } x,y\in X.$$

Then d is a b-metric on X. The proof of this Corollary follows from the Proposition 2.9. □

Example 3.8. Let $X=\mathbb{R}$ and define $d: X\times X \to [0,+\infty)$ by $d(x,y)=\mid x-y\mid^2$. Then d is a b-metric on X with coefficient $b=2^{2-1}=2$. Define a mapping $T: X\to X$ by

$$Tx=\frac{x}{3}$$

Now,

$$d(Tx,Ty) = \mid Tx - Ty \mid^2 = \left|\frac{x}{3} - \frac{y}{3}\right|^2 = \frac{1}{9} \mid x - y \mid^2 .$$

Choose the constants

$$\alpha_1 = \frac{1}{4},\ \alpha_2 = 0 = \alpha_3 \ \text{ such that } \ b(\alpha_1 + \alpha_2) + \alpha_3 = 2\left(\frac{1}{4} + 0 + 0\right) = \frac{1}{2} < 1.$$

Let the functions $\psi, \phi : [0, +\infty) \to [0, +\infty)$ be defined by

$$\psi(t) = t \ \text{ and } \ \phi(t) = \frac{t}{2}.$$

Therefore,

$$\psi\left(d(Tx,Ty)\right) = d(Tx,Ty) = \frac{1}{9} \mid x - y \mid^2 . \tag{12}$$

Moreover, we observe that

$$\alpha_1 d(x,y) + \alpha_2 d(x,Tx) + \alpha_3 d(y,Ty) = \alpha_1 d(x,y) + 0 + 0 = \frac{1}{4} \mid x - y \mid^2 .$$

Hence,

$$\begin{aligned}
&\psi\left(\alpha_1 d(x,y) + \alpha_2 d(x,Tx) + \alpha_3 d(y,Ty)\right) - \phi\left(\alpha_1 d(x,y) + \alpha_2 d(x,Tx) + \alpha_3 d(y,Ty)\right)\\
&= \psi\left(\frac{1}{4} \mid x - y \mid^2\right) - \phi\left(\frac{1}{4} \mid x - y \mid^2\right)\\
&= \frac{1}{4} \mid x - y \mid^2 - \frac{1}{2}\left(\frac{1}{4} \mid x - y \mid^2\right)\\
&= \frac{1}{8} \mid x - y \mid^2 > \frac{1}{9} \mid x - y \mid^2 = \psi\left(d(Tx,Ty)\right).
\end{aligned}$$

Therefore,

$$\psi\left(d(Tx,Ty)\right) \le \psi\left(\alpha_1 d(x,y) + \alpha_2 d(x,Tx) + \alpha_3 d(y,Ty)\right) - \phi\left(\alpha_1 d(x,y) + \alpha_2 d(x,Tx) + \alpha_3 d(y,Ty)\right).$$

Thus, all the conditions of Theorem 3.1 are satisfied. Consequently, T has a unique fixed point $x = 0 \in X$.

Example 3.9. Let $X = \mathbb{R}^n$ and let $\|\cdot\|$ be a norm on X. Consider an S-metric on X defined by

$$S(x,y,z) = \|x - z\| + \|y - z\| \ \text{ for all } x,y,z \in \mathbb{R}^n.$$

Define the mapping $T : X \to X$ by

$$T(x) = \frac{x}{5}.$$

Choose constants

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = \alpha_3 = \frac{1}{8}, \text{ so that } \ \alpha_1 + \alpha_2 + \alpha_3 = \frac{3}{4} < 1.$$

Let the functions $\phi, \psi : [0 + \infty) \to [0, +\infty)$ be defined by

$$\psi(t) = t \quad \text{and} \quad \phi(t) = \frac{t}{2}.$$

Now,

$$\begin{aligned}
S(Tx,Tx,Ty) &= \|Tx - Ty\| + \|Tx - Ty\|\\
&= \left\|\frac{x}{5} - \frac{y}{5}\right\| + \left\|\frac{x}{5} - \frac{y}{5}\right\|\\
&= \frac{2}{5}\|x - y\|.
\end{aligned}$$

