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On periodic solutions for implicit problem with nonlinear fractional differential equation involving the Riesz-Caputo fractional derivative

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Abstract

The main goal of this paper is to study the existence and uniqueness of periodic solutions for implicit problem with nonlinear fractional differential equation (NFDEs) involving the Riesz-Caputo fractional derivative. The proofs are based upon the coincidence degree theory of Mawhin. An example is constructed to authenticate and affirm the main findings.

Keywords: Coincidence degree theory, existence, uniqueness, Riesz-Caputo fractional derivative, coupled system
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1. Introduction

Fractional calculus has become a very important tool in modeling of many phenomena in applications and sciences such as physics, biology, finance, engineering, stability, controllability and rheology. It can better describe the memory properties of the physical process than the standard integer order calculus. For more details on the applications of fractional calculus, the reader is directed to the books of Baleanu *et al.* [3], Benchohra *et al.* [1, 6, 7, 8], Graef *et al.* [11] and [26, 27]. In [2, 4, 5, 12, 13], the authors presented some results on the fractional differential equations. Salim *et al.* [15, 16, 22, 23, 24, 25] addressed the existence, stability, and uniqueness of solutions for diverse problems with fractional differential equations using various fractional derivatives and different types of conditions.

The authors of [4] studied the existence of solution for the following boundary value problem:

$$\begin{cases} {}_0^{RC}D_{\varkappa}^{\nu}\varphi(\theta) = g(\theta, \varphi(\theta)), \quad \theta \in \Theta := [0, \varkappa], \\ \varphi(0) = \varphi_0, \quad \varphi(\varkappa) = \varphi_{\varkappa}, \end{cases}$$

where ${}_0^{RC}D_{\varkappa}^{\nu}$ is a Riesz-Caputo derivative of order $0 < \nu \leq 1$, $g : \Theta \times \mathbb{R} \to \mathbb{R}$ a continuous function and $\varphi_0, \varphi_{\varkappa} \in \mathbb{R}$. Their arguments are based on Leray-Schauder fixed point theorem, and Schauder's fixed point theorem.

In [17], Li and Wang discussed the following fractional problem:

$$\begin{gathered} {}_0^{RC}D_1^{\gamma}\varphi(\theta) = \Psi(\theta, \varphi(\theta)), \quad \theta \in [0, 1], \quad 0 < \gamma \leq 1, \\ \varphi(0) = a, \quad \varphi(1) = b\varphi(\eta), \end{gathered}$$

where ${}_0^{RC}D_1^{\gamma}$ is the Riesz-Caputo derivative, $\Psi \in C([0, 1]\times\ [0, +\infty), [0, +\infty)), 0 < \eta < 1, a > 0, 0 < b < 2$. They found the positive solutions by applying the technique of monotone iterative.

Naas *et al.* [19] investigated the existence and uniqueness results of the following fractional differential equation with the Riesz-Caputo derivative:

$$\begin{cases} {}_0^{RC}D_T^{\vartheta}\varkappa(\theta) + \mathfrak{F}\left(\theta, \varkappa(\theta), {}_0^{RC}D_T^{\varsigma}\varkappa(\theta)\right) = 0, \theta \in \mathcal{J} := [0, T], \\ \varkappa(0) + \varkappa(T) = 0, \quad \mu\varkappa'(0) + \sigma\varkappa'(T) = 0, \end{cases}$$

where $1 < \vartheta \leq 2$ and $, 0 < \varsigma \leq 1, {}_0^{RC}D_T^{\kappa}$ is the Riesz-Caputo fractional derivative of order $\kappa \in \{\vartheta, \varsigma\}$, $\mathfrak{F} : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is a continuous function, and μ, σ are nonnegative constants with $\mu > \sigma$. The existence and uniqueness of solutions of the above cited problem are demonstrated with the Riesz-Caputo derivatives via Banach's, Schaefer's, and Krasnoselskii's fixed point theorems.

In this paper, we study the existence and uniqueness of periodic solutions for the problem with the Riesz-Caputo fractional derivative:

$${}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta) = f\left(\theta, \chi(\theta), {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta)\right) \ \theta \in \mathfrak{J} := [0, \mathfrak{b}], \tag{1}$$

$$\chi(0) = \chi(\mathfrak{b}), \tag{2}$$

where $0 < \alpha < 1,, {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}$, is the Riesz-Caputo fractional derivative, $f : \mathfrak{J} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous function.

The structure of this paper is as follows: Section 2 presents certain notations and preliminaries about the Riesz-Caputo fractional derivative used throughout this manuscript. In Section 3, we present existence and uniqueness result for the problem (1)-(2) that are based upon the coincidence degree theory of Mawhin. Ultimately, numerical examples are furnished to demonstrate our outcomes.

2. Preliminaries

First, we give the definitions and the notations that we will use throughout this paper. We denote by $C(\mathfrak{J}, \mathbb{R})$ the Banach space of all continuous functions from $\mathfrak{J}$ into $\mathbb{R}$ with the following norm

$$\|f\|_{\infty} = \sup_{\theta \in \mathfrak{J}}\{|f(\theta)|\}.$$

Definition 2.1. [14] Let $\alpha > 0$. The left and right Riemann-Liouville fractional integrals of a function $\chi \in C(\mathfrak{J}, \mathbb{R})$, of order α are given respectively by

$${}_0\mathcal{I}_{\theta}^{\alpha}\chi(\theta) = \frac{1}{\Gamma(\alpha)}\int_0^{\theta}(\theta - s)^{\alpha-1}\chi(s)ds, \tag{3}$$

and

$${}_{\theta}\mathcal{I}_{\mathfrak{b}}^{\alpha}\chi(\theta) = \frac{1}{\Gamma(\alpha)}\int_{\theta}^{\mathfrak{b}}(s - \theta)^{\alpha-1}\chi(s)ds. \tag{4}$$

where ${}_0\mathcal{I}_{\theta}^{\alpha}$ and ${}_{\theta}\mathcal{I}_{\mathfrak{b}}^{\alpha}$, are the left and right fractional integrals of Riemann-Liouville.

Definition 2.2. [9] Let $\alpha > 0$. The Riesz fractional integral of a function $\chi \in C(\mathfrak{J}, \mathbb{R})$, of order α is defined by

$${}_0^R\mathcal{I}_{\mathfrak{b}}^{\alpha}\chi(\theta) = \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}}|\theta - s|^{\alpha-1}\chi(s)ds$$

Remark 2.3. From equations (3) and (4) it follows that

$${}_0^R\mathcal{I}_{\mathfrak{b}}^{\alpha}\chi(\theta) = \frac{1}{2}\left({}_0\mathcal{I}_{\theta}^{\alpha}\chi(\theta) + {}_{\theta}\mathcal{I}_{\mathfrak{b}}^{\alpha}\chi(\theta)\right),\ \theta \in \mathfrak{J} \tag{5}$$

Definition 2.4. [14] Let $\alpha \in (n, n+1]$, $n \in \mathbb{N}$. The left and right Caputo fractional derivatives of a function $\chi \in C^{n+1}(\mathfrak{J}, \mathbb{R})$, of order α are given respectively by

$${}_0^C\mathfrak{D}_{\theta}^{\alpha}\chi(\theta) = \frac{1}{\Gamma(n+1-\alpha)}\int_0^{\theta}(\theta - s)^{n-\alpha}\chi^{(n+1)}(s)ds, \tag{6}$$

and

$${}_{\theta}^C\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta) = \frac{(-1)^{n+1}}{\Gamma(n+1-\alpha)}\int_{\theta}^{\mathfrak{b}}(s - \theta)^{n-\alpha}\chi^{(n+1)}(s)ds. \tag{7}$$

Lemma 2.5. *([14] See Corollary 2.3) Let $\chi \in C^n(\mathfrak{J}, \mathbb{R})$, and $\alpha \in (n, n+1]$, $n \in \mathbb{N}$, then the Caputo fractional differentiation operators ${}_0^C\mathfrak{D}_{\theta}^{\alpha}$ and ${}_{\theta}^C\mathfrak{D}_{\mathfrak{b}}^{\alpha}$, are bounded from the space $C^n(\mathfrak{J}, \mathbb{R})$. Moreover,*

$$\|{}_0^C\mathfrak{D}_{\theta}^{\alpha}\chi\|_{C^n} \leq k_{\alpha}\|\chi\|_{C^n}$$

and

$$\|{}_{\theta}^C\mathfrak{D}_{\mathfrak{b}}^{\alpha}\|_{C^n} \leq k_{\alpha}\|\chi\|_{C^n},$$

where

$$k_{\alpha} = \frac{\mathfrak{b}^{n-\alpha}}{\Gamma(n-\alpha)(n-\alpha+1)}.$$

In the particular case $0 < \alpha < 1$, we have:

$$\|{}_0^C\mathfrak{D}_{\theta}^{\alpha}\chi\|_{\infty} \leq k_{\alpha}\|\chi\|_{\infty}$$

and

$$\|{}_{\theta}^C\mathfrak{D}_{\mathfrak{b}}^{\alpha}\|_{\infty} \leq k_{\alpha}\|\chi\|_{\infty},$$

where

$$k_{\alpha} = \frac{\mathfrak{b}^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}.$$

Remark 2.6. ([21] See Section 2.4) The Caputo derivative of a constant is equal to zero.

