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A Unified Framework for Common Fixed Point Results in Suprametric Spaces with Applications to Volterra Integral Equations

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Abstract

This paper develops a systematic common fixed point theory within the framework of suprametric spaces, a recently introduced generalization of metric spaces that accommodates a broader class of distance structures through a nonlinear relaxation of the triangle inequality. We establish several common fixed point theorems for pairs of self-mappings satisfying distinct contractive conditions, specifically Banach-type, Reich-type, Ćirić-type, and rational Gupta–Saxena type contractions, in the setting of complete suprametric spaces. Each theoretical result is accompanied by a concrete example that explicitly verifies the respective contractive hypothesis and confirms the existence and uniqueness of the common fixed point. To demonstrate the practical utility of the developed framework, the main results are applied to establish the existence and uniqueness of solutions to Volterra integral equations of the second kind, modeled via associated integral operators on the space of continuous functions endowed with a suprametric. The suprametric structure, by encompassing classical metric and b-metric spaces as special cases, renders the presented results both comprehensive in scope and broadly applicable to nonlinear problems in analysis and mathematical physics.

Keywords: common fixed point, suprametric space, Volterra integral equation

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1. Introduction and Basic Concepts

Fixed point theory is a fundamental area of nonlinear analysis with wide-ranging applications in mathematics, economics, and engineering. The foundation of this theory was laid by Banach [8], whose celebrated contraction principle established that every contraction mapping on a complete metric space possesses a unique fixed point, providing a powerful tool for proving the existence and uniqueness of solutions to various equations. Building upon this landmark result, Kannan [16] introduced a new class of mappings that extended the scope of fixed point theorems beyond the continuity requirement imposed by Banach's original condition. Reich [18] subsequently offered further generalizations by considering a broader class of contraction-type mappings that unified several existing results under a single framework. In the same period, Chatterjea [12] proposed a symmetric contractive condition that complemented the work of Kannan and enriched the landscape of fixed point theory, while Ćirić [13] investigated generalized multi-valued contractions, extending fixed point results to set-valued mappings. Shortly thereafter, Hardy and Rogers [15] presented a generalization of Reich's theorem that encompassed a wider family of contractive conditions, offering a more flexible and comprehensive framework. The culmination of these early developments came with the seminal contribution of Ćirić [14], who introduced the notion of a quasi-contraction one of the most general single-valued contractive conditions, thereby unifying and extending nearly all previously known results in the field.

Suprametric spaces develop into a new mathematical field that builds upon traditional metric spaces and becomes increasingly popular. Berzig [9] established fundamental research for his field in 2022 when he presented the formal definition of suprametric spaces together with their essential topological and analytical characteristics. The research established two major findings that described the structural framework of suprametric spaces while demonstrating their practical applications through initial research. Developing these ideas in 2023, Berzig [11] deeply explains such points in a paper that is entirely dedicated to nonlinear contractions on the b , suprametric spaces. Following this research, several delicate contractive behaviors were revealed, which are in fact very different from those of the classical metric and the b -metric spaces and these serve as pretty good illustrations of the richness and the versatility of the suprametric framework. Simultaneously, S. K. Panda et al. [17] proposed the concept of extended suprametric spaces and provided correlations with Stone-type results. In this work, the authors have firmly established the mathematical bases of suprametric spaces by linking them to well-known topological results, thus enhancing their mathematical utility and leading to abstract analysis. The progress of this study was continued by Berzig [10], who provided additional fixed-point results in generalized suprametric spaces. Those results were improvements on previous works and opened the way towards a more profound understanding of the existence and behavior when structural conditions are relaxed. Such results underscore the effectiveness of suprametric systems to address nonlinear problems that cannot be addressed through traditional measures of metrics. Ahmad et al. [1, 2, 6] examined the concept of multivalued and proximal contraction in suprametric contexts, where convergence and the closest proximity point were proved, and they were directly applied to nonlinear fractional differential equations and two-dimensional Volterra integral equations. Younis et al. [19] employ a fixed point approach to discuss the existence and uniqueness of the solutions of Chua's attractor model within the framework of suprametric spaces, incorporating the Atangana–Baleanu derivative and a two-step Lagrange polynomial approximation. Aldosari et al. [7] introduced a strong extended s -suprametric space as a new mathematical concept, which extends existing suprametric systems and provides a framework to study fixed points. This advancement enables rigorous treatment of nonlinear boundary value problems, including applications in chemical diffusion and satellite coupling systems, which show both theoretical and practical importance.

Despite the rapid progress in fixed point theory within suprametric spaces, the study of common fixed points for pairs of self-mappings under diverse generalized contractive conditions in this setting remains largely unexplored. In particular, Reich-type, Ćirić-type, and rational Gupta–Saxena type contractions have not

yet been systematically investigated in the suprametric framework, and their potential to address applied problems, such as integral equations arising in mathematical physics and engineering — has not been fully realized. Motivated by this gap, the present paper aims to establish a unified collection of common fixed point theorems in complete suprametric spaces, covering a spectrum of contractive conditions that generalize the classical Banach contraction principle and its well-known extensions. Specifically, we prove existence and uniqueness results under Banach-type (Theorem 2.1), Reich-type (Theorem 2.2), Ćirić-type (Theorem 2.4), and rational Gupta-Saxena type (Theorem 2.5) contractions, each supported by a detailed illustrative example. The theoretical framework is then applied to establish the existence and uniqueness of solutions to Volterra integral equations of the second kind, confirming the analytical depth and practical relevance of suprametric fixed point theory.

Barzig [9] introduced the notion of a *suprametric space*, which is defined as follows:

Definition 1.1. [9] Consider a mapping $d : X \times X \rightarrow \mathbb{R}^+$ on a nonempty set X satisfying the following conditions:

1. $d(x, y) \geq 0$,
2. $d(x, y) = 0 \iff x = y$,
3. $d(x, y) = d(y, x)$,
4. $d(x, z) \leq d(x, y) + d(y, z) + \hbar d(x, y)d(y, z)$

for all $x, y, z \in X$ and $\hbar \in \mathbb{R}^+$. Then the pair (X, d) is called a suprametric space.

Every metric naturally satisfies the conditions of a suprametric, but one can also generate suprametrics from existing metrics by suitably altering or weakening the triangle inequality. Various constructions allow defining such functions that maintain the essential properties of a distance, while providing more flexibility. This approach facilitates the investigation of fixed point results in spaces more general than conventional metric spaces.

Example 1.2. Let (X, d) be a metric space, and let $\eta, \mu > 0$ be fixed positive numbers. Define the following two functions on X :

1. $d_1^\eta(x, y) = d(x, y) (d(x, y) + \eta)$,
2. $d_2^\mu(x, y) = \mu (e^{d(x, y)} - 1)$.

Then d_1^η and d_2^μ are suprametrics with constants $\hbar = \frac{2}{\eta}$ and $\hbar = \frac{1}{\mu}$, respectively.

For a specific instance, take $X = \mathbb{R}$ with $d(x, y) = |x - y|$. By choosing $d = d_1^1$ or $d = d_2^1$, these functions fail to satisfy the usual triangle inequality, as

$$d(0, 1) + d(1, 2) < d(0, 2).$$

Additionally, in this example, d is not a suprametric with constant $\hbar = \frac{1}{3}$, demonstrating that any suprametric with a given constant \hbar is automatically a suprametric for all constants $\hbar' > \hbar$.

Definition 1.3. Let (X, d) be a suprametric space.

(i) A sequence $\{x_n\}$ is said to converge to $x \in X$ if, for any $\epsilon > 0$, there exists a positive integer N_ϵ such that $d(x_n, x) < \epsilon$ for all $n \geq N_\epsilon$. Symbolically,

$$\lim_{n \rightarrow \infty} x_n = x.$$

(ii) The sequence $\{x_n\}$ is called a Cauchy sequence if, for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $m, n \geq N_\epsilon$.

(iii) The space is said to be complete if every Cauchy sequence in X converges to a point in X .

(iv) If d is continuous, then each convergent sequence has a unique limit.

The study of fixed point theory has been significantly shaped by a series of classical contractive conditions introduced by several mathematicians over the decades. Beginning with the foundational work of Banach, numerous generalizations have been proposed to extend the applicability of fixed point results to broader classes of mappings. The following definition presents some of the most celebrated contractive conditions in metric spaces.