Therefore,

$$\psi\left(S(Tx,Tx,Ty)\right)=\psi\left(\frac{2}{5}\|x-y\|\right)=\frac{2}{5}\|x-y\|. \tag{13}$$

Now,

$$\begin{aligned}S(x,x,Tx)&=\|x-Tx\|+\|x-Tx\|\\&=2\left\|x-\frac{x}{5}\right\|\\&=\frac{8}{5}\|x\|.\end{aligned}$$

Similarly, we can show that $S(y,y,Ty)=\frac{8}{5}\|y\|$. Consequently,

$$\begin{aligned}\alpha_1S(x,x,y)+\alpha_2S(x,x,Tx)+\alpha_3S(y,y,Ty)&=\frac{1}{2}\left(2\|x-y\|\right)+\frac{1}{8}\left(\frac{8}{5}\|x\|\right)+\frac{1}{8}\left(\frac{8}{5}\|y\|\right)\\&=\|x-y\|+\frac{1}{5}\left(\|x\|+\|y\|\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\psi\left(\alpha_1S(x,x,y)+\alpha_2S(x,x,Tx)+\alpha_3S(y,y,Ty)\right)-\phi\left(\alpha_1S(x,x,y)+\alpha_2S(x,x,Tx)+\alpha_3S(y,y,Ty)\right)\\&=\psi\left(\|x-y\|+\frac{1}{5}\left(\|x\|+\|y\|\right)\right)-\phi\left(\|x-y\|+\frac{1}{5}\left(\|x\|+\|y\|\right)\right)\\&=\|x-y\|+\frac{1}{5}\left(\|x\|+\|y\|\right)-\frac{1}{2}\left(\|x-y\|+\frac{1}{5}\left(\|x\|+\|y\|\right)\right)\\&=\frac{1}{2}\|x-y\|+\frac{1}{10}\left(\|x\|+\|y\|\right)\end{aligned} \tag{14}$$

Using inequalities (13) and (14), we deduce that

$$\begin{aligned}\psi\left(S(Tx,Tx,Ty)\right)&=\frac{2}{5}\|x-y\|\\&\leq\frac{1}{2}\|x-y\|+\frac{1}{10}\left(\|x\|+\|y\|\right)\\&=\psi\left(\alpha_1S(x,x,y)+\alpha_2S(x,x,Tx)+\alpha_3S(y,y,Ty)\right)\\&\quad-\phi\left(\alpha_1S(x,x,y)+\alpha_2S(x,x,Tx)+\alpha_3S(y,y,Ty)\right).\end{aligned}$$

That is,

$$\begin{aligned}\psi\left(S(Tx,Tx,Ty)\right)\leq{}&\psi\left(\alpha_1S(x,x,y)+\alpha_2S(x,x,Tx)+\alpha_3S(y,y,Ty)\right)\\&-\phi\left(\alpha_1S(x,x,y)+\alpha_2S(x,x,Tx)+\alpha_3S(y,y,Ty)\right).\end{aligned}$$

Hence, all the assumptions of Corollary 3.3 are satisfied and it follows that the mapping T has a unique fixed point $x=0\in X$.

4. Application to Integral Equation

Consider the integral equation

$$x(t)=\int_0^1 K\big(t,s,x(s)\big)\,ds+g(t),\quad t\in[0,1]$$

where $K : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$ are continuous functions. Let $X = C\big([0,1], \mathbb{R}\big)$ be the space of all continuous real-valued functions on $[0,1]$, equipped with the supremum norm

$$\|x\| = \sup_{t\in[0,1]} |x(t)|.$$

Let $X = \mathbb{R}^n$ and consider the S-metric on X defined by

$$S(x,y,z) = \|x-z\| + \|y-z\| \;\text{ for all } x,y,z \in X.$$

Define the operator $T : X \to X$ by

$$(Tx)(t) = \int_0^1 K\big(t,s,x(s)\big)\, ds + g(t) \;\text{ where } t \in [0,1].$$