Definition 2.7. [9](fractional derivatives in the sense of Riesz and Riesz-Caputo). Let $\chi \in C^{n+1}(\mathfrak{J}, \mathbb{R})$, and $\alpha \in (n, n+1]$, $n \in \mathbb{N}$, the Riesz fractional derivative ${}_0^R\mathfrak{D}_{\mathfrak{b}}^{\alpha}$ and the Riesz-Caputo fractional derivative ${}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}$ of order α are defined by

$$ {}_0^R\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta) = \mathfrak{D}^n {}_0^R\mathfrak{J}_{\mathfrak{b}}^{n-\alpha}\chi(\theta) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{d\theta}\right)^n \int_0^{\mathfrak{b}} |\theta - s|^{n-\alpha-1}\chi(s)ds, \tag{8} $$

and

$$ {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta) = {}_0^R\mathfrak{J}_{\mathfrak{b}}^{n-\alpha}\mathfrak{D}^n\chi(\theta) = \frac{1}{\Gamma(n-\alpha)}\int_0^{\mathfrak{b}} |\theta - s|^{n-\alpha-1}\left(\frac{d}{d\theta}\right)^n \chi(s)ds, \tag{9} $$

where $\mathfrak{D}^n = \left(\frac{d}{d\theta}\right)^n$.

Remark 2.8. [9] Using equations (5) and (6)-(9) it follows that

$$ {}_0^R\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta) = \frac{1}{2}\left({}_0\mathfrak{D}_{\theta}^{\alpha}\chi(\theta) + (-1)^n {}_{\theta}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta)\right) $$

and

$$ {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta) = \frac{1}{2}\left({}_0^C\mathfrak{D}_{\theta}^{\alpha}\chi(\theta) + (-1)^n {}_{\theta}^C\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta)\right). $$

In the particular case $0 < \alpha < 1$, we have:

$$ {}_0^R\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta) = \frac{1}{2}\left({}_0\mathfrak{D}_{\theta}^{\alpha}\chi(\theta) - {}_{\theta}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta)\right) $$

and

$$ {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta) = \frac{1}{2}\left({}_0^C\mathfrak{D}_{\theta}^{\alpha}\chi(\theta) - {}_{\theta}^C\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta)\right). \tag{10} $$

Remark 2.9. By Lemma 2.5 and (10) we have

$$ \begin{aligned} \|{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi\|_{\infty} &\leq \frac{1}{2}\left[\|{}_0^C\mathfrak{D}_{\theta}^{\alpha}\chi\|_{\infty} + \|{}_{\theta}^C\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi\|_{\infty}\right] \\ &\leq \frac{\mathfrak{b}^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\|\chi\|_{\infty}. \end{aligned} $$

Lemma 2.10. *[14] If $\chi \in C^{n+1}(\mathfrak{J}, \mathbb{R})$, and $\alpha \in (n, n+1]$, then we have*

$$ {}_0\mathfrak{J}_{\theta}^{\alpha} {}_0^C\mathfrak{D}_{\theta}^{\alpha}\chi(\theta) = \chi(\theta) - \sum_{k=0}^{n} \frac{\chi^{(k)}(0)}{k!}(\theta - 0)^k, $$

and

$$ {}_{\theta}\mathfrak{J}_{\mathfrak{b}}^{\alpha} {}_{\theta}^C\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta) = (-1)^{n+1}\left[\chi(\theta) - \sum_{k=0}^{n} \frac{(-1)^k\chi^{(k)}(\mathfrak{b})}{k!}(\mathfrak{b} - \theta)^k\right]. $$

Consequently, we may have

$$ {}_0^R\mathfrak{J}_{\mathfrak{b}}^{\alpha} {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta) = \frac{1}{2}\left[{}_0\mathfrak{J}_{\theta}^{\alpha} {}_0^C\mathfrak{D}_{\theta}^{\alpha}\chi(\theta) + (-1)^{n+1} {}_{\theta}\mathfrak{J}_{\mathfrak{b}}^{\alpha} {}_{\theta}^C\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta)\right]. $$

In particular, if $0 < \alpha < 1$, then we obtain

$$ {}_0^R\mathfrak{J}_{\mathfrak{b}}^{\alpha} {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta) = \chi(\theta) - \frac{1}{2}\left[\chi(0) + \chi(\mathfrak{b})\right]. $$

Lemma 2.11. *If $\chi \in C^1(\mathfrak{J}, \mathbb{R})$ and $0 < \alpha < 1$, then by (9) we have*

$$\begin{aligned} {}^{RC}_{0}\mathfrak{D}^{\alpha}_{\mathfrak{b}}\, {}^{R}_{0}\mathfrak{J}^{\alpha}_{\mathfrak{b}}\chi(\theta) &= {}^{R}_{0}\mathfrak{J}^{1-\alpha}_{\mathfrak{b}}\mathfrak{D}\, {}^{R}_{0}\mathfrak{J}^{\alpha}_{\mathfrak{b}}\chi(\theta) \\ &= {}^{R}_{0}\mathfrak{J}^{1-\alpha}_{\mathfrak{b}}\, {}^{R}_{0}\mathfrak{J}^{\alpha-1}_{\mathfrak{b}}\chi(\theta) \\ &= \chi(\theta). \end{aligned}$$

Lemma 2.12. *If $\chi \in C^1(\mathfrak{J}, \mathbb{R})$ and $0 < \alpha < 1$, then we have*

$$ {}^{RC}_{0}\mathfrak{D}^{\alpha}_{\mathfrak{b}}\chi(\theta) = 0 \Leftrightarrow \chi(\theta) = \textit{constant}, \ \theta \in \mathfrak{J}.$$

Proof. Let $\chi \in C^1(\mathfrak{J}, \mathbb{R})$ and $\theta \in \mathfrak{J}$, then by Definitions 2.4 and Remark 2.8, we have

$$\begin{aligned} {}^{RC}_{0}\mathfrak{D}^{\alpha}_{\mathfrak{b}}\chi(\theta) = 0 \ &\Rightarrow \ {}^{C}_{0}\mathfrak{D}^{\alpha}_{\theta}\chi(\theta) - {}^{C}_{\theta}\mathfrak{D}^{\alpha}_{\mathfrak{b}}\chi(\theta) = 0 \\ &\Rightarrow \ \frac{1}{\Gamma(1-\alpha)}\left[\int_0^{\theta}(\theta - s)^{-\alpha}\chi'(s)ds + \int_{\theta}^{\mathfrak{b}}(s-\theta)^{-\alpha}\chi'(s)ds\right] = 0 \\ &\Rightarrow \ \int_0^{\mathfrak{b}}(\theta - s)^{-\alpha}(1 + (-1)^{-\alpha})\chi'(s)ds = 0 \\ &\Rightarrow \ \chi'(\theta) = 0, \ \theta \in \mathfrak{J}, \\ &\Rightarrow \ \chi = \text{constant}. \end{aligned}$$

Conversly, if $\chi = \text{constant}$ for all $\theta \in \mathfrak{J}$, then by Remark 2.9 we have

$$ {}^{RC}_{0}\mathfrak{D}^{\alpha}_{\mathfrak{b}}\chi(\theta) = 0, \ \theta \in \mathfrak{J}.$$

Which completes the proof.