Definition 1.4. Let $T : X \rightarrow X$ be a mapping on a metric space (X, d) . For all $x, y \in X$, the mapping T is said to satisfy one of the following contractive conditions:

i **Banach Contraction** [8]: There exists $\alpha \in [0, 1)$ such that

$$d(T(x), T(y)) \leq \alpha d(x, y).$$

ii **Kannan Contraction** [16]: There exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(T(x), T(y)) \leq \alpha [d(x, T(x)) + d(y, T(y))].$$

iii **Chatterjea Contraction** [12]: There exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(T(x), T(y)) \leq \alpha [d(x, T(y)) + d(y, T(x))].$$

iv **Reich Contraction** [18]: There exist $\alpha_1, \alpha_2, \alpha_3 \in [0, 1)$ with $\alpha_1 + \alpha_2 + \alpha_3 < 1$ such that

$$d(T(x), T(y)) \leq \alpha_1 d(x, y) + \alpha_2 d(x, T(x)) + \alpha_3 d(y, T(y)).$$

v **Hardy–Rogers Contraction** [15]: There exists nonnegative constants with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$ such that

$$d(T(x), T(y)) \leq \alpha_1 d(x, y) + \alpha_2 d(x, T(x)) + \alpha_3 d(y, T(y)) + \alpha_4 d(y, T(x)) + \alpha_5 d(x, T(y)).$$

vi **Ćirić Contraction** [13]: There exists $\alpha \in [0, 1)$ such that

$$d(T(x), T(y)) \leq \alpha \max \left\{ d(x, y), d(x, T(x)), d(y, T(y)), d(x, T(y)), d(y, T(x)) \right\}.$$

If T satisfies any one of the above conditions, then it admits a fixed point.

2. Common Fixed Point Results

In this section, we establish the main results of the paper by proving several common fixed point theorems in the setting of complete suprametric spaces. We consider different classes of contractive conditions, including Reich-type, maximum-type, and rational Gupta–Saxena type contractions. These results extend and generalize existing fixed point principles to a broader framework. The obtained theorems ensure the existence and uniqueness of common fixed points for pairs of self-mappings under suitable assumptions.

Theorem 2.1. Let (X, d) be a complete suprametric space and $F, G : X \rightarrow X$ be a pair of self-mappings is said to be Banach type contraction if there exists a constant $\lambda \in (0, 1)$ such that

$$d(Fx, Gy) \leq \lambda d(x, y), \quad \forall u, v \in X. \quad (1)$$

Then F and G possess a unique common fixed point in X .

Proof. Choose an arbitrary point $x_0 \in X$ and define a sequence $\{x_n\}_{n=0}^\infty$ by

$$x_{2n+1} = Fx_{2n}, \quad x_{2n+2} = Gx_{2n+1}, \quad n = 0, 1, 2, \dots$$

Applying (1) repeatedly yields

$$\begin{aligned} d(x_1, x_2) &= d(Fx_0, Gx_1) \leq \lambda d(x_0, x_1), \\ d(x_2, x_3) &= d(Fx_2, Gx_1) \leq \lambda d(x_2, x_1) = \lambda d(x_1, x_2) \leq \lambda^2 d(x_0, x_1), \\ d(x_3, x_4) &= d(Fx_2, Gx_3) \leq \lambda d(x_2, x_3) \leq \lambda^3 d(x_0, x_1), \end{aligned}$$

and by induction

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1), \quad \text{for all } n \geq 0. \tag{2}$$

This implies that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$, so there exists $k \in \mathbb{N}$ such that for all $n \geq k$, we have

$$d(x_n, x_{n+1}) \leq 1. \tag{3}$$

To show that $\{x_n\}$ is Cauchy, let $m, n \in \mathbb{N}$ with $m > n > k$. Using the suprametric inequality and (2),

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_m) + \hbar d(x_n, x_{n+1}) d(x_{n+1}, x_m) \\ &\leq \lambda^n d(x_0, x_1) + d(x_{n+1}, x_m) + \hbar \lambda^n d(x_0, x_1) d(x_{n+1}, x_m) \\ &= \lambda^n d(x_0, x_1) + [1 + \hbar \lambda^n d(x_0, x_1)] d(x_{n+1}, x_m). \end{aligned}$$

Apply the same estimate to $d(x_{n+1}, x_m)$:

$$d(x_{n+1}, x_m) \leq \lambda^{n+1} d(x_0, x_1) + [1 + \hbar \lambda^{n+1} d(x_0, x_1)] d(x_{n+2}, x_m).$$

Substituting back,

$$\begin{aligned} d(x_n, x_m) &\leq \lambda^n d(x_0, x_1) + [1 + \hbar \lambda^n d(x_0, x_1)] \lambda^{n+1} d(x_0, x_1) \\ &\quad + [1 + \hbar \lambda^n d(x_0, x_1)] [1 + \hbar \lambda^{n+1} d(x_0, x_1)] d(x_{n+2}, x_m). \end{aligned}$$

Continuing this process and using (3) in every term of the sum, until we obtain

$$d(x_n, x_m) \leq d(x_0, x_1) \sum_{i=0}^{m-n-1} \lambda^{n+i} \prod_{j=0}^{i-1} [1 + \hbar \lambda^{n+j} d(x_0, x_1)]. \tag{4}$$

Since $\lambda \in (0, 1)$, it follows that

$$d(x_n, x_m) \leq d(x_0, x_1) \lambda^n \sum_{i=0}^{m-n-1} \lambda^i \prod_{j=0}^{i-1} [1 + \hbar \lambda^j d(x_0, x_1)]. \tag{5}$$

Now, it is not difficult to see that the series $\sum_{i=0}^\infty u_i$ converges, where

$$u_i = \lambda^i \prod_{j=0}^{i-1} [1 + \hbar \lambda^j d(x_0, x_1)].$$

Hence, we deduce that $d(x_n, x_m)$ tends to zero as n, m tend to infinity, which implies that the sequence $\{x_n\}$ is Cauchy. By completeness of (X, d) , there exists $z \in X$ such that $\lim_{n \rightarrow \infty} x_n = z$. In particular

$$\lim_{n \rightarrow \infty} x_{2n} = z, \quad \lim_{n \rightarrow \infty} x_{2n+1} = z. \tag{6}$$

We now prove that z is a common fixed point of F and G . Using the suprametric inequality and (1),

$$\begin{aligned} d(z, Fz) &\leq d(z, x_{2n+2}) + d(Fz, Gx_{2n+1}) + \hbar d(z, x_{2n+2}) d(Fz, Gx_{2n+1}) \\ &\leq d(z, x_{2n+2}) + \lambda d(z, x_{2n+1}) + \hbar \lambda d(z, x_{2n+2}) d(z, x_{2n+1}). \end{aligned}$$

Letting $n \rightarrow \infty$ and invoking (6) yields $d(z, Fz) = 0$; hence $Fz = z$. Similarly,

$$\begin{aligned} d(z, Gz) &\leq d(z, x_{2n+1}) + d(Fx_{2n}, Gz) + \hbar d(z, x_{2n+1}) d(Fx_{2n}, Gz) \\ &\leq d(z, x_{2n+1}) + \lambda d(x_{2n}, z) + \hbar \lambda d(z, x_{2n+1}) d(x_{2n}, z) \end{aligned}$$

gives, after taking the limit, $d(z, Gz) = 0$ and therefore $Gz = z$.

For **uniqueness**, if $w \in X$ is another common fixed point of F and G , then (1) implies

$$d(z, w) = d(Fz, Gw) \leq \lambda d(z, w),$$

which forces $d(z, w) = 0$ because $\lambda \in (0, 1)$; hence $z = w$. Thus z is the unique common fixed point of F and G . □

Reich-type contractions generalize the classical Banach contraction principle by incorporating additional control terms involving the images of the mappings. Such conditions are particularly useful in the study of nonlinear operators where standard contractive assumptions may fail. In the setting of suprametric spaces, these generalized contractions retain sufficient structure to ensure convergence of iterative schemes. The following result establishes the existence and uniqueness of a common fixed point for a pair of mappings under a Reich-type contractive condition.

Theorem 2.2. *Let (X, d) be a complete suprametric space and $F, G : X \rightarrow X$ be a pair of-mappings satisfying the Reich-type contraction condition, there exist constants $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$ such that*

$$d(Fu, Gv) \leq \alpha d(u, v) + \beta d(u, Fu) + \gamma d(v, Gv), \quad \forall u, v \in X. \tag{7}$$

Then F and G have a unique common fixed point in X .

