



Letters in Nonlinear Analysis and its Application

Peer Review Scientific Journal

ISSN: 2958-874x

Coupled fixed point theorems in partially ordered G_b -metric space

Dnyaneshwar Raghunath Nhavi^{a,*}, Chintaman Tukaram Aage^b

^aDepartment of First Year Engineering, G.H.Raisoni College of Engineering and management, Jalgaon, India.

^bDepartment of Mathematics, Kavayitri Bahinabai Chaudhari North Maharashtra University, Jalgaon, India.

Abstract

In this paper, we introduce some coupled fixed point theorems in a partial order G_b -metric space for various contractions. We provide some examples for supporting the results..

Keywords: G_b -metric spaces; Coupled fixed point; mixed monotone property; contraction mappings.

2010 MSC: 47H10; 54H25, 54E50

1. Introduction

The fixed point theory has wide applications in various fields such as physics, computer science, applied mathematics, economics, social sciences, engineering, etc. For the certain mappings in functional analysis to investigate the existence and uniqueness of the fixed points in different metric spaces which are generalized by many authors by different ways. Czerwik [25] introduced b-metric space. Zead Mustafa and Brailey Sims [29] discovered the new generalization of the metric space called G -metric space. Some authors has proved the existence and uniqueness of a fixed point of a contraction mapping in a partially ordered complete metric space. Aghajani and et al. [2] coined the new structure of metric space, called G_b -metric space which is

*Corresponding author

Email addresses: dnyaneshwar.nhavi@gmail.com (Dnyaneshwar Raghunath Nhavi), ctaage@gmail.com (Chintaman Tukaram Aage)

the mixture of G -metric space and b -metric space and obtained the results with G_b -metric space [1]. Also, many authors [3], [9], [10], [11], [14], [15], [16], [17], [21], [22] proved the fixed point theorems in G -metric space. Bhaskar and Laxmikantham et. al developed the concept of a coupled fixed point result in a partially ordered metric spaces. In this paper, we prove the coupled fixed point results for mappings with mixed monotone property and mixed g -monotone property in partially ordered G_b -metric space. The existence of fixed point theorems in partially ordered sets has been discovered by D. O'Regan and A. Petrusel [7] and others proved the coupled fixed point theorems [4], [5], [6], [8], [12], [18], [19], [20],[23], [24], [27], [28]. We illustrate the examples for given theorems.

2. Preliminaries

Definition 2.1. [2] Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G_b : X \times X \times X \rightarrow R^+$ satisfies:

- (i) $G_b(x, y, z) = 0$ if $x = y = z$ for all $x, y, z \in X$;
- (ii) $0 < G_b(x, x, y)$ for all $x, y, z \in X$ with $x \neq y$;
- (iii) $G_b(x, x, y) \leq G_b(x, y, z)$ for all x, y, z with $y \neq z$;
- (iv) $G_b(x, y, z) = G_b(px, z, y)$, where p is a permutation of x, y, z (symmetry);
- (v) $G_b(x, y, z) \leq s[G_b(x, a, a) + G_b(a, y, z)]$, for all $x, y, z, a \in X$.

Then G_b is called a generalized b -metric or G_b -metric on X . The ordered pair (X, G_b) is called generalized b -metric or G_b -metric space.

Following example shows that a G_b -metric on X need not be a G -metric on X .

Example 2.2. [2] Let (X, G) be a G -metric space and for all $x, y, z \in R$, $G_*(x, y, z) = G(x, y, z)^p$; where $p > 1$ is a real number. Note that G_* is a G_b -metric with $s = 2^{p-1}$.

Also in the above example, (X, G_*) is not necessarily a G -metric space. For example, let $X = R$ and G -metric be defined by

$$G(x, y, z) = \frac{1}{3}(|x - y| + |y - z| + |x - z|).$$

for all $x, y, z \in R$.

Example 2.3. Let $X = R$ and a mapping $G_b : X \times X \times X \rightarrow R^+$ is defined as

$$G_b(x, y, z) = \max\{|x - y|^2, |y - z|^2, |x - z|^2\}.$$

for all $x, y, z \in R$. Then (X, G_b) is a G_b -metric space with the coefficient $s = 2$.

Definition 2.4. [2] Let X be a G_b -metric space. A sequence $\{x_n\}$ in X is said to be :

- (i) G_b -Cauchy sequence if, for each $\epsilon > 0$, there exists a positive integer $n_0 \in \mathbb{N}$ such that, for all $m, n, l \geq n_0$, $G(x_n, x_m, x_l) < \epsilon$;
- (ii) G_b -convergent to a point $x \in X$ if, for each $\epsilon > 0$, there exists a positive integer $n_0 \in \mathbb{N}$ such that, for all $m, n \geq n_0$, $G(x_n, x_m, x) < \epsilon$.

Proposition 2.5. [2] Let (X, G_b) be a G_b -metric space. Then the following are equivalent:

- (i) x_n is G_b -convergent to x .

(ii) $G_b(x_n, x_n, x) \rightarrow 0$, as $n \rightarrow \infty$.

(iii) $G_b(x_n, x, x) \rightarrow 0$, as $n \rightarrow \infty$.

Proposition 2.6. [2] Let (X, G_b) be a G_b -metric space. Then the following are equivalent.

(i) The sequence $\{x_n\}$ is G_b -Cauchy.

(ii) For every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G_b(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq n_0$.

Definition 2.7. [2] A G_b -metric space X is called G_b -complete if every G_b -Cauchy sequence is G_b -convergent in X .