We will present definitions and the coincidence degree theory that are essential in proofs of our results, see [10, 18].

Definition 2.13. We consider the normed spaces $\mathcal{X}$ and $\mathcal{Y}$. A Fredholm operator of index zero is a linear operator $\mathfrak{L} : Dom(\mathfrak{L}) \subset \mathcal{X} \to \mathcal{Y}$ such that

a) $\mathfrak{dim} \ker \mathfrak{L} = \mathfrak{codim}\mathfrak{Img}\mathfrak{L} < +\infty$.

b) $\mathfrak{Img}\mathfrak{L}$ is a closed subset of $\mathcal{Y}$.

By Definition 2.13, there exist continuous projectors $\mathcal{Q} : \mathcal{Y} \to \mathcal{Y}$ and $\mathcal{P} : \mathcal{X} \to \mathcal{X}$ satisfying

$$\mathfrak{Img}\mathfrak{L} = \ker \mathcal{Q}, \qquad \ker \mathfrak{L} = \mathfrak{Img}\mathcal{P}, \qquad \mathcal{Y} = \mathfrak{Img}\mathcal{Q} \oplus \mathfrak{Img}\mathfrak{L}, \qquad \mathcal{X} = \ker \mathcal{P} \oplus \ker \mathfrak{L}.$$

Thus, the restriction of $\mathfrak{L}$ to $Dom\mathfrak{L} \cap \ker \mathcal{P}$, denoted by $\mathfrak{L}_{\mathcal{P}}$, is an isomorphism onto its image.

Definition 2.14. Let $\Omega \subseteq \mathcal{X}$ be a bounded subset and $\mathfrak{L}$ be a Fredholm operator of index zero with $Dom\mathfrak{L} \cap \Omega \neq \emptyset$. Then, the operator $\mathcal{N} : \overline{\Omega} \to \mathcal{Y}$ is called to be $\mathfrak{L}-$compact in $\overline{\Omega}$ if

a) the mapping $\mathcal{Q}\mathcal{N} : \overline{\Omega} \to \mathcal{Y}$ is continuous and $\mathcal{Q}\mathcal{N}\left(\overline{\Omega}\right) \subseteq \mathcal{Y}$ is bounded.

b) the mapping $(\mathfrak{L}_{\mathcal{P}})^{-1}(id - \mathcal{Q})\mathcal{N} : \overline{\Omega} \to \mathcal{X}$ is completely continuous.

Lemma 2.15. *[20] Let $\mathcal{X}, \mathcal{Y}$ be a Banach spaces, $\Omega \subset \mathcal{X}$ a bounded open set and symmetric with $0 \in \Omega$. Suppose that $\mathfrak{L} : Dom\mathfrak{L} \subset \mathcal{X} \to \mathcal{Y}$ is a Fredholm operator of index zero with $Dom\mathfrak{L} \cap \overline{\Omega} \neq \emptyset$ and $\mathcal{N} : \mathcal{X} \to \mathcal{Y}$ is a $\mathfrak{L}-$compact operator on $\overline{\Omega}$. Assume, moreover, that*

$$\mathfrak{L}x - \mathcal{N}x \neq -\zeta(\mathfrak{L}x + \mathcal{N}(-x)),$$

for any $x \in Dom\mathfrak{L} \cap \partial\Omega$ and any $\zeta \in (0, 1]$, where $\partial\Omega$ is the boundary of Ω with respect to $\mathcal{X}$. If these conditions are verified, then there exist at least one solution of the equation $\mathfrak{L}x = \mathcal{N}x$ on $Dom\mathfrak{L} \cap \overline{\Omega}$.

3. Main Results

Let the spaces

$$\mathcal{X} = \left\{\chi \in C(\mathfrak{J}, \mathfrak{R}) : \chi(\theta) = {}_0^R\mathfrak{I}_{\mathfrak{b}}^{\alpha}\upsilon(\theta) : \upsilon \in C(\mathfrak{J}, \mathfrak{R}) \text{ and } \theta \in \mathfrak{J}\right\},$$

and

$$\mathcal{Y} = C(\mathfrak{J}, \mathfrak{R}),$$

be endowed with the norms

$$\|\chi\|_{\mathcal{X}} = \|\chi\|_{\mathcal{Y}} = \sup_{\theta \in \mathfrak{J}} |\chi(\theta)|.$$

We give now the definition of the operator $\mathfrak{L} : Dom\mathfrak{L} \subseteq \mathcal{X} \to \mathcal{Y}$

$$\mathfrak{L}\chi := {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi, \tag{11}$$

where

$$Dom\mathfrak{L} = \{\chi \in \mathcal{X} : \mathfrak{L}\chi \in \mathcal{Y} \text{ and } \chi(0) = \chi(\mathfrak{b})\}.$$

Lemma 3.1. *Using the definition of $\mathfrak{L}$ given in (11). Then*

$$\ker \mathfrak{L} = \{\chi \in \mathcal{X} : \chi(\theta) = \chi(0),\ \theta \in \mathfrak{J}\},$$

and

$$\mathfrak{Img}\mathfrak{L} = \left\{\upsilon \in \mathcal{Y} : \int_0^{\mathfrak{b}} \left[(\mathfrak{b} - s)^{\alpha-1} - s^{\alpha-1}\right] \upsilon(s)ds = 0\right\}.$$

Proof. By Lemma 2.12, we have for all $\chi \in Dom\mathfrak{L} \subset \mathcal{X}$ the equation $\mathfrak{L}\chi(\theta) = {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta) = 0$ in $\mathfrak{J}$, has a solution of the form

$$\chi(\theta) = \chi(0),\ \theta \in \mathfrak{J},$$

then

$$\ker \mathfrak{L} = \{\chi \in \mathcal{X} : \chi(\theta) = \chi(0),\ \theta \in \mathfrak{J}\}.$$

For $\upsilon \in \mathfrak{Img}\mathfrak{L}$, there exists $\chi \in Dom\mathfrak{L}$ such that $\upsilon = \mathfrak{L}\chi \in \mathcal{Y}$. Using Lemma 2.10, we obtain for every $\theta \in \mathfrak{J}$:

$$\begin{aligned} \chi(\theta) &= \frac{1}{2}(\chi(0) + \chi(\mathfrak{b})) + {}_0^R\mathcal{I}_{\mathfrak{b}}^{\alpha}\upsilon(\theta) \\ &= \frac{1}{2}(\chi(0) + \chi(\mathfrak{b})) + \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}} |\theta - s|^{\alpha-1}\upsilon(s)ds. \end{aligned}$$

Since $\chi \in Dom\mathfrak{L}$, then we have $\chi(0) = \chi(\mathfrak{b})$. Thus

$$\int_0^{\mathfrak{b}} \left[(\mathfrak{b} - s)^{\alpha-1} - s^{\alpha-1}\right] \upsilon(s)ds = 0.$$

Furthermore, if $\upsilon \in \mathcal{Y}$, and satisfies

$$\int_0^{\mathfrak{b}} \left[(\mathfrak{b} - s)^{\alpha-1} - s^{\alpha-1}\right] \upsilon(s)ds = 0, \tag{12}$$

for any $\chi(\theta) \in \mathcal{X}$ and by the definition of the space $\mathcal{X}$, then $\chi(\theta) = {}_0^R\mathcal{I}_{\mathfrak{b}}^{\alpha}\upsilon(\theta)$, using Lemma 2.11, we get

$$\upsilon(\theta) = {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta) = \mathfrak{L}\chi(\theta).$$

On the other hand, for each $\theta \in \mathfrak{J}$, we have

$$\chi(\theta) = {}_0^R\mathcal{I}_{\mathfrak{b}}^{\alpha}\upsilon(\theta) = \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}} |\theta - s|^{\alpha-1}\upsilon(s)ds.$$

Which implies that

$$\chi(0)=\frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}} s^{\alpha-1}\upsilon(s)ds$$

and

$$\chi(\mathfrak{b})=\frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}} (\mathfrak{b}-s)^{\alpha-1}\upsilon(s)ds,$$

then

$$\chi(\mathfrak{b})-\chi(0)=\frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}} \left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]\upsilon(s)ds.$$

And by (12), we obtain

$$\chi(0)=\chi(\mathfrak{b}).$$

Therefore $\chi\in Dom\mathfrak{L}$. So $\upsilon\in\mathfrak{Img}\mathfrak{L}$.
Hence

$$\mathfrak{Img}\mathfrak{L}=\left\{\upsilon\in\mathcal{Y}:\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]\upsilon(s)ds=0\right\}.$$

Which completes the proof.