Proof. Choose an arbitrary point $x_0 \in X$ and define a sequence $\{x_n\}_{n=0}^\infty$ by

$$x_{2n+1} = Fx_{2n}, \quad x_{2n+2} = Gx_{2n+1}, \quad n = 0, 1, 2, \dots$$

Applying (7) repeatedly yields

$$\begin{aligned} d(x_1, x_2) &= d(Fx_0, Gx_1) \\ &\leq \alpha d(x_0, x_1) + \beta d(x_0, Fx_0) + \gamma d(x_1, Gx_1) \\ &= \alpha d(x_0, x_1) + \beta d(x_0, x_1) + \gamma d(x_1, x_2). \end{aligned}$$

Rearranging, we obtain

$$(1 - \gamma) d(x_1, x_2) \leq (\alpha + \beta) d(x_0, x_1),$$

so that

$$d(x_1, x_2) \leq \frac{\alpha + \beta}{1 - \gamma} d(x_0, x_1).$$

Set $k_1 = \frac{\alpha + \beta}{1 - \gamma}$ (note that $k_1 < 1$ since $\alpha + \beta + \gamma < 1$). Similarly,

$$\begin{aligned} d(x_2, x_3) &= d(Fx_2, Gx_1) \\ &\leq \alpha d(x_2, x_1) + \beta d(x_2, Fx_2) + \gamma d(x_1, Gx_1) \\ &= \alpha d(x_1, x_2) + \beta d(x_2, x_3) + \gamma d(x_1, x_2), \end{aligned}$$

which gives

$$(1 - \beta) d(x_2, x_3) \leq (\alpha + \gamma) d(x_1, x_2),$$

hence

$$d(x_2, x_3) \leq \frac{\alpha + \gamma}{1 - \beta} d(x_1, x_2).$$

Set $k_2 = \frac{\alpha + \gamma}{1 - \beta}$ (also $k_2 < 1$). Define $k = \max\{k_1, k_2\}$. Then $k < 1$ and we have

$$d(x_1, x_2) \leq k d(x_0, x_1), \quad d(x_2, x_3) \leq k d(x_1, x_2) \leq k^2 d(x_0, x_1).$$

By induction, we obtain

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1), \quad \text{for all } n \geq 0. \tag{8}$$

To show that $\{x_n\}$ is a Cauchy sequence for $m, n \in \mathbb{N}$ with $m > n$, we will follow Theorem 2.1. By completeness of (X, d) , there exists $z \in X$ such that $\lim_{n \rightarrow \infty} x_n = z$. In particular

$$\lim_{n \rightarrow \infty} x_{2n} = z, \quad \lim_{n \rightarrow \infty} x_{2n+1} = z. \tag{9}$$

We now prove that z is a common fixed point of F and G . Using the suprametric inequality and (7),

$$\begin{aligned} d(z, Fz) &\leq d(z, x_{2n+2}) + d(Fz, Gx_{2n+1}) + \hbar d(z, x_{2n+2}) d(Fz, Gx_{2n+1}) \\ &\leq d(z, x_{2n+2}) + \alpha d(z, x_{2n+1}) + \beta d(z, Fz) + \gamma d(x_{2n+1}, x_{2n+2}) \\ &\quad + \hbar d(z, x_{2n+2}) [\alpha d(z, x_{2n+1}) + \beta d(z, Fz) + \gamma d(x_{2n+1}, x_{2n+2})]. \end{aligned}$$

Letting $n \rightarrow \infty$ and invoking (9) together with $d(x_{2n+1}, x_{2n+2}) \rightarrow 0$, we obtain

$$d(z, Fz) \leq \beta d(z, Fz).$$

Since $\beta \leq \alpha + \beta + \gamma < 1$, this implies $d(z, Fz) = 0$; hence $Fz = z$. Similarly,

$$\begin{aligned} d(z, Gz) &\leq d(z, x_{2n+1}) + d(Fx_{2n}, Gz) + \hbar d(z, x_{2n+1}) d(Fx_{2n}, Gz) \\ &\leq d(z, x_{2n+1}) + \alpha d(x_{2n}, z) + \beta d(x_{2n}, x_{2n+1}) + \gamma d(z, Gz) \\ &\quad + \hbar d(z, x_{2n+1}) [\alpha d(x_{2n}, z) + \beta d(x_{2n}, x_{2n+1}) + \gamma d(z, Gz)]. \end{aligned}$$

Passing to the limit $n \rightarrow \infty$ yields

$$d(z, Gz) \leq \gamma d(z, Gz),$$

and since $\gamma < 1$, we deduce $d(z, Gz) = 0$; hence $Gz = z$.

For **uniqueness**, if $w \in X$ is another common fixed point of F and G , then (7) implies

$$d(z, w) = d(Fz, Gw) \leq \alpha d(z, w) + \beta d(z, Fz) + \gamma d(w, Gw) = \alpha d(z, w).$$

Because $\alpha < 1$, this forces $d(z, w) = 0$; hence $z = w$. Thus z is the unique common fixed point of F and G . □

To illustrate the applicability of Theorem 2.2, we present the following example. It demonstrates how the Reich-type contraction condition can be verified in a concrete setting within a suprametric space. Moreover, it confirms the existence and uniqueness of a common fixed point for the given pair of mappings.

Example 2.3. Let $X = [0, 5]$ and define $d : X \times X \rightarrow [0, \infty)$ by

$$d(x, y) = |x - y|(|x - y| + \hbar), \quad \hbar = 2.$$

For a metric space, the triangle inequality must hold:

$$d(x, z) \leq d(x, y) + d(y, z).$$

Take $x = 0, y = 1, w = 2$. Then

$$d(0, 2) = 2(2 + \hbar) = 4 + 2\hbar,$$

$$d(0, 1) + d(1, 2) = 1(1 + \hbar) + 1(1 + \hbar) = 2 + 2\hbar.$$

Thus,

$$d(0, 2) = 4 + 2\hbar > 2 + 2\hbar = d(0, 1) + d(1, 2),$$

so the triangle inequality fails. Hence (X, d) is not a metric space. Let $a = |x - y|$ and $b = |y - z|$. Since $|x - z| \leq a + b$, we have

$$d(x, w) \leq (a + b)(a + b + \hbar).$$

We show that

$$(a + b)(a + b + \hbar) \leq a(a + \hbar) + b(b + \hbar) + \hbar a(a + \hbar) b(b + \hbar).$$

Expanding:

$$(a + b)(a + b + \hbar) = a^2 + 2ab + b^2 + \hbar(a + b),$$

$$a(a + \hbar) + b(b + \hbar) = a^2 + b^2 + \hbar(a + b).$$

Thus, we arrive at

$$d(x, z) \leq d(x, y) + d(y, w) + 2ab.$$

Now, we have

$$2ab \leq \hbar a(a + \hbar) b(b + \hbar) = \hbar d(x, y) d(y, z),$$

since $a + \hbar \geq \hbar$ and $b + \hbar \geq \hbar$. Hence, we achieve

$$d(x, z) \leq d(x, y) + d(y, z) + \hbar d(x, y) d(y, z),$$

so (X, d) is a suprametric space. Define $F, G : X \rightarrow X$ by

$$F(x) = \begin{cases} 0, & 0 \leq x < 2, \\ \frac{x}{4}, & 2 \leq x \leq 5, \end{cases} \quad G(y) = \begin{cases} 0, & 0 \leq y < 3, \\ \frac{y}{5}, & 3 \leq y \leq 5. \end{cases}$$

Now, we have to verify the Reich contraction mentioned in 2.2, as follows Take

$$\alpha = \frac{1}{5}, \quad \beta = \frac{1}{4}, \quad \gamma = \frac{1}{6}, \quad \alpha + \beta + \gamma < 1.$$

We verify

$$d(Fx, Gy) \leq \alpha d(x, y) + \beta d(x, Fx) + \gamma d(y, Gy).$$

Case 1: For $x \in [0, 2)$ and $y \in [0, 3)$, we have $F(x) = 0$ and $G(y) = 0$. Consequently, it follows that:

$$d(Fx, Gy) = d(0, 0) = 0,$$

$$d(x, y) = |x - y|(|x - y| + 2),$$

$$d(x, Fx) = x(x + 2),$$

$$d(y, Gy) = y(y + 2).$$

Thus, after substituting we get

$$0 \leq \alpha d(x, y) + \beta x(x + 2) + \gamma y(y + 2),$$

which shows above inequality holds as shown in the Figure 1.

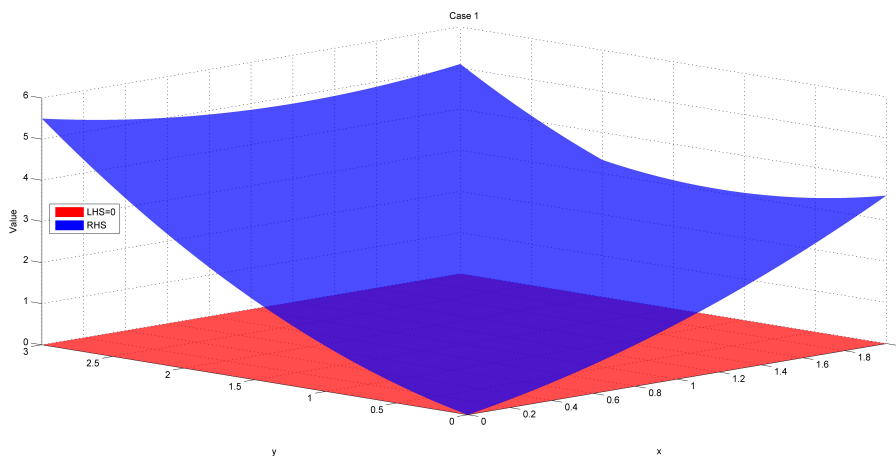


Figure 1: Graph related to Case 1 of Example 2.3

Case 2: For $x \in [0, 2)$ and $y \in [3, 5]$, we have $F(x) = 0$ and $G(y) = \frac{y}{5}$. Since, we arrive at

$$d(Fx, Gy) = d\left(0, \frac{y}{5}\right) = \frac{y}{5} \left(\frac{y}{5} + 2\right) = \frac{y^2}{25} + \frac{2y}{5},$$

$$d(x, y) = |x - y|(|x - y| + 2),$$

$$d(x, Fx) = x(x + 2),$$

$$d(y, Gy) = \left|\frac{4y}{5}\right| \left(\frac{4y}{5} + 2\right) = \frac{16y^2}{25} + \frac{8y}{5}.$$

After substituting we get

$$\left(\frac{y^2 + 10y}{25}\right) \leq \frac{1}{5}|x - y|(|x - y| + 2) + \frac{1}{4}x(x + 2) + \frac{1}{6} \left(\frac{16y^2}{25} + \frac{8y}{5}\right).$$

the above inequality holds as shown in the Figure 2.