Assume that there exists a constant $L \in (0,1)$ such that

$$|K(t,s,u) - K(t,s,v)| \leq L|u-v|$$

for all $t,s \in [0,1]$ and $u,v \in \mathbb{R}$. Then for any $x,y \in X$, we have

$$\begin{aligned} |Tx(t) - Ty(t)| &= \left| \int_0^1 \Big(K\big(t,s,x(s)\big) - K\big(t,s,y(s)\big) \Big)\, ds \right| \\ &\leq \int_0^1 \big| K\big(t,s,x(s)\big) - K\big(t,s,y(s)\big) \big|\, ds \\ &\leq \int_0^1 L|x(s)-y(s)|\, ds \\ &\leq L\|x-y\|. \end{aligned}$$

Taking supremum over $t \in [0,1]$, we obtain

$$\|Tx - Ty\| \leq L\|x-y\|.$$

Hence,

$$\begin{aligned} S(Tx,Tx,Ty) &= \|Tx-Ty\| + \|Tx-Ty\| \\ &= 2\|Tx-Ty\| \\ &\leq 2L\|x-y\| \\ &= LS(x,x,y). \end{aligned}$$

Thus, $S(Tx,Tx,Ty) \leq LS(x,x,y)$. Now, consider the functions $\phi, \psi : [0+\infty) \to [0,+\infty)$ defined by

$$\psi(t) = t \;\text{ and }\; \phi(t) = \delta t,$$

where $\delta \in (0, 1-L)$. Clearly, ψ and ϕ satisfy all the required conditions of the Corollary 3.3. Choose the constants

$$\alpha_1 = \frac{L}{1-\delta}, \quad \alpha_2 = 0, \quad \alpha_3 = 0.$$

Then clearly $\alpha_1 + \alpha_2 + \alpha_3 = \dfrac{L}{1-\delta} < 1$. Now,

$$\begin{aligned} \psi\big(\alpha_1 S(x,x,y)\big) - \phi\big(\alpha_1 S(x,x,y)\big) &= \alpha_1 S(x,x,y) - \delta\alpha_1 S(x,x,y) \\ &= (1-\delta)\alpha_1 S(x,x,y) \\ &= LS(x,x,y). \end{aligned}$$

Thus,

$$\psi\big(S(Tx,Tx,Ty)\big) \leq \psi\big(\alpha_1 S(x,x,y)\big) - \phi\big(\alpha_1 S(x,x,y)\big),$$

which shows that all the conditions of Corollary 3.3 are satisfied. Therefore, the operator T has a unique fixed point in X. Consequently, the given integral equation has a unique solution in $C\big([0,1], \mathbb{R}\big)$.

5. Conclusion

In this paper, we have established several fixed point results for generalized weak contractions of Type I and Type II in b-metric spaces. Also, we have derived analogous results in S-metric spaces. Some examples are given to support the results. Furthermore, an application to integral equation shows the usefulness of our result in ensuring the existence and uniqueness of solution.

References

[1] I. A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal. **30** (1989), 26–37.

[2] M. Boriceanu, *Fixed point theory for multivalued generalized contractions on a set with two b-metrics*, Stud. Univ. Babeș-Bolyai Math. **54**(3) (2009), 1–14.

[3] M. Bota, A. Molnar, C. Varga, *On Ekeland's variational principle in b-metric spaces*, Fixed Point Theory **12** (2011), no. 2, 21–28.

[4] R. Chugh, T. Kadian, A. Rani, B. E. Rhoades, *Property P in G-metric spaces*, Fixed Point Theory Appl. **2010** (2010), Article ID 401684.

[5] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostrav. **1**(1) (1993), 5–11.

[6] S. Z. Degefa, V. Singh, K. Singh, A. Bamel, Y. G. Aemro, *Fixed point results in the fuzzy multiplicative metric spaces and application*, Int. J. Math. Sci. Comput. **11**(3) (2025), 58–72.