Lemma 3.2. *Let $\mathfrak{L}$ be defined by (11). Then $\mathfrak{L}$ is a Fredholm operator of index zero, and the linear continuous projector operators $\mathcal{Q}:\mathcal{Y}\to\mathcal{Y}$ and $\mathcal{P}:\mathcal{X}\to\mathcal{X}$ can be written as*

$$\mathcal{Q}\upsilon=\frac{1}{\varpi(\mathfrak{b})}\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]\upsilon(s)ds,$$

where

$$\varpi(\mathfrak{b})=\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]ds.$$

And

$$\mathcal{P}(\chi)=\chi(0).$$

Furthermore, the operator $\mathfrak{L}_{\mathcal{P}}^{-1}:\mathfrak{Img}\mathfrak{L}\to\mathcal{X}\cap\ker\mathcal{P}$ can be written by

$$\mathfrak{L}_{\mathcal{P}}^{-1}(\upsilon)(\theta)={}_0^R\mathcal{I}_{\mathfrak{b}}^{\alpha}\upsilon(\theta),\ \theta\in\mathfrak{J}.$$

Proof. Obviously, for each $\upsilon\in\mathcal{Y}$, $\mathcal{Q}^2\upsilon=\mathcal{Q}\upsilon$ and $\upsilon=\mathcal{Q}(\upsilon)+(\upsilon-\mathcal{Q}(\upsilon))$, where $(\upsilon-\mathcal{Q}(\upsilon))\in\ker\mathcal{Q}=\mathfrak{Img}\mathfrak{L}$.
Using the fact that $\mathfrak{Img}\mathfrak{L}=\ker\mathcal{Q}$ and $\mathcal{Q}^2=\mathcal{Q}$ then $\mathfrak{Img}\mathfrak{L}\cap\mathfrak{Img}\mathcal{Q}=0$. So,

$$\mathcal{Y}=\mathfrak{Img}\mathfrak{L}\oplus\mathfrak{Img}\mathcal{Q}.$$

By the same way we get that $\mathfrak{Img}\mathcal{P}=\ker\mathfrak{L}$ and $\mathcal{P}^2=\mathcal{P}$. It follows for each $\chi\in\mathcal{X}$, that $\chi=(\chi-\mathcal{P}(\chi))+\mathcal{P}(\chi)$ then $\mathcal{X}=\ker\mathcal{P}+\ker\mathfrak{L}$. Clearly we have $\ker\mathcal{P}\cap\ker\mathfrak{L}=0$. So

$$\mathcal{X}=\ker\mathcal{P}\oplus\ker\mathfrak{L}.$$

Using Rank−nullity theorem, we get :

$$\begin{aligned}\mathfrak{codim}\mathfrak{Img}\mathfrak{L}&=\dim\mathcal{Y}-\dim\mathfrak{Img}\mathfrak{L}\\&=[\dim\ker\mathcal{Q}+\dim\mathfrak{Img}\mathcal{Q}]-\dim\mathfrak{Img}\mathfrak{L},\end{aligned}$$

and since $\mathfrak{Img}\mathfrak{L}=\ker\mathcal{Q}$, then

$$\mathfrak{codim}\mathfrak{Img}\mathfrak{L}=\dim\mathfrak{Img}\mathcal{Q}. \tag{13}$$

Using also Rank−nullity theorem, we obtain

$$\dim\ker\mathfrak{L} = \dim\mathcal{X} - \dim\mathfrak{Img}\mathfrak{L} = \mathfrak{codim}\mathfrak{Img}\mathfrak{L},$$

which implies that

$$\dim\ker\mathfrak{L} = \mathfrak{codim}\mathfrak{Img}\mathfrak{L}. \tag{14}$$

By (13) and (14) we have:

$$\dim\ker\mathfrak{L} = \mathfrak{codim}\mathfrak{Img}\mathfrak{L} = \dim\mathfrak{Img}\mathcal{Q},$$

and since $\dim\mathfrak{Img}\mathcal{Q} < \infty$, then

$$\dim\ker\mathfrak{L} = \mathfrak{codim}\mathfrak{Img}\mathfrak{L} < \infty.$$

And since $\mathfrak{Img}\mathfrak{L}$ is a closed subset of $\mathcal{Y}$, then $\mathfrak{L}$ is a Fredholm operator of index zero.

Now, we will show that the inverse of $\mathfrak{L}|_{Dom\mathfrak{L}\cap\ker\mathcal{P}}$ is $\mathfrak{L}_{\mathcal{P}}^{-1}$. Effectively, for $\upsilon \in \mathfrak{Img}\mathfrak{L}$, by Lemma 2.11, we have

$$\mathfrak{L}\mathfrak{L}_{\mathcal{P}}^{-1}(\upsilon) = {}_{0}^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\left({}_{0}^{R}\mathcal{I}_{\mathfrak{b}}^{\alpha}\upsilon\right) = \upsilon. \tag{15}$$

Furthermore, for $\chi \in Dom\mathfrak{L}\cap\ker\mathcal{P}$ we get

$$\mathfrak{L}_{\mathcal{P}}^{-1}\mathfrak{L}\chi(\theta) = {}_{0}^{R}\mathcal{I}_{\mathfrak{b}}^{\alpha}{}_{0}^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta) = \chi(\theta) - \frac{1}{2}(\chi(0)+\chi(\mathfrak{b})),\ \theta\in\mathfrak{J}.$$

Using the fact that $\chi \in Dom\mathfrak{L}\cap\ker\mathcal{P}$, then

$$\chi(0) = \chi(\mathfrak{b}) = 0.$$

Thus,

$$\mathfrak{L}_{\mathcal{P}}^{-1}\mathfrak{L}(\chi) = \chi. \tag{16}$$

Using (15) and (16) together, we get $\mathfrak{L}_{\mathcal{P}}^{-1} = \left(\mathfrak{L}|_{Dom\mathfrak{L}\cap\ker\mathcal{P}}\right)^{-1}$. Which completes the demonstration.

Let the following hypothesis:

(*H1*) There exist positive constants γ, and β with

$$|f(\theta,\chi,\upsilon) - f(\theta,\bar{\chi},\bar{\upsilon})| \leqslant \gamma|\chi-\bar{\chi}| + \beta|\bar{\upsilon}-\upsilon|,$$

for every $\theta\in\mathfrak{J}$ and $\chi,\bar{\chi},\upsilon,\bar{\upsilon}\in\mathfrak{R}$.

Define $\mathcal{N}:\mathcal{X}\to\mathcal{Y}$ by

$$\mathcal{N}\chi(\theta) = f\left(\theta,\chi(\theta),{}_{0}^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta)\right),\ \theta\in\mathfrak{J}.$$

Then the problem (1)-(2) is equivalent to the problem $\mathfrak{L}\chi = \mathcal{N}\chi$.