Case 3: For $x \in [2, 5]$ and $y \in [0, 3)$, we have $F(x) = \frac{x}{4}$ and $G(y) = 0$. Consequently, it follows that:

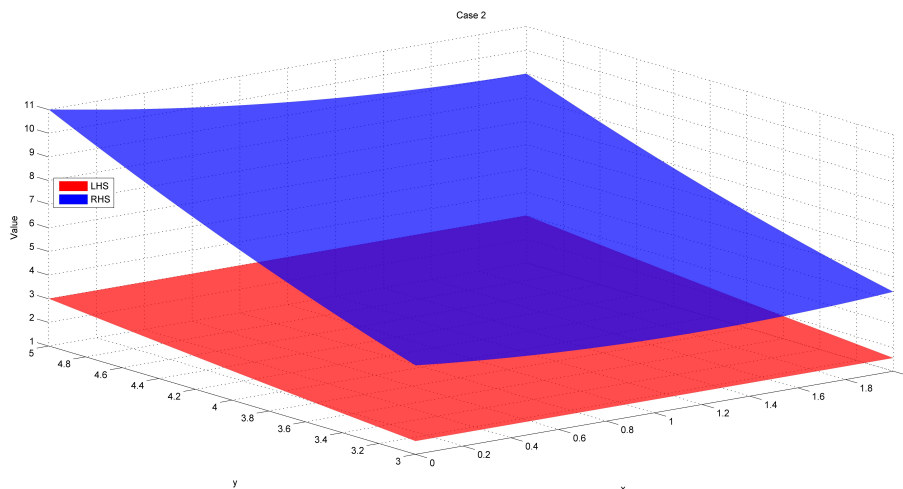


Figure 2: Graph related to Case 2 of Example 2.3

$$d(Fx, Gy) = \frac{x}{4} \left(\frac{x}{4} + 2 \right) = \frac{x^2}{16} + \frac{x}{2},$$

$$d(x, y) = |x - y|(|x - y| + 2),$$

$$d(x, Fx) = \frac{3x}{4} \left(\frac{3x}{4} + 2 \right) = \frac{9x^2}{16} + \frac{3x}{2},$$

$$d(y, Gy) = y(y + 2).$$

After substituting, we get

$$\left(\frac{x^2 + 8x}{16} \right) \leq \frac{1}{5}|x - y|(|x - y| + 2) + \frac{1}{4} \left(\frac{9x^2}{16} + \frac{3x}{2} \right) + \frac{1}{6}y(y + 2).$$

the above inequality holds for the given domain as shown in the Figure 3.

Case 4: For $x \in [2, 5]$ and $y \in [3, 5]$, we have $F(x) = \frac{x}{4}$ and $G(y) = \frac{y}{5}$. Since, we arrive at

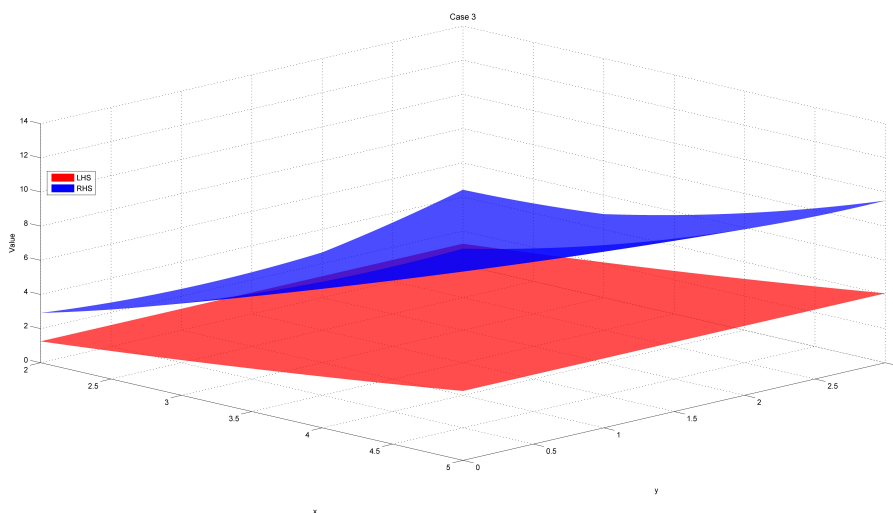


Figure 3: Graph related to Case 3 of Example 2.3

$$d(Fx, Gy) = \left| \frac{x}{4} - \frac{y}{5} \right| \left(\left| \frac{x}{4} - \frac{y}{5} \right| + 2 \right),$$

$$d(x, y) = |x - y|(|x - y| + 2),$$

$$d(x, Fx) = \frac{9x^2}{16} + \frac{3x}{2},$$

$$d(y, Gy) = \frac{16y^2}{25} + \frac{8y}{5}.$$

After substituting, we get

$$\left| \frac{x}{4} - \frac{y}{5} \right| \left(\left| \frac{x}{4} - \frac{y}{5} \right| + 2 \right) \leq \frac{1}{5}|x - y|(|x - y| + 2) + \frac{1}{4} \left(\frac{9x^2}{16} + \frac{3x}{2} \right) + \frac{1}{6} \left(\frac{16y^2}{25} + \frac{8y}{5} \right),$$

which shows above inequality holds for all given domain as shown in the Figure 4.

All conditions of Theorem 2.2 are satisfied. The mappings F and G admit a unique common fixed point 0 in the suprametric space (X, d) .

We now consider another important class of generalized contractions involving a Ćirić-type control. Such conditions are useful in capturing the dominant behavior among several distance terms and often

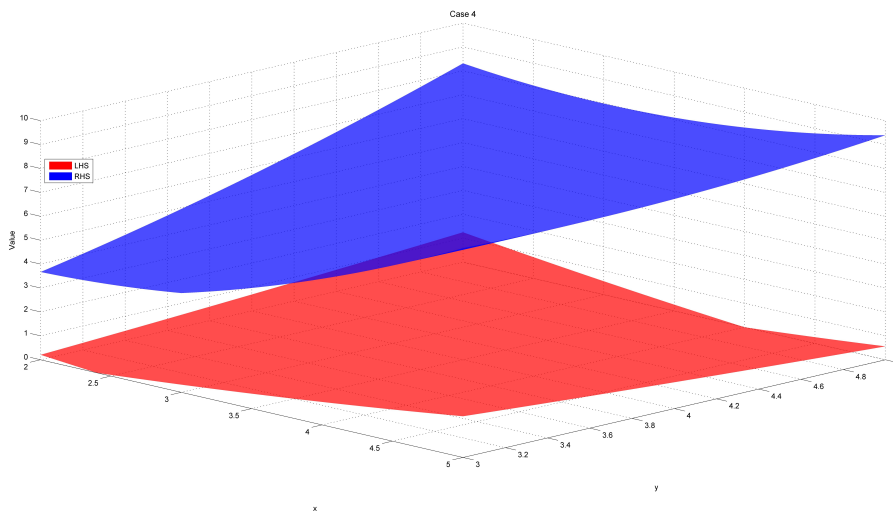


Figure 4: Graph related to Case 4 of Example 2.3

arise in nonlinear analysis. In the framework of suprametric spaces, these contractions still guarantee strong convergence properties. The following theorem establishes a unique common fixed point under this maximum-type contractive condition.

Theorem 2.4. *Let (X, d) be a complete suprametric space and $F, G : X \rightarrow X$ be a pair of-mappings satisfying is said to be Ćirić-type contraction there exists a constant $\lambda \in (0, 1)$ such that for all $u, v \in X$,*

$$d(Fx, Gy) \leq \lambda \max \{ d(x, y), d(x, Fx), d(y, Gy) \}. \tag{10}$$

Then F and G possess a unique common fixed point in X .