[7] S. Z. Degefa, V. Singh, Y. G. Aemro, *Fixed point theorems in the fuzzy supermetric spaces and application*, Bangmod Int. J. Math. Comput. Sci. **11** (2025), 282–304.

[8] B. C. Dhage, *Generalized metric spaces and mappings with fixed point*, Bull. Calcutta Math. Soc. **84** (1992), 329–336.

[9] S. Duraj, S. Liftaj, *A common fixed-point theorem of mappings on S-metric spaces*, Asian J. Probab. Stat. **20**(2) (2022), 40–45.

[10] S. Gähler, *2-metrische Räume und ihre topologische Struktur*, Math. Nachr. **26** (1963), 115–148.

[11] M. Jovanović, Z. Kadelburg, S. Radenović, *Common fixed point results in metric-type spaces*, Fixed Point Theory Appl. **2010** (2010), 1–15.

[12] E. Karapınar, *A short survey on the recent fixed point results on b-metric spaces*, Constr. Math. Anal. **1** (2018), 15–44.

[13] A. Kumar, *Banach fixed point theorem in extended bv(s)-metric spaces*, Comput. Sci. Math. Forum **7**(1) (2023), 58.

[14] M. A. Khamsi, N. Hussain, *KKM mappings in metric type spaces*, Nonlinear Anal. **7** (2010), 3123–3129.

[15] Z. D. Mitrović, *Fixed point results in b-metric space*, Fixed Point Theory **20**(2) (2019), 559–566.

[16] N. Mlaiki, *Common fixed points in complex S-metric space*, Adv. Fixed Point Theory **4**(4) (2014), 509–524.

[17] N. Mlaiki, *α-ψ-contractive mapping on S-metric space*, Math. Sci. Lett. **4**(1) (2015), 9–12.

[18] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal. **7** (2006), 289–297.

[19] M. Păcurar, *Sequences of almost contractions and fixed points in b-metric spaces*, An. Univ. Vest. Timiș., Ser. Mat.-Inform. **48** (2010), 125–137.

[20] K. Prudhvi, *Fixed point results in S-metric spaces*, Univer. J. Comput. Math. **3** (2015), 19–21.

[21] S. Sedghi, N. Shobe, H. Zhou, *A common fixed point theorem in D^*-metric spaces*, Fixed Point Theory Appl. (2007), 1–13.

[22] S. Sedghi, N. Shobe, A. Aliouche, *A generalization of fixed point theorems in S-metric spaces*, Mat. Vesnik **64**(3) (2012), 258–266.

[23] S. Sedghi, N. Van Dung, *Fixed point theorems on S-metric spaces*, Mat. Vesnik **66**(1) (2014), 113–124.

[24] P. S. Sharma, C. T. Aage, *Results on common fixed points in S_b-metric spaces*, e-J. Anal. Appl. Math. **2025** (2025), 57–78.

[25] W. Shatanawi, *Fixed point theory for contractive mappings satisfying Φ-maps in G-metric spaces*, Fixed Point Theory Appl. **2010** (2010), Article ID 181650.

[26] A. Singh, N. Hooda, *Coupled fixed point theorems in S-metric spaces*, Int. J. Math. Stat. Invention **2**(4) (2014), 33–39.

[27] B. Singh, V. Singh, A. Ahlawat, *Fixed point theorem for* B-*type contraction in partial metric spaces*, Mapana J. Sci. **21**(1) (2022), 39–50.

[28] V. Singh, B. Singh, *Fixed point theorems in M-metric spaces via rational expression and application*, Palest. J. Math. **13**(2) (2024), 297–307.

[29] K. Singh, V. Singh, B. Singh, *Fixed point of generalized F-Suzuki contraction mapping on complete extended b-metric spaces with application*, Filomat **39**(3) (2025), 935–947.

[30] N. Taş, N. Özgür, *On the geometry of fixed points for self-mappings on S-metric spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **69**(2) (2020), 1184–1192.

[31] R. K. Vats, *Weakly compatible maps in metric spaces*, J. Indian Math. Soc. **69**(1) (2002), 139–143.