Lemma 3.3. *Suppose that (H1) is satisfied then, for any bounded open set $\Omega\subset\mathcal{X}$, the operator $\mathcal{N}$ is $\mathfrak{L}-$compact.*

Proof. We consider for $\mathcal{M}>0$ the bounded open set $\Omega = \{\chi\in\mathcal{X} : \|\chi\|_{\mathcal{X}} < \mathcal{M}\}$. We split the proof into three steps:

Step 1: *$\mathcal{QN}$ is continuous.*

Let $(\upsilon_{\mathfrak{n}})_{\mathfrak{n}\in\mathbb{N}}$ be a sequence such that $\upsilon_{\mathfrak{n}}\longrightarrow\upsilon$ in $\mathcal{Y}$, thus for $\theta\in\mathfrak{J}$, we have

$$|\mathcal{QN}(\upsilon_{\mathfrak{n}})(\theta) - \mathcal{QN}(\upsilon)(\theta)|$$

$$\leqslant \frac{1}{\varpi(\mathfrak{b})}\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]|\mathcal{N}(\upsilon_{\mathfrak{n}})(s)-\mathcal{N}(\upsilon)(s)|\,ds.$$

By (*H1*) and Remark 2.9 we have

$$\begin{aligned}
&|\mathcal{QN}(\upsilon_{\mathfrak{n}})(\theta)-\mathcal{QN}(\upsilon)(\theta)|\\
\leqslant\ &\frac{\gamma}{\varpi(\mathfrak{b})}\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]|\upsilon_{\mathfrak{n}}(s)-\upsilon(s)|\,ds\\
&+\frac{\beta}{\varpi(\mathfrak{b})}\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]\left|{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\upsilon_{\mathfrak{n}}(s)-{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\upsilon(s)\right|ds\\
\leqslant\ &\left(\gamma+\frac{\beta\mathfrak{b}^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right)\|\upsilon_{\mathfrak{n}}-\upsilon\|_{\mathcal{Y}}.
\end{aligned}$$

Thus

$$\sup_{\theta\in\mathfrak{J}}|\mathcal{QN}(\upsilon_{\mathfrak{n}})(\theta)-\mathcal{QN}(\upsilon)(\theta)|\leqslant\left(\gamma+\frac{\beta\mathfrak{b}^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right)\|\upsilon_{\mathfrak{n}}-\upsilon\|_{\mathcal{Y}},$$

and hence

$$\|\mathcal{QN}(\upsilon_{\mathfrak{n}})-\mathcal{QN}(\upsilon)\|_{\mathcal{Y}}\longrightarrow 0 \text{ as } n\longrightarrow+\infty.$$

We deduce that $\mathcal{QN}$ is continuous.

Step 2: $\mathcal{QN}(\overline{\Omega})$ *is bounded*

For $\theta\in\mathfrak{J}$ and $\upsilon\in\overline{\Omega}$, we have

$$\begin{aligned}
|\mathcal{QN}(\upsilon)(\theta)| &\leqslant \frac{1}{\varpi(\mathfrak{b})}\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]|\mathcal{N}(\upsilon)(s)|ds\\
&\leqslant \frac{1}{\varpi(\mathfrak{b})}\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]\left|f\left(s,\upsilon(s),{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\upsilon(s)\right)-f(s,0,0)\right|ds\\
&\quad+\frac{1}{\varpi(\mathfrak{b})}\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]|f(s,0,0)|ds\\
&\leqslant f^*+\frac{\gamma}{\varpi(\mathfrak{b})}\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]|\upsilon(s)|\,ds\\
&\quad+\frac{\beta}{\varpi(\mathfrak{b})}\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]\left|{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\upsilon(s)\right|ds\\
&\leqslant f^*+\left(\gamma+\frac{\beta\mathfrak{b}^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right)\mathcal{M},
\end{aligned}$$

where $f^*=\sup_{\theta\in\mathfrak{J}}|f(\theta,0,0)|$.

Thus

$$\sup_{\theta\in\mathfrak{J}}|\mathcal{QN}(\upsilon)(\theta)|\leqslant f^*+\left(\gamma+\frac{\beta\mathfrak{b}^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right)\mathcal{M},$$

which implies that

$$\|\mathcal{QN}(\upsilon)\|_{\mathcal{Y}}\leqslant f^*+\left(\gamma+\frac{\beta\mathfrak{b}^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right)\mathcal{M}.$$

So, $\mathcal{QN}(\overline{\Omega})$ is a bounded set in $\mathcal{Y}$.

Step 3: $\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}:\overline{\Omega}\to\mathcal{X}$ *is completely continuous.*
We will use the Arzelà-Ascoli theorem, so we have to show that $\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}(\overline{\Omega})\subset\mathcal{X}$ is equicontinuous and bounded. Firstly, for any $\chi\in\overline{\Omega}$ and $\theta\in\mathfrak{J}$, we get

$$
\begin{aligned}
&\mathfrak{L}_{\mathcal{P}}^{-1}\Big(\mathcal{N}\chi(\theta)-\mathcal{Q}\mathcal{N}\chi(\theta)\Big)\\
&={}_0^R\mathcal{I}_{\mathfrak{b}}^{\alpha}\Big[f\left(\theta,\chi(\theta),{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta)\right)\\
&\quad-\frac{1}{\varpi(\mathfrak{b})}\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]f\left(s,\chi(s),{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(s)\right)ds\Big]\\
&=\frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}}|\theta-s|^{\alpha-1}f\left(s,\chi(s),{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(s)\right)ds\\
&\quad-\frac{[\theta^{\alpha}+(\mathfrak{b}-\theta)^{\alpha}]}{2\Gamma(\alpha+1)\varpi(\mathfrak{b})}\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]f\left(s,\chi(s),{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(s)\right)ds.
\end{aligned}
$$

For all $\chi\in\overline{\Omega}$, and $\theta\in\mathfrak{J}$ we get

$$
\begin{aligned}
&|\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}\chi(\theta)|\\
\leqslant\ &\frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}}|\theta-s|^{\alpha-1}\Big|f\left(s,\chi(s),{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(s)\right)-f(s,0,0)\Big|ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}}|\theta-s|^{\alpha-1}|f(s,0,0)|ds\\
&+\frac{\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)\varpi(\mathfrak{b})}\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]\\
&\quad\Big|f\left(s,\chi(s),{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(s)\right)-f(s,0,0)\Big|ds\\
&+\frac{\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)\varpi(\mathfrak{b})}\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]|f(s,0,0)|ds,\\
\leqslant\ &\frac{2f^*\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)}+\frac{\gamma}{\Gamma(\alpha)}\int_0^{\mathfrak{b}}|\theta-s|^{\alpha-1}\left|\chi(s)\right|ds\\
&+\frac{\beta}{\Gamma(\alpha)}\int_0^{\mathfrak{b}}|\theta-s|^{\alpha-1}\left|{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(s)\right|ds\\
&+\frac{\gamma\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)\varpi(\mathfrak{b})}\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]\left|\chi(s)\right|ds\\
&+\frac{\beta\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)\varpi(\mathfrak{b})}\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]\left|{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(s)\right|ds\\
\leqslant\ &\frac{2f^*\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)}+\frac{2\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)}\left(\gamma+\frac{\beta\mathfrak{b}^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right)\mathcal{M},
\end{aligned}
$$

Which implies that

$$
\sup_{\theta\in\mathfrak{J}}\left|\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}\chi(\theta)\right|\ \leqslant\ \frac{2f^*\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)}+\frac{2\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)}\left(\gamma+\frac{\beta\mathfrak{b}^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right)\mathcal{M}.
$$

Therefore

$$
\|\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}\chi\|_{\mathcal{X}}\ \leqslant\ \frac{2f^*\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)}+\frac{2\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)}\left(\gamma+\frac{\beta\mathfrak{b}^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right)\mathcal{M}.
$$

This means that $\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}(\overline{\Omega})$ is uniformly bounded in $\mathcal{X}$.
It remains to show that $\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}(\overline{\Omega})$ is equicontinuous.