Proof. Choose an arbitrary point $x_0 \in X$ and define a sequence $\{x_n\}_{n=0}^\infty$ by

$$x_{2n+1} = Fx_{2n}, \quad x_{2n+2} = Gx_{2n+1}, \quad n = 0, 1, 2, \dots$$

For each $n \geq 0$, consider the distances $d(x_n, x_{n+1})$. Applying (10) with $x = x_{2n}$ and $y = x_{2n+1}$ yields

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Fx_{2n}, Gx_{2n+1}) \\ &\leq \lambda \max \{ d(x_{2n}, x_{2n+1}), d(x_{2n}, Fx_{2n}), d(x_{2n+1}, Gx_{2n+1}) \} \\ &= \lambda \max \{ d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \} \\ &= \lambda \max \{ d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \}. \end{aligned}$$

Similarly, with $x = x_{2n+2}$ and $y = x_{2n+1}$, we estimate

$$\begin{aligned} d(x_{2n+2}, x_{2n+3}) &= d(Fx_{2n+2}, Gx_{2n+1}) \\ &\leq \lambda \max \{ d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, Fx_{2n+2}), d(x_{2n+1}, Gx_{2n+1}) \} \\ &= \lambda \max \{ d(x_{2n+1}, x_{2n+2}), d(x_{2n+2}, x_{2n+3}), d(x_{2n+1}, x_{2n+2}) \} \\ &= \lambda \max \{ d(x_{2n+1}, x_{2n+2}), d(x_{2n+2}, x_{2n+3}) \}. \end{aligned}$$

Thus, for every $n \geq 0$ we have

$$d(x_{n+1}, x_{n+2}) \leq \lambda \max \{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}) \}. \tag{11}$$

The above inequality (11) have two possibilities as follows:

Case 1: If we choose $\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_n, x_{n+1})$. Then, we have

$$d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1}).$$

Case 2: If we pick $\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_{n+1}, x_{n+2})$. Then, we arrive at

$$d(x_{n+1}, x_{n+2}) \leq \lambda d(x_{n+1}, x_{n+2}),$$

which forces $d(x_{n+1}, x_{n+2}) = 0$ because $\lambda < 1$. Consequently $d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1})$ holds trivially. Hence in all above two cases, we obtain

$$d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1}) \quad (n \geq 0).$$

By induction, we conclude

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1) \quad \text{for all } n \geq 0. \tag{12}$$

To show that $\{x_n\}$ is a Cauchy sequence for $m, n \in \mathbb{N}$ with $m > n$, we will follow Theorem 2.1. By completeness of (X, d) , there exists $z \in X$ with $x_n \rightarrow z$; in particular

$$\lim_{n \rightarrow \infty} x_{2n} = z, \quad \lim_{n \rightarrow \infty} x_{2n+1} = z. \tag{13}$$

Now, we will show that z is a common fixed point of F and G . For this, we will utilize the suprametric inequality and the contraction condition (10),

$$\begin{aligned} d(z, Fz) &\leq d(z, x_{2n+2}) + d(Fz, Gx_{2n+1}) + \hbar d(z, x_{2n+2}) d(Fz, Gx_{2n+1}) \\ &\leq d(z, x_{2n+2}) + \lambda \max\left\{d(z, x_{2n+1}), d(z, Fz), d(x_{2n+1}, Gx_{2n+1})\right\} \\ &\quad + \hbar \lambda d(z, x_{2n+2}) \max\left\{d(z, x_{2n+1}), d(z, Fz), d(x_{2n+1}, Gx_{2n+1})\right\}. \end{aligned}$$

Now $d(x_{2n+1}, Gx_{2n+1}) = d(x_{2n+1}, x_{2n+2}) \leq \lambda^{2n+1} d(x_0, x_1) \rightarrow 0$ by equation (12), and $d(z, x_{2n+1}) \rightarrow 0$ by equation (13). Hence the maximum tends to $d(z, Fz)$. Passing to the limit as $n \rightarrow \infty$ gives

$$d(z, Fz) \leq 0 + \lambda d(z, Fz) + 0,$$

so that $(1 - \lambda)d(z, Fz) \leq 0$. Since $\lambda < 1$, we must have $d(z, Fz) = 0$ that is $Fz = z$. Now we show that $Gz = z$ for this we will utilize the suprametric inequality and equation (10) with $u = x_{2n}$ and $v = z$, we conclude

$$\begin{aligned} d(z, Gz) &\leq d(z, x_{2n+1}) + d(Fx_{2n}, Gz) + \hbar d(z, x_{2n+1}) d(Fx_{2n}, Gz) \\ &\leq d(z, x_{2n+1}) + \lambda \max\left\{d(x_{2n}, z), d(x_{2n}, Fx_{2n}), d(z, Gz)\right\} \\ &\quad + \hbar \lambda d(z, x_{2n+1}) \max\left\{d(x_{2n}, z), d(x_{2n}, Fx_{2n}), d(z, Gz)\right\}. \end{aligned}$$

Now $d(x_{2n}, Fx_{2n}) = d(x_{2n}, x_{2n+1}) \leq \lambda^{2n} d(x_0, x_1) \rightarrow 0$ by equation (12), and $d(x_{2n}, z) \rightarrow 0$ by equation (13). Hence the maximum tends to $d(z, Gz)$. Passing to the limit as $n \rightarrow \infty$ gives

$$d(z, Gz) \leq 0 + \lambda d(z, Gz) + 0,$$

so that $(1 - \lambda)d(z, Gz) \leq 0$. Since $\lambda < 1$, we must have $d(z, Gz) = 0$ that is $Gz = z$.

For **uniqueness**, assume that $w \in X$ is another common fixed point of F and G . Then

$$\begin{aligned} d(z, w) &= d(Fz, Gw) \leq \lambda \max\left\{d(z, w), d(z, Fz), d(w, Gw)\right\} \\ &= \lambda d(z, w), \end{aligned}$$

which forces $d(z, w) = 0$ because $\lambda < 1$; hence $z = w$. Thus F and G have a unique common fixed point in X . □

We further extend the discussion to rational-type contractions, which provide a more refined control by incorporating nonlinear combinations of distance terms. These contractions, inspired by the Gupta–Saxena framework, are particularly effective in handling mappings with complex interaction behavior. In suprametric spaces, such rational expressions still preserve the contractive structure required for convergence. The following theorem ensures the existence and uniqueness of a common fixed point under this rational Gupta–Saxena type condition.

Theorem 2.5. *Let (X, d) be a complete suprametric space and let $F, G : X \rightarrow X$ be a pair of-mappings referred to as rational Gupta-Saxena contraction. Suppose there exists a constant $\lambda \in (0, 1)$ such that for all $x, y \in X$,*

$$d(Fx, Gy) \leq \lambda \max \left\{ d(x, y), \frac{d(x, Fx)(1 + d(y, Gy))}{1 + d(x, y)}, \frac{d(y, Gy)(1 + d(x, Fx))}{1 + d(x, y)} \right\}. \tag{14}$$

Then F and G possess a unique common fixed point in X .

Proof. Choose an arbitrary point $x_0 \in X$ and define a sequence $\{x_n\}_{n=0}^\infty$ by

$$x_{2n+1} = Fx_{2n}, \quad x_{2n+2} = Gx_{2n+1}, \quad n = 0, 1, 2, \dots$$

We first derive a recursive estimate for $d(x_n, x_{n+1})$. For even indices, set $x = x_{2n}$ and $y = x_{2n+1}$. Then by utilizing the equation (14) as follows

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Fx_{2n}, Gx_{2n+1}) \\ &\leq \lambda \max \left\{ d(x_{2n}, x_{2n+1}), \frac{d(x_{2n}, Fx_{2n})(1 + d(x_{2n+1}, Gx_{2n+1}))}{1 + d(x_{2n}, x_{2n+1})}, d(x_{2n+1}, Gx_{2n+1}) \right\} \\ &= \lambda \max \left\{ d(x_{2n}, x_{2n+1}), \frac{d(x_{2n}, x_{2n+1})(1 + d(x_{2n+1}, x_{2n+2}))}{1 + d(x_{2n}, x_{2n+1})}, d(x_{2n+1}, x_{2n+2}) \right\}. \end{aligned}$$

For odd indices, set $x = x_{2n+2}$ and $y = x_{2n+1}$, we estimate

$$\begin{aligned} d(x_{2n+2}, x_{2n+3}) &= d(Fx_{2n+2}, Gx_{2n+1}) \\ &\leq \lambda \max \left\{ d(x_{2n+2}, x_{2n+1}), \frac{d(x_{2n+2}, Fx_{2n+2})(1 + d(x_{2n+1}, Gx_{2n+1}))}{1 + d(x_{2n+2}, x_{2n+1})}, \right. \\ &\quad \left. \frac{d(x_{2n+1}, Gx_{2n+1})(1 + d(x_{2n+2}, Fx_{2n+2}))}{1 + d(x_{2n+2}, x_{2n+1})} \right\} \\ &= \lambda \max \left\{ d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n+2}, x_{2n+3})(1 + d(x_{2n+1}, x_{2n+2}))}{1 + d(x_{2n+1}, x_{2n+2})}, \right. \\ &\quad \left. \frac{d(x_{2n+1}, x_{2n+2})(1 + d(x_{2n+2}, x_{2n+3}))}{1 + d(x_{2n+1}, x_{2n+2})} \right\} \\ &= \lambda \max \left\{ d(x_{2n+1}, x_{2n+2}), d(x_{2n+2}, x_{2n+3}), \frac{d(x_{2n+1}, x_{2n+2})(1 + d(x_{2n+2}, x_{2n+3}))}{1 + d(x_{2n+1}, x_{2n+2})} \right\}. \end{aligned}$$