For $0<\theta_1<\theta_2\leqslant\mathfrak{b}$ and $\chi\in\overline{\Omega}$, we have

$$\begin{aligned}
&|\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}\chi(\theta_2)-\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}\chi(\theta_1)|\\
\leqslant\ & \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}}\left|(\theta_2-s)^{\alpha-1}-(\theta_1-s)^{\alpha-1}\right|\left|f\left(s,\chi(s),{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(s)\right)\right|ds\\
&+\frac{\left[|\theta_2^{\alpha}-\theta_1^{\alpha}|+|(\mathfrak{b}-\theta_2)^{\alpha}-(\mathfrak{b}-\theta_1)^{\alpha}|\right]}{2\Gamma(\alpha+1)\varpi(\mathfrak{b})}\\
&\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]\left|f\left(s,\chi(s),{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(s)\right)\right|ds\\
\leqslant\ & \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}}\left|(\theta_2-s)^{\alpha-1}-(\theta_1-s)^{\alpha-1}\right|\\
&\left|f\left(s,\chi(s),{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(s)\right)-f(s,0,0)\right|ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}}\left|(\theta_2-s)^{\alpha-1}-(\theta_1-s)^{\alpha-1}\right|\left|f(s,0,0)\right|ds\\
&+\frac{\left[|\theta_2^{\alpha}-\theta_1^{\alpha}|+|(\mathfrak{b}-\theta_2)^{\alpha}-(\mathfrak{b}-\theta_1)^{\alpha}|\right]}{2\Gamma(\alpha+1)\varpi(\mathfrak{b})}\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]\\
&\left|f\left(s,\chi(s),{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(s)\right)-f(s,0,0)\right|ds\\
&+\frac{\left[|\theta_2^{\alpha}-\theta_1^{\alpha}|+|(\mathfrak{b}-\theta_2)^{\alpha}-(\mathfrak{b}-\theta_1)^{\alpha}|\right]}{2\Gamma(\alpha+1)\varpi(\mathfrak{b})}\int_0^{\mathfrak{b}}\left[(\mathfrak{b}-s)^{\alpha-1}-s^{\alpha-1}\right]\left|f(s,0,0)\right|ds\\
\leqslant\ & \frac{f^*}{\Gamma(\alpha)}\int_0^{\mathfrak{b}}\left|(\theta_2-s)^{\alpha-1}-(\theta_1-s)^{\alpha-1}\right|ds+\frac{f^*\left[|\theta_2^{\alpha}-\theta_1^{\alpha}|+|(\mathfrak{b}-\theta_2)^{\alpha}-(\mathfrak{b}-\theta_1)^{\alpha}|\right]}{2\Gamma(\alpha+1)}\\
&+\left(\gamma+\frac{\beta\mathfrak{b}^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right)\left[\frac{\mathcal{M}}{\Gamma(\alpha)}\int_0^{\mathfrak{b}}\left|(\theta_2-s)^{\alpha-1}-(\theta_1-s)^{\alpha-1}\right|ds\right.\\
&\left.+\frac{\mathcal{M}\left[|\theta_2^{\alpha}-\theta_1^{\alpha}|+|(\mathfrak{b}-\theta_2)^{\alpha}-(\mathfrak{b}-\theta_1)^{\alpha}|\right]}{2\Gamma(\alpha+1)}\right].
\end{aligned}$$

The operator $\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}(\overline{\Omega})$ is equicontinuous in $\mathcal{X}$ because the right-hand side of the above inequality tends to zero as $\theta_1\to\theta_2$ and the limit is independent of χ. The Arzelà-Ascoli theorem implies that $\mathfrak{L}_{\mathcal{P}}^{-1}(id-\mathcal{Q})\mathcal{N}(\overline{\Omega})$ is relatively compact in $\mathcal{X}$. As a consequence of steps 1 to 3, we get that $\mathcal{N}$ is $\mathfrak{L}-$compact in $\overline{\Omega}$. Which completes the demonstration.

Lemma 3.4. *Assume (H1). If the condition*

$$\frac{\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)}\left(\gamma+\frac{\beta\mathfrak{b}^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right)<\frac{1}{2}, \tag{17}$$

is satisfied, then there exists $\mathcal{A}>0$, which is independent of ζ, such that,

$$\mathfrak{L}(\chi)-\mathcal{N}(\chi)=-\zeta[\mathfrak{L}(\chi)+\mathcal{N}(-\chi)]\Longrightarrow\|\chi\|_{\mathcal{X}}\leqslant\mathcal{A},\ \zeta\in(0,1].$$

Proof. Let $\chi\in\mathcal{X}$ satisfies

$$\mathfrak{L}(\chi)-\mathcal{N}(\chi)=-\zeta\mathfrak{L}(\chi)-\zeta\mathcal{N}(-\chi),$$

then

$$\mathfrak{L}(\chi)=\frac{1}{1+\zeta}\mathcal{N}(\chi)-\frac{\zeta}{1+\zeta}\mathcal{N}(-\chi).$$

So, we obtain for any $\theta \in \mathfrak{J}$:

$$\begin{aligned}\mathfrak{L}\chi(\theta) &= {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta) \\ &= \frac{1}{1+\zeta} f\left(\theta, \chi(\theta), {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta)\right) - \frac{\zeta}{1+\zeta} f\left(\theta, -\chi(\theta), -{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta)\right).\end{aligned}$$

By Lemma 2.10 we get

$$\chi(\theta) = c_0 + \frac{1}{\zeta+1}\left[{}_0^{R}\mathcal{I}_{\mathfrak{b}}^{\alpha}\left(f\left(\theta, \chi(s), {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(s)\right)\right)(\theta) - \zeta\, {}_0^{R}\mathcal{I}_{\mathfrak{b}}^{\alpha}\left(f\left(\theta, -\chi(s), -{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(s)\right)\right)(\theta)\right],$$

where $c_0 = \frac{1}{2}[\chi(0) + \chi(\mathfrak{b})]$. Thus for every $\theta \in \mathfrak{J}$ we obtain

$$\begin{aligned}|\chi(\theta)| &\leqslant |c_0| + \frac{1}{(\zeta+1)\Gamma(\alpha)}\int_0^{\mathfrak{b}} |\theta - s|^{\alpha-1}\left|f\left(\theta, \chi(s), {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(s)\right)\right| ds \\ &\quad + \frac{\zeta}{(\zeta+1)\Gamma(\alpha)}\int_0^{\mathfrak{b}} |\theta - s|^{\alpha-1}\left|f\left(\theta, -\chi(s), -{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(s)\right)\right| ds \\ &\leqslant |c_0| + \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}} |\theta - s|^{\alpha-1}\left|f\left(s, \chi(s), {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(s)\right) - f(s,0,0)\right| ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}} |\theta - s|^{\alpha-1}|f(s,0,0)| ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}} |\theta - s|^{\alpha-1}\left|f\left(s, -\chi(s), -{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(s)\right) - f(s,0,0)\right| ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{b}} |\theta - s|^{\alpha-1}|f(s,0,0)| ds \\ &\leqslant \frac{2f^*\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)} + |c_0| + \frac{2\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)}\left(\gamma + \frac{\beta\mathfrak{b}^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right)\|\chi\|_{\mathcal{X}},\end{aligned}$$

thus

$$\sup_{\theta\in\mathfrak{J}}|\chi(\theta)| \leqslant \frac{2f^*\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)} + |c_0| + \frac{2\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)}\left(\gamma + \frac{\beta\mathfrak{b}^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right)\|\chi\|_{\mathcal{X}}.$$

We deduce that

$$\|\chi\|_{\mathcal{X}} \leqslant \frac{\dfrac{2f^*\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)} + |c_0|}{\left[1 - \dfrac{2\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)}\left(\gamma + \dfrac{\beta\mathfrak{b}^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right)\right]} := \mathcal{A}.$$

The demonstration is completed.

Lemma 3.5. *If conditions (H1) and (17) are verified, then there exist a bounded open set $\Omega \subset \mathcal{X}$ with*

$$\mathfrak{L}(\chi) - \mathcal{N}(\chi) \neq -\zeta[\mathfrak{L}(\chi) + \mathcal{N}(-\chi)], \tag{18}$$

for any $\chi \in \partial\Omega$ and any $\zeta \in (0,1]$.

Proof. Using Lemma 3.4, then there exists a positive constant $\mathcal{A}$ which is independent of ζ such that, if χ verify

$$\mathfrak{L}(\chi) - \mathcal{N}(\chi) = -\zeta[\mathfrak{L}(\chi) + \mathcal{N}(-\chi)],\ \zeta \in (0,1],$$

thus $\|\chi\|_{\mathcal{X}} \leqslant \mathcal{A}$. So, if

$$\Omega = \{\chi \in \mathcal{X}; \|\chi\|_{\mathcal{X}} < \vartheta\}, \tag{19}$$

such that $\vartheta > \mathcal{A}$, we deduce that

$$\mathfrak{L}(\chi) - \mathcal{N}(\chi) \neq -\zeta[\mathfrak{L}(\chi) - \mathcal{N}(-\chi)],$$

for all $\chi \in \partial\Omega = \{\chi \in \mathcal{X}; \|\chi\|_{\mathcal{X}} = \vartheta\}$ and $\zeta \in (0,1]$.