Thus, for every $n \geq 0$ we obtain an inequality of the form

$$d(x_{n+1}, x_{n+2}) \leq \lambda \max \left\{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+1})(1 + d(x_{n+1}, x_{n+2}))}{1 + d(x_n, x_{n+1})} \right\}. \tag{15}$$

Now, we analyse the behaviour of the third term from the above inequality. For brevity, denote

$$A = d(x_n, x_{n+1}), \quad B = d(x_{n+1}, x_{n+2}).$$

Then the third term is

$$\frac{A(1+B)}{1+A}.$$

We show that this term always lies between A and B . Compute the differences:

$$\frac{A(1+B)}{1+A} - A = \frac{A(1+B) - A(1+A)}{1+A} = \frac{A(B-A)}{1+A},$$

and

$$B - \frac{A(1+B)}{1+A} = \frac{B(1+A) - A(1+B)}{1+A} = \frac{B-A}{1+A}.$$

We consider the two possible orderings of A and B :

Case $B \geq A$: Then $B - A \geq 0$, so

$$\frac{A(1+B)}{1+A} - A = \frac{A(B-A)}{1+A} \geq 0 \implies \frac{A(1+B)}{1+A} \geq A,$$

and

$$B - \frac{A(1+B)}{1+A} = \frac{B-A}{1+A} \geq 0 \implies \frac{A(1+B)}{1+A} \leq B.$$

Hence $A \leq \frac{A(1+B)}{1+A} \leq B$.

Case $B \leq A$: Then $B - A \leq 0$, so

$$\frac{A(1+B)}{1+A} - A = \frac{A(B-A)}{1+A} \leq 0 \implies \frac{A(1+B)}{1+A} \leq A,$$

and

$$B - \frac{A(1+B)}{1+A} = \frac{B-A}{1+A} \leq 0 \implies \frac{A(1+B)}{1+A} \geq B.$$

Hence $B \leq \frac{A(1+B)}{1+A} \leq A$.

In both cases the third term lies between A and B . In either case, the third term is between A and B . Consequently we can simplify the maximum in (15):

$$d(x_{n+1}, x_{n+2}) \leq \lambda \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}. \tag{16}$$

Consider the two possibilities for the maximum in equation (16), we have:

Case 1: If we choose $\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_{n+1}, x_{n+2})$. Then (16) becomes $d(x_{n+1}, x_{n+2}) \leq \lambda d(x_{n+1}, x_{n+2})$, which forces $d(x_{n+1}, x_{n+2}) = 0$ because $\lambda < 1$. Consequently, the inequality $d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1})$ holds trivially.

Case 2: If we pick $\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_n, x_{n+1})$. Then (16) yields directly $d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1})$.

Thus, in all cases

$$d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1}) \quad (n \geq 0).$$

By induction, we obtain

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1) \quad \text{for all } n \geq 0. \tag{17}$$

To show that $\{x_n\}$ is a Cauchy sequence for $m, n \in \mathbb{N}$ with $m > n$, we will follow Theorem 2.1. By completeness of (X, d) there exists $z \in X$ such that $\lim_{n \rightarrow \infty} x_n = z$; in particular

$$\lim_{n \rightarrow \infty} x_{2n} = z, \quad \lim_{n \rightarrow \infty} x_{2n+1} = z. \tag{18}$$

Now, we will show that z is a common fixed point of F and G . For this, we employ the suprametric inequality and contractive condition (14),

$$\begin{aligned} d(z, Fz) &\leq d(z, x_{2n+2}) + d(Fz, Gx_{2n+1}) + \hbar d(z, x_{2n+2}) d(Fz, Gx_{2n+1}) \\ &\leq d(z, x_{2n+2}) + \lambda M_n + \hbar \lambda d(z, x_{2n+2}) M_n, \end{aligned}$$

where

$$M_n = \max \left\{ d(z, x_{2n+1}), \frac{d(z, Fz)(1 + d(x_{2n+1}, Gx_{2n+1}))}{1 + d(z, x_{2n+1})}, \frac{d(x_{2n+1}, Gx_{2n+1})(1 + d(z, Fz))}{1 + d(z, x_{2n+1})} \right\}.$$

Now $d(x_{2n+1}, Gx_{2n+1}) = d(x_{2n+1}, x_{2n+2}) \leq \lambda^{2n+1} d(x_0, x_1) \rightarrow 0$ by equation (17), and $d(z, x_{2n+1}) \rightarrow 0$ by equation (18). Therefore

$$\frac{d(z, Fz)(1 + d(x_{2n+1}, x_{2n+2}))}{1 + d(z, x_{2n+1})} \rightarrow d(z, Fz), \quad \frac{d(x_{2n+1}, x_{2n+2})(1 + d(z, Fz))}{1 + d(z, x_{2n+1})} \rightarrow 0.$$

Hence $M_n \rightarrow d(z, Fz)$ as $n \rightarrow \infty$. Passing to the limit in the estimate for $d(z, Fz)$ yields

$$d(z, Fz) \leq 0 + \lambda d(z, Fz) + 0,$$

so that $(1 - \lambda)d(z, Fz) \leq 0$. Since $\lambda < 1$, we must have $d(z, Fz) = 0$ that is $Fz = z$.

Similarly, we will show that z is a fixed point of G . For this, we employ the suprametric inequality and equation (14) with $x = x_{2n}$ and $y = z$,

$$\begin{aligned} d(z, Gz) &\leq d(z, x_{2n+1}) + d(Fx_{2n}, Gz) + \hbar d(z, x_{2n+1}) d(Fx_{2n}, Gz) \\ &\leq d(z, x_{2n+1}) + \lambda N_n + \hbar \lambda d(z, x_{2n+1}) N_n, \end{aligned}$$

where

$$N_n = \max \left\{ d(x_{2n}, z), \frac{d(x_{2n}, Fx_{2n})(1 + d(z, Gz))}{1 + d(x_{2n}, z)}, \frac{d(z, Gz)(1 + d(x_{2n}, Fx_{2n}))}{1 + d(x_{2n}, z)} \right\}.$$

Now, $d(x_{2n}, Fx_{2n}) = d(x_{2n}, x_{2n+1}) \leq \lambda^{2n} d(x_0, x_1) \rightarrow 0$ by equation (17), and $d(x_{2n}, z) \rightarrow 0$ by equation (18). Therefore,

$$\frac{d(x_{2n}, x_{2n+1})(1 + d(z, Gz))}{1 + d(x_{2n}, z)} \rightarrow 0, \quad \frac{d(z, Gz)(1 + d(x_{2n}, x_{2n+1}))}{1 + d(x_{2n}, z)} \rightarrow d(z, Gz).$$

Hence $N_n \rightarrow \max\{0, 0, d(z, Gz)\} = d(z, Gz)$ as $n \rightarrow \infty$. Passing to the limit in the estimate for $d(z, Gz)$ and noting that $d(z, x_{2n+1}) \rightarrow 0$ yields

$$d(z, Gz) \leq 0 + \lambda d(z, Gz) + 0,$$

so that $(1 - \lambda)d(z, Gz) \leq 0$. Since $\lambda < 1$, we must have $d(z, Gz) = 0$ that is $Gz = z$.

For **uniqueness**, we assume that $w \in X$ is another common fixed point of F and G . Applying (14) with $x = z, y = w$ we obtain

$$\begin{aligned} d(z, w) = d(Fz, Gw) &\leq \lambda \max \left\{ d(z, w), \frac{d(z, Fz)(1 + d(w, Gw))}{1 + d(z, w)}, \frac{d(w, Gw)(1 + d(z, Fz))}{1 + d(z, w)} \right\} \\ &= \lambda d(z, w). \end{aligned}$$

Because $\lambda < 1$, this forces $d(z, w) = 0$; hence $z = w$. Thus F and G have a unique common fixed point in X . □

To demonstrate the applicability of Theorem 2.5, we present the following example. It verifies that the given mappings satisfy the rational Gupta–Saxena type contraction condition in a suprametric space. Consequently, the existence and uniqueness of a common fixed point are ensured. This example highlights the effectiveness of the rational framework in dealing with nonlinear mappings.

Example 2.6. Let $X = [0, 5]$ and define $d : X \times X \rightarrow [0, \infty)$ by

$$d(x, y) = |x - y|(|x - y| + 2).$$

Then (X, d) is a complete suprametric space (as discussed in 2.3). Define two discontinuous self-mappings $F, G : X \rightarrow X$ by

$$F(x) = \begin{cases} \frac{x}{3}, & 0 \leq x < 3, \\ 1, & 3 \leq x \leq 5, \end{cases} \quad G(y) = \begin{cases} 0, & 0 \leq y \leq 3, \\ \frac{y}{2} - 1, & 3 < y \leq 5. \end{cases}$$

Now, we will verify the rational Gupta–Saxena contraction as mentioned in 2.5:

$$d(Fx, Gy) \leq \lambda \max\{A(x, y), B(x, y), C(x, y)\}, \quad \lambda \in (0, 1),$$

where

$$A = d(x, y), \\ B = \frac{d(x, Fx)(1 + d(y, Gy))}{1 + d(x, y)}, \quad C = \frac{d(y, Gy)(1 + d(x, Fx))}{1 + d(x, y)}.$$

We compute all terms in all four cases and determine the maximum.

Case 1: For $x \in [0, 3)$ and $y \in [0, 3]$ we have $F(x) = \frac{x}{3}$ and $G(y) = 0$. So we conclude that

$$d(Fx, Gy) = \left| \frac{x}{3} - 0 \right| \left(\left| \frac{x}{3} - 0 \right| + 2 \right) = \frac{x}{3} \left(\frac{x}{3} + 2 \right)$$

$$A = |x - y|(|x - y| + 2)$$

$$d(x, Fx) = \frac{2x}{3} \left(\frac{2x}{3} + 2 \right), \quad d(y, Gy) = y(y + 2)$$

$$B = \frac{\frac{2x}{3} \left(\frac{2x}{3} + 2 \right) (1 + y(y + 2))}{1 + A}$$

$$C = \frac{y(y + 2) \left(1 + \frac{2x}{3} \left(\frac{2x}{3} + 2 \right) \right)}{1 + A}$$

The term A grows directly with $|x - y|$ in the given domain, while B and C are damped by $1 + A$ in the denominator. Hence,

$$\boxed{M(x, y) = C}$$

After substituting in the contraction, we get

$$\frac{x}{3} \left(\frac{x}{3} + 2 \right) \leq \lambda |x - y|(|x - y| + 2).$$

The above inequality holds for all given domain as shown in the Figure 5.

Case 2: For $x \in [0, 3)$ and $y \in (3, 5]$ we have $F(x) = \frac{x}{3}$ and $G(y) = \frac{y}{2} - 1$. So we conclude that

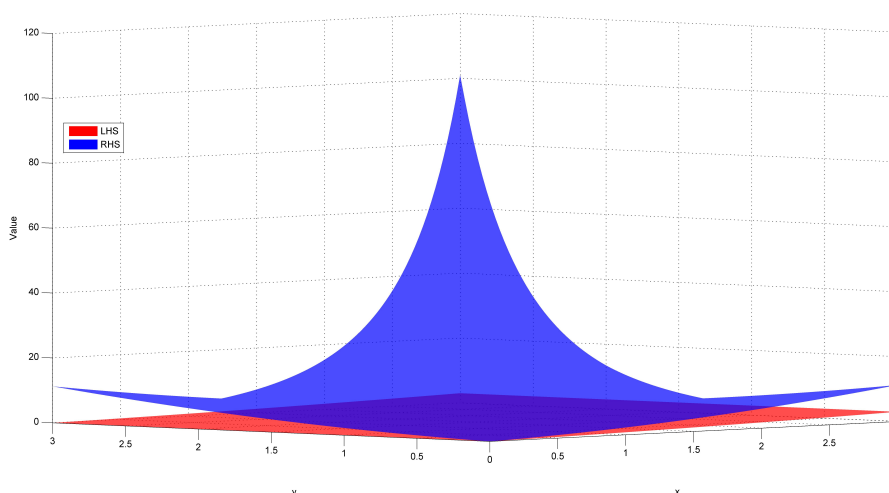


Figure 5: Graph related to Case 1 of Example 2.6

$$d(Fx, Gy) = \left| \frac{x}{3} - \left(\frac{y}{2} - 1 \right) \right| \left(\left| \frac{x}{3} - \left(\frac{y}{2} - 1 \right) \right| + 2 \right)$$

$$A = |x - y|(|x - y| + 2)$$

$$d(y, Gy) = \left(\frac{y}{2} + 1 \right) \left(\frac{y}{2} + 3 \right)$$

$$B = \frac{\frac{2x}{3} \left(\frac{2x}{3} + 2 \right) (1 + d(y, Gy))}{1 + A}$$

$$C = \frac{d(y, Gy) \left(1 + \frac{2x}{3} \left(\frac{2x}{3} + 2 \right) \right)}{1 + A}$$

Since $x < 3 < y$, the separation $|x - y|$ is large throughout this region, making the term A dominant over B and C . Hence,

$$M(x, y) = A$$

After substituting in the contraction, we get

$$\left| \frac{x}{3} - \left(\frac{y}{2} - 1 \right) \right| \left(\left| \frac{x}{3} - \left(\frac{y}{2} - 1 \right) \right| + 2 \right) \leq \lambda |x - y|(|x - y| + 2).$$

The above inequality holds for all given domain as shown in the Figure 6.

Case 3: For $x \in [3, 5]$ and $y \in [0, 3]$ we have $F(x) = 1$ and $G(y) = 0$. Consequently, we arrive at

$$d(Fx, Gy) = |1 - 0|(|1 - 0| + 2) = 3$$

$$A = |x - y|(|x - y| + 2)$$

$$d(x, Fx) = |x - 1|(|x - 1| + 2), \quad d(y, Gy) = y(y + 2)$$

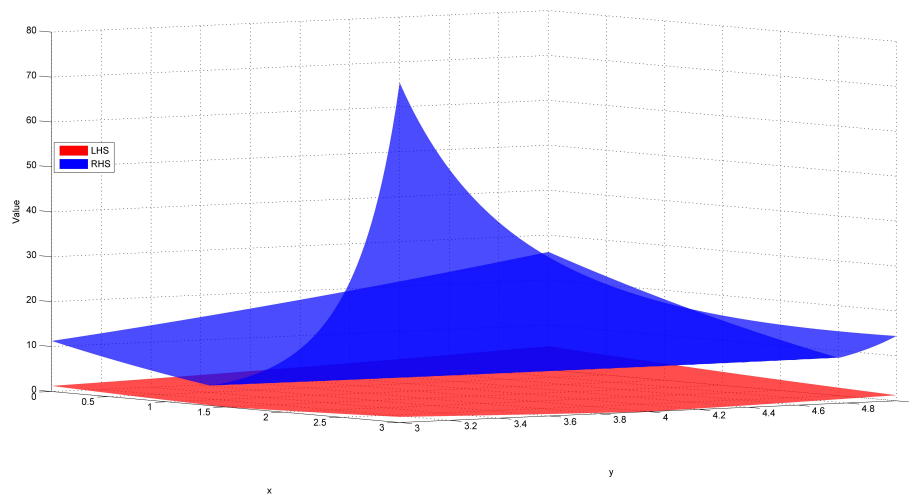


Figure 6: Graph related to Case 2 of Example 2.6

$$B = \frac{|x - 1|(|x - 1| + 2)(1 + y(y + 2))}{1 + A}$$

$$C = \frac{y(y + 2)(1 + |x - 1|(|x - 1| + 2))}{1 + A}$$

Since $x \geq 3$, the quantity $d(x, Fx) = |x - 1|(|x - 1| + 2) \geq 8$ is large and the factor $(1 + d(y, Gy))$ in the numerator of B grows with y , making B the dominant term throughout this region.

$$M(x, y) = B$$

After substituting in the contraction, we get

$$3 \leq \lambda \left(\frac{|x - 1|(|x - 1| + 2)(1 + y(y + 2))}{1 + A} \right).$$

The above inequality holds for all given domain as shown in the Figure 7.

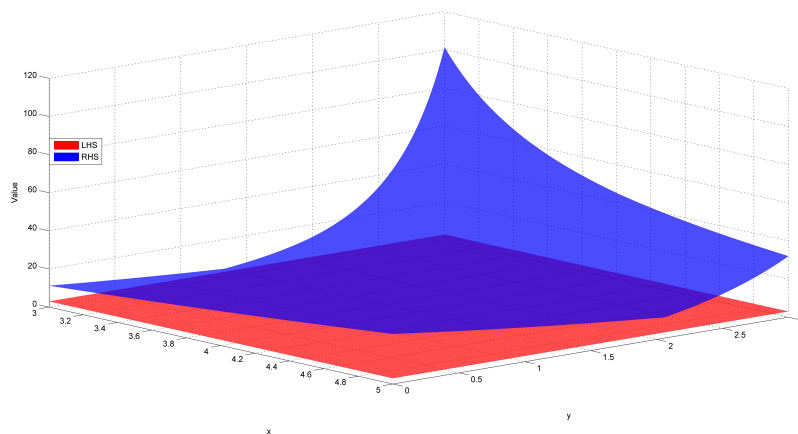


Figure 7: Graph related to Case 3 of Example 2.6

Case 4: For $x \in [3, 5]$ and $y \in (3, 5]$, we have $F(x) = 1$ and $G(y) = \frac{y}{2} - 1$. We estimate

$$d(Fx, Gy) = \left| 1 - \left(\frac{y}{2} - 1\right) \right| \left(\left| 1 - \left(\frac{y}{2} - 1\right) \right| + 2 \right)$$

$$A = |x - y|(|x - y| + 2)$$

$$d(x, Fx) = |x - 1|(|x - 1| + 2)$$

$$d(y, Gy) = \left(\frac{y}{2} + 1\right) \left(\frac{y}{2} + 3\right)$$

$$B = \frac{|x - 1|(|x - 1| + 2) (1 + d(y, Gy))}{1 + A}$$

$$C = \frac{d(y, Gy) (1 + |x - 1|(|x - 1| + 2))}{1 + A}$$

Since both x and y are large in this region, the quadratic growth of $d(y, Gy) = \left(\frac{y}{2} + 1\right) \left(\frac{y}{2} + 3\right)$ in y makes C the dominant term.

$$M(x, y) = B$$

After substituting in the contraction, we get

$$\left| 1 - \left(\frac{y}{2} - 1\right) \right| \left(\left| 1 - \left(\frac{y}{2} - 1\right) \right| + 2 \right) \leq \lambda \left(\frac{d(y, Gy) (1 + |x - 1|(|x - 1| + 2))}{1 + A} \right).$$

The above inequality holds for all given domain as shown in the Figure 8.