Theorem 3.6. *Assume (H1) and (17), then there exist at least one solution for the problem (1)-(2) in* $Dom\mathfrak{L} \cap \overline{\Omega}$.

Proof. It is clear that the set Ω defined in (19) is symmetric, $0 \in \Omega$ and $\mathcal{X} \cap \overline{\Omega} = \overline{\Omega} \neq \emptyset$. In addition, By Lemma 3.5, assume (H1) and (17), then

$$\mathfrak{L}(\chi) - \mathcal{N}(\chi) \neq -\zeta[\mathfrak{L}(\chi) - \mathcal{N}(-\chi)],$$

for each $\chi \in \mathcal{X} \cap \partial\Omega = \partial\Omega$ and each $\zeta \in (0,1]$. By Lemma 2.15, problem (1)-(2) has at least one solution in $Dom\mathfrak{L} \cap \overline{\Omega}$. Which completes the demonstration.

Theorem 3.7. *thm1 Let (H1) and (17) satisfied. Moreover we assume that*

(H2) *There exist positive constants* $\overline{\gamma}$ *and* $\overline{\beta}$ *with*

$$|f(\theta,\chi,\upsilon) - f(\theta,\bar{\chi},\bar{\upsilon})| \geq \overline{\gamma}|\chi - \bar{\chi}| - \overline{\beta}|\upsilon - \bar{\upsilon}|,$$

for every $\theta \in \mathfrak{J}$ *and* $\chi, \bar{\chi}, \upsilon, \bar{\upsilon} \in \mathfrak{R}$.

If one has

$$\frac{2\mathfrak{b}}{\Gamma(1-\alpha)(2-\alpha)\Gamma(\alpha+1)} + \frac{\overline{\beta}\mathfrak{b}^{1-\alpha}}{\overline{\gamma}\Gamma(1-\alpha)(2-\alpha)} < 1, \tag{20}$$

then the problem (1)-(2) has a unique solution in $Dom\mathfrak{L} \cap \overline{\Omega}$.

Proof. by Theorem 3.6 the problem (1)-(2) has at least one solution in $Dom\mathfrak{L} \cap \overline{\Omega}$.
Now, we prove the uniqueness result. Suppose that the problem (1)-(2) has two different solutions $\chi, \overline{\chi} \in Dom\mathfrak{L} \cap \overline{\Omega}$. Then, for each $\theta \in \mathfrak{J}$ we have

$${}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta) = f\left(\theta, \chi(\theta), {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta)\right),$$

$${}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\overline{\chi}(\theta) = f\left(\theta, \overline{\chi}(\theta), {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\overline{\chi}(\theta)\right),$$

and

$$\chi(0) = \chi(\mathfrak{b}), \ \overline{\chi}(0) = \overline{\chi}(\mathfrak{b}).$$

Let $\mathfrak{U}(\theta) = \chi(\theta) - \overline{\chi}(\theta)$, for all $\theta \in \mathfrak{J}$.
Then

$$\begin{aligned} \mathfrak{L}\mathfrak{U}(\theta) &= {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\mathfrak{U}(\theta) \\ &= {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta) - {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\overline{\chi}(\theta) \\ &= f\left(\theta, \chi(\theta), {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(\theta)\right) - f\left(\theta, \overline{\chi}(\theta), {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\overline{\chi}(\theta)\right). \end{aligned}$$

Using the fact that $\mathfrak{Img}\mathfrak{L} = \ker \mathcal{Q}$, then we have

$$\int_0^{\mathfrak{b}} \left[(\mathfrak{b}-s)^{\alpha-1} - s^{\alpha-1}\right] \left[f\left(s, \chi(s), {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(s)\right) - f\left(s, \overline{\chi}(s), {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\overline{\chi}(s)\right)\right] ds = 0.$$

Since f is continuou function then, there exist $t_0 \in [0, \mathfrak{b}]$ such that

$$f\left(t_0, \chi(t_0), {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(t_0)\right) - f\left(t_0, \overline{\chi}(t_0), {}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\overline{\chi}(t_0)\right) = 0.$$

In view of (H2) we have

$$\left|\chi(t_0)-\overline{\chi}(t_0)\right| \leqslant \frac{\overline{\beta}}{\overline{\gamma}}\left|{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\chi(t_0)-{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\overline{\chi}(t_0)\right|,$$

then

$$|\mathfrak{U}(t_0)| \leqslant \frac{\overline{\beta}\mathfrak{b}^{1-\alpha}}{\overline{\gamma}\Gamma(1-\alpha)(2-\alpha)}\|\mathfrak{U}\|_{\mathcal{X}}. \tag{21}$$

On the other hand, by Lemma 2.10, for each $\theta\in[0,\mathfrak{b}]$, we have

$${}_0^{R}\mathfrak{I}_{\mathfrak{b}}^{\alpha}{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\mathfrak{U}(\theta)=\mathfrak{U}(\theta)-\frac{1}{2}\left(\mathfrak{U}(0)+\mathfrak{U}(\mathfrak{b})\right),$$

wich implies that

$$\frac{1}{2}\left(\mathfrak{U}(0)+\mathfrak{U}(\mathfrak{b})\right)=\mathfrak{U}(t_0)-{}_0^{R}\mathfrak{I}_{\mathfrak{b}}^{\alpha}{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\mathfrak{U}(t_0),$$

and therfore

$$\mathfrak{U}(\theta)={}_0^{R}\mathfrak{I}_{\mathfrak{b}}^{\alpha}{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\mathfrak{U}(\theta)+\mathfrak{U}(t_0)-{}_0^{R}\mathfrak{I}_{\mathfrak{b}}^{\alpha}{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\mathfrak{U}(t_0).$$

Using (21) and Remark 2.9, for every $\theta\in\mathfrak{J}$, we have

$$\begin{aligned}|\mathfrak{U}(\theta)| &\leqslant \left|{}_0^{R}\mathfrak{I}_{\mathfrak{b}}^{\alpha}{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\mathfrak{U}(\theta)\right|+|\mathfrak{U}(t_0)|+\left|{}_0^{R}\mathfrak{I}_{\mathfrak{b}}^{\alpha}{}_0^{RC}\mathfrak{D}_{\mathfrak{b}}^{\alpha}\mathfrak{U}(t_0)\right|\\ &\leqslant \left[\frac{2\mathfrak{b}}{\Gamma(1-\alpha)(2-\alpha)\Gamma(\alpha+1)}+\frac{\overline{\beta}\mathfrak{b}^{1-\alpha}}{\overline{\gamma}\Gamma(1-\alpha)(2-\alpha)}\right]\|\mathfrak{U}\|_{\mathcal{X}}.\end{aligned}$$

Then

$$\|\mathfrak{U}\|_{\mathcal{X}}\leqslant\left[\frac{2\mathfrak{b}}{\Gamma(1-\alpha)(2-\alpha)\Gamma(\alpha+1)}+\frac{\overline{\beta}\mathfrak{b}^{1-\alpha}}{\overline{\gamma}\Gamma(1-\alpha)(2-\alpha)}\right]\|\mathfrak{U}\|_{\mathcal{X}}.$$

Hence, by (20), we conclude that

$$\|\mathfrak{U}\|_{\mathcal{X}}=0.$$

As a result, for any $\theta\in\mathfrak{J}$, we get

$$\mathfrak{U}(\theta)=0\Longrightarrow\chi(\theta)=\overline{\chi}(\theta).$$

This completes the proof.

4. An example

Consider the following system

$$\begin{cases}{}_0^{RC}\mathfrak{D}_1^{\frac{7}{8}}\chi(\theta)=f\left(\theta,\chi(\theta),{}_0^{RC}\mathfrak{D}_1^{\frac{7}{8}}\chi(\theta)\right), & \theta\in\mathfrak{J}:=[0,1],\\ \chi(0)=\chi(1),\end{cases}$$

where

$$f\left(\theta,\chi,{}_0^{RC}\mathfrak{D}_1^{\frac{7}{8}}\chi\right) = \ln(\theta+7)+\frac{3}{73\sqrt{\pi}}\left(\sin\chi+\frac{1}{3}\chi\right)+\frac{e^{-11-\theta}}{17\left(1+{}_0^{RC}\mathfrak{D}_1^{\frac{7}{8}}\chi\right)}.$$

Here $\alpha=\frac{7}{8}$ and $\mathfrak{b}=1$.
It is easy to see that $f\in C\big([0,1]\times\mathfrak{R}^2,\mathfrak{R}\big)$. Let $\chi,\bar{\chi},\upsilon,\bar{\upsilon}\in\mathfrak{R}$ and $\theta\in\mathfrak{J}$, then

$$|f(\theta,\chi,\upsilon)-f(\theta,\bar{\chi},\overline{\upsilon})|\leqslant\frac{12}{219\sqrt{\pi}}|\chi-\bar{\chi}|+\frac{1}{17e^{11}}|\upsilon-\overline{\upsilon}|.$$

Hence, the assumption (H1) is satisfied with $\gamma = \dfrac{12}{219\sqrt{\pi}}$ and $\beta = \dfrac{1}{17e^{11}}$,
which implies that $\dfrac{\mathfrak{b}^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)} \approx 0.11798$ and $\dfrac{\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)} \approx 1.0488$.
By simple calculations, we see that

$$\frac{\mathfrak{b}^{\alpha}}{\Gamma(\alpha+1)}\left(\gamma + \frac{\beta\mathfrak{b}^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)}\right) \approx 0,032423 < \frac{1}{2}.$$

With the use of Theorem 3.6 our problem has at least one solution.

Otherwise for each $\chi, \bar{\chi}, \upsilon, \bar{\upsilon} \in \mathfrak{R}$ and $\theta \in \mathfrak{J}$, we have

$$|f(\theta,\chi,\upsilon) - f(\theta,\bar{\chi},\overline{\upsilon})| \geq \frac{3}{73\sqrt{\pi}}|\chi - \bar{\chi}| - \frac{1}{17e^{11}}|\upsilon - \overline{\upsilon}|,$$

then (H2) is satisfied with $\overline{\gamma} = \frac{3}{73\sqrt{\pi}}$ and $\overline{\beta} = \dfrac{1}{17e^{11}}$.
Which implies that

$$\frac{2\mathfrak{b}}{\Gamma(1-\alpha)(2-\alpha)\Gamma(\alpha+1)} + \frac{\overline{\beta}\mathfrak{b}^{1-\alpha}}{\overline{\gamma}\Gamma(1-\alpha)(2-\alpha)} \approx 0,24751 < 1.$$

With the use of Theorem 3.7 our problem has a unique solution.

References

[1] S. Abbas, B. Ahmad, M. Benchohra and A. Salim, *Fractional Difference, Differential Equations and Inclusions: Analysis and Stability*, Morgan Kaufmann, Cambridge, 2024. https://doi.org/10.1016/C2023-0-00030-9

[2] R. S. Adiguzel, U. Aksoy, E. Karapinar, I. M. Erhan, On The Solutions Of Fractional Differential Equations Via Geraghty Type Hybrid Contractions, *Appl. Comput. Math.* **20** (2021), 313-333.

[3] D. Baleanu, Z. B. Güvenç, and J. A. T. Machado *New Trends in Nanotechnology and Fractional Calculus Applications*, Springer, New York, 2010.

[4] F. Chen, D. Baleanu, and G. Wu, Existence results of fractional differential equations with Riesz-Caputo derivative, *Eur. Phys. J.* **226** (2017), 3411-3425.

[5] F. Chen, A. Chen, and X. Wu, Anti-periodic boundary value problems with Riesz-Caputo derivative, *Adv. Difference Equ.* **2019** (2019). https://doi.org/10.1186/s13662-019-2001-z

[6] M. Benchohra, S. Bouriah, A. Salim and Y. Zhou, *Fractional Differential Equations: A Coincidence Degree Approach*, De Gruyter, Berlin, Boston, 2024. https://doi.org/10.1515/9783111334387

[7] M. Benchohra, E. Karapınar, J. E. Lazreg and A. Salim, *Advanced Topics in Fractional Differential Equations: A Fixed Point Approach*, Springer, Cham, 2023. https://doi.org/10.1007/978-3-031-26928-8

[8] M. Benchohra, E. Karapınar, J. E. Lazreg and A. Salim, *Fractional Differential Equations: New Advancements for Generalized Fractional Derivatives*, Springer, Cham, 2023. https://doi.org/10.1007/978-3-031-34877-8

[9] J. Dehong, M.Yuan and G. Weigao, A Singular Fractional Differential Equation with Riesz-Caputo Derivative, *Journal of Applied Analysis and Computation*, Volume 14, Number 2,(April 2024), 642-656.

[10] R.E. Gaines and J. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Lecture Notes in Math., vol. 568, Springer-Verlag, Berlin, 1977.

[11] J. R. Graef, J. Henderson and A. Ouahab, *Impulsive Differential Inclusions. A Fixed Point Approch*, De Gruyter, Berlin/Boston, 2013.

[12] C. Y. Gu, G. C. Wu, Positive solutions of fractional differential equations with the Riesz space derivative. *Appl. Math. Lett.* **95** (2019), 59-64.

[13] E. Karapınar, R. Sevinik-Adıgüzel, Ü. Aksoy, İ.M. Erhan, A new approach to the existence and uniqueness of solutions for a class of nonlinear q-fractional boundary value problems, *Appl. Comput. Math.* (2025), 235-249. https://doi.org/10.30546/1683-6154.24.2.2025.235

[14] A. A. Kilbas, Hari M. Srivastava and Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[15] S. Krim, A. Salim and M. Benchohra, Nonlinear contractions and Caputo tempered implicit fractional differential equations in b-metric spaces with infinite delay. *Filomat.* **37** (22) (2023), 7491-7503. https://doi.org/10.2298/FIL2322491K

[16] N. Laledj, A. Salim, J. E. Lazreg, S. Abbas, B. Ahmad and M. Benchohra, On implicit fractional q-difference equations: Analysis and stability. *Math. Meth. Appl. Sci.* **2** (2022), 1-23. https://doi.org/10.1002/mma.8417

[17] M. Li and Y. Wang, Existence and iteration of monotone positive solutions for fractional boundary value problems with Riesz-Caputo derivative. *Engineering Letters* **29** (2021), 1-5.

[18] J. Mawhin, *NSFCBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, 1979.

[19] A. Naas, M. Benbachir, M. S. Abdo and A. Boutiara, Analysis of a fractional boundary value problem involving Riesz-Caputo fractional derivative. *ATNAA.* **1** (2022), 14-27.

[20] D. O'Regan, Y.J. Chao and Y.Q. Chen, *Topological Degree Theory and Application,* Taylor and Francis Group, Boca Raton, London, New York, 2006.

[21] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[22] A. Salim, S. Abbas, M. Benchohra and E. Karapinar, A Filippov's theorem and topological structure of solution sets for fractional q-difference inclusions. *Dynam. Syst. Appl.* **31** (2022), 17-34. https://doi.org/10.46719/dsa202231.01.02

[23] A. Salim, S. Abbas, M. Benchohra and E. Karapinar, Global stability results for Volterra-Hadamard random partial fractional integral equations. *Rend. Circ. Mat. Palermo (2).* (2022), 1-13. https://doi.org/10.1007/s12215-022-00770-7

[24] A. Salim, M. Benchohra and J. E. Lazreg, Nonlocal k-generalized ψ-Hilfer impulsive initial value problem with retarded and advanced arguments, *Appl. Anal. Optim.* **6** (2022), 21-47.

[25] A. Salim, S. Bouriah, M. Benchohra, J. E. Lazreg and E. Karapınar, A study on k-generalized ψ-Hilfer fractional differential equations with periodic integral conditions. *Math. Methods Appl. Sci.* (2023), 1-18. https://doi.org/10.1002/mma.9056

[26] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Amsterdam, 1987, Engl. Trans. from the Russian.

[27] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.