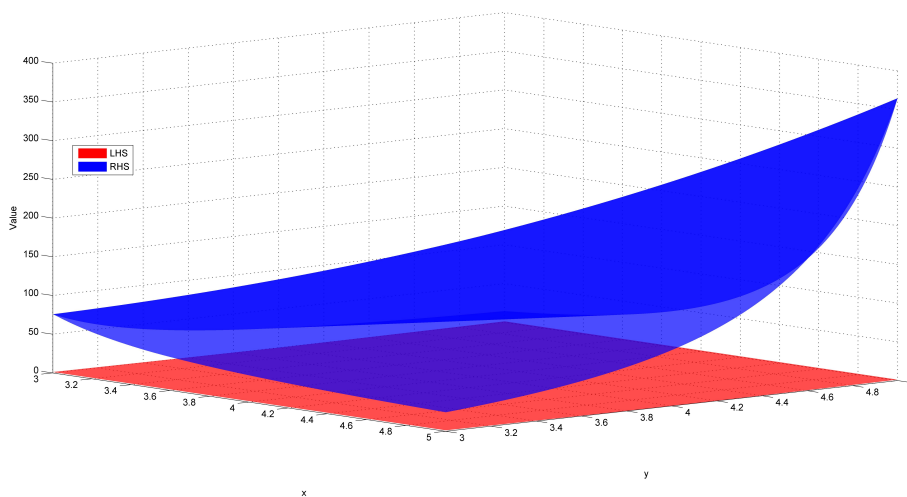


Figure 8: Graph related to Case 4 of Example 2.6

Since all the four cases holds for $\lambda = \frac{3}{4}$. Hence, all conditions of Theorem 2.5 are satisfied. Therefore, F and G admit a unique common fixed point in X , which is 0.

3. Application to Volterra Integral Equation

Volterra integral equations arise naturally in a wide range of applied mathematical problems, including population dynamics, viscoelasticity, and control theory, where the state of a system at a given time depends on its entire past history. The existence and uniqueness of solutions to such equations is therefore a question of fundamental importance, and fixed point theory has long served as one of the principal tools for addressing it. In the classical setting, Banach’s contraction principle provides the standard framework; however, when the underlying space carries a non-standard metric structure, more refined arguments are required.

In this section, we demonstrate that the suprametric framework developed above is well suited to the analysis of Volterra integral equations of the second kind. Specifically, we show that under natural Lipschitz-type conditions on the kernel and the nonlinearity, the associated integral operators F and G satisfy a contraction inequality with respect to the suprametric d , from which the existence and uniqueness of a common fixed point and hence of a solution to the integral equation follows directly from Theorem 2.1.

Let $X = C([0, 1], \mathbb{R})$ be endowed with the suprametric

$$d(u, v) = \sup_{t \in [0,1]} |u(t) - v(t)|(|u(t) - v(t)| + \hbar), \quad \hbar \in \mathbb{R}^+. \tag{19}$$

Then (X, d) is a complete suprametric space. Consider the operators $F, G : X \rightarrow X$ defined by

$$(Fu)(t) = \int_0^t K(t, s)f(s, u(s)) ds, \quad (Gu)(t) = \int_0^t K(t, s)g(s, u(s)) ds,$$

where $K : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is continuous. Assume that there exist constants $L_1 > 0$ and $\delta \in (0, 1)$ such that:

1. For all $t \in [0, 1]$ and all $u, v \in \mathbb{R}$,

$$|f(t, u) - g(t, v)| \leq L_1 |u - v|. \tag{20}$$

The constant L_1 is *uniform* (independent of t).

2. For all $t \in [0, 1]$,

$$\int_0^t |K(t, s)| ds \leq \delta. \tag{21}$$

Additionally, we assume that the constants satisfy

$$\delta L_1 < 1. \tag{22}$$

By using the suprametric distance mention in (19), we compute

$$\begin{aligned} |Fu(t) - Gv(t)|(|Fu(t) - Gv(t)| + \hbar) &= \left| \int_0^t K(t, s)(f(s, u(s)) - g(s, v(s))) ds \right| \\ &\times \left(\left| \int_0^t K(t, s)(f(s, u(s)) - g(s, v(s))) ds \right| + \hbar \right), \end{aligned}$$

where we utilize the given simplification in the above inequality

$$|Fu(t) - Gv(t)| \leq \int_0^t |K(t, s)| |f(s, u(s)) - g(s, v(s))| ds.$$

Applying condition (20), we get

$$\begin{aligned} |Fu(t) - Gv(t)|(|Fu(t) - Gv(t)| + \hbar) &\leq \left(L_1 |u(t) - v(t)| \int_0^t |K(t, s)| ds \right) \\ &\times \left(L_1 |u(t) - v(t)| \int_0^t |K(t, s)| ds + \hbar \right). \end{aligned}$$

Applying condition (21), we obtain

$$|Fu(t) - Gv(t)|(|Fu(t) - Gv(t)| + \hbar) \leq \left(\delta L_1 |u(t) - v(t)|\right) \left(\delta L_1 |u(t) - v(t)| + \hbar\right).$$

Since $\delta L_1 < 1$, we have $\delta L_1 |u(t) - v(t)| \leq |u(t) - v(t)|$, so the map φ gives

$$\left(\delta L_1 |u(t) - v(t)|\right) \left(\delta L_1 |u(t) - v(t)| + \hbar\right) \leq \delta L_1 |u(t) - v(t)| \left(|u(t) - v(t)| + \hbar\right).$$

Combining the two inequalities above yields

$$|Fu(t) - Gv(t)|(|Fu(t) - Gv(t)| + \hbar) \leq \delta L_1 |u(t) - v(t)| \left(|u(t) - v(t)| + \hbar\right).$$

Taking the supremum over $t \in [0, 1]$ and setting $\lambda = \delta L_1 \in (0, 1)$, we obtain

$$d(Fu, Gv) \leq \lambda d(u, v).$$

By Theorem 2.1, the operators F and G satisfy the hypotheses for the existence of a unique common fixed point $u^* \in X$, i.e. $Fu^* = Gu^* = u^*$. Hence, the Volterra integral equation admits a unique solution.

4. Conclusion and Future Directions

This paper successfully establishes a unified fixed point theory in complete suprametric spaces by proving four distinct common fixed point theorems for pairs of self-mappings under Banach-type, Reich-type, Ćirić-type, and rational Gupta-Saxena type contractive conditions. Each theorem is supported by detailed examples that verify the contractive hypotheses and confirm the existence and uniqueness of common fixed points. The suprametric framework, characterized by the parameter $\hbar \geq 0$ that relaxes the classical triangle inequality through a nonlinear correction term, provides a significantly more general and flexible setting than traditional metric spaces while still preserving essential convergence properties. This structure naturally encompasses both classical metric spaces and b-metric spaces as special cases, making the results broadly applicable to nonlinear problems where standard distance measures prove too restrictive. The practical utility of this theoretical framework is demonstrated through its application to Volterra integral equations of the second kind, where the integral operators are shown to satisfy Banach-type contractions under appropriate Lipschitz and kernel boundedness conditions, thereby establishing existence and uniqueness of solutions for equations commonly arising in mathematical physics, engineering, and biological systems with memory effects. The paper opens promising avenues for future research, particularly regarding extensions to multivalued contractions and applications to fractional-order dynamical systems including the fractional Lorenz system and fractional King Cobra population models, where the flexibility of suprametric spaces could provide valuable new analytical tools. Overall, this work demonstrates that suprametric spaces offer a natural, powerful, and practically relevant setting for fixed point theory, successfully unifying classical contraction principles while extending their reach to more general mathematical structures and real-world applications.

Open Problems:

- i Can Theorems 2.1-2.5 be generalized to the framework of multivalued contractions in order to obtain analogous fixed point or best proximity point results?
- ii To what extent can Theorems 2.1-2.5 be applied to establish the existence and uniqueness of solutions for fractional-order dynamical systems, including models such as the fractional-order Lorenz system [4], the fractional-order King cobra model [5], and other related fractional-order systems [3]?

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Ethical approval

This research complies with all relevant ethical guidelines regarding the use of human and animal subjects, as none were involved.

Conflicts of Interest

The author declares no competing interests that may have influenced the research or its presentation.

Data Availability

Data Usage: No data was utilized in the research presented in this article.

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