Definition 2.8. [2] Let $(X, G_b), (X', G'_b)$ be two G_b -metric spaces. Then a function $T : X \rightarrow X$ is G_b -continuous at a point $x \in X$ if and only if it is G_b sequentially continuous at x , that is $\{T(x_n)\}$ is G_b -convergent to $T(x)$, whenever $\{x_n\}$ is G_b -convergent to x .

Definition 2.9. [26] Let (X, \preceq) be a partially ordered set and $F : X \times X \rightarrow X$ be a mapping. The mapping F is said to have the mixed monotone property if F is monotone non-decreasing in x and is monotone non-increasing in y , that is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

Definition 2.10. [13] Let (X, \preceq) be a partially ordered set and $F : X \times X \rightarrow X$ and $g : X \rightarrow X$. The function F is said to have the mixed g -monotone property if $F(x, y)$ is monotone g -nondecreasing in x and is monotone g -nonincreasing in y , that is, for any $x, y \in X$,

$$g(x_1) \leq g(x_2) \Rightarrow F(x_1, y) \leq F(x_2, y), \quad \text{for } x_1, x_2 \in X$$

and

$$g(y_1) \leq g(y_2) \Rightarrow F(x, y_1) \geq F(x, y_2), \quad \text{for } y_1, y_2 \in X.$$

It is clear that Definition 2.10 reduces to Definition 2.9 when g is the identity.

Definition 2.11. [26] An element $(x, y) \in X \times X$, when X is any non-empty set, is called a coupled fixed point of the mapping $F : X \times X \rightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.12. [13] An element $(x, y) \in X \times X$, when X is any non-empty set, is called a coupled coincidence point of the mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 2.13. [13] An element $(x, y) \in X \times X$, when X is any non-empty set, is called a coupled common fixed point of the mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ if $F(x, y) = gx = x$ and $F(y, x) = gy = y$.

Definition 2.14. [13] Let X be a non-empty set, then we say that the mappings $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ are commutative if $gF(x, y) = F(gx, gy)$.

Definition 2.15. Let (X, G_b) be a G_b -metric space. A mapping $F : X \times X \rightarrow X$ is said to be continuous if for any two G_b -convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y respectively, $\{F(x_n, y_n)\}$ is G_b -convergent to $F(x, y)$.

Theorem 2.16. [30] Let (X, G) be complete G -metric space and $T : X \rightarrow X$ be a given mapping satisfying the following condition for all $x, y, z \in X$;

$$G(Tx, Ty, Tz) \leq kG(x, y, z),$$

where $k \in [0, 1)$. Then T has a unique fixed point.

Theorem 2.17. [6] Let (X, \preceq) be a partially ordered set and G be a G -metric on X such that (X, G) is complete G -metric space. Let $F : X \times X \rightarrow X$ be a continuous mapping having the mixed monotone property on X . Assume that there exists a $k \in [0, 1)$ such that for $x, y, z, u, v, w \in X$, the mappings satisfying

$$G(F(x, y), F(u, v), F(w, z)) \leq \frac{k}{2} [G(x, u, w) + G(y, v, z)] \quad (1)$$

for all $x \geq u \geq w$ and $y \leq v \leq z$, where $u \neq w$ or $v \neq z$. if there exists $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point in X , that is, there exists $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

3. Main Results

We generalized Theorem 2.17 in G_b -metric space.

Theorem 3.1. Let (X, \preceq) be a partially ordered set and suppose that there exists G_b -metric in X such that (X, G_b) is a complete metric space. Let $F : X \times X \rightarrow X$ be a continuous mapping having the mixed monotone property on X . Assume that there exist $sk \in [0, 1)$ and $s \geq 1$ such that the mappings satisfy

$$G_b(F(x, y), F(u, v), F(w, z)) \leq \frac{k}{2} [G_b(x, u, w) + G_b(y, v, z)] \quad (2)$$

for $x, u, w \in X$ with $x \geq u \geq w$ and for $y, v, z \in X$ with $y \leq v \leq z$, where either $u \neq w$ or $v \neq z$. if there exists $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point $(x, y) \in X \times X$, that is, there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Proof: Given that there exists $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. We define $x_1, y_1 \in X$ such that $x_1 = F(x_0, y_0) \geq x_0$ and $y_1 = F(y_0, x_0) \leq y_0$. Let $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$, we write

$$F^2(x_0, y_0) = F(F(x_0, y_0), F(y_0, x_0)) = F(x_1, y_1) = x_2$$

and

$$F^2(y_0, x_0) = F(F(y_0, x_0), F(x_0, y_0)) = F(y_1, x_1) = y_2.$$

By the mixed monotone property of F , we have

$$x_2 = F^2(x_0, y_0) = F(x_1, y_1) \geq F(x_0, y_0) = x_1 \geq x_0$$

and

$$y_2 = F^2(y_0, x_0) = F(y_1, x_1) \leq F(y_0, x_0) = y_1 \leq y_0,$$

continuing in this way, we have for all $n \geq 0$

$$x_{n+1} = F^{n+1}(x_0, y_0) = F(F^n(x_0, y_0), F^n(y_0, x_0))$$

and

$$y_{n+1} = F^{n+1}(y_0, x_0) = F(F^n(y_0, x_0), F^n(x_0, y_0)).$$

Then for all $n \geq 0$,

$$x_0 \leq F(x_0, y_0) = x_1 \leq F^2(x_0, y_0) = x_2 \leq \cdots \leq F^{n+1}(x_0, y_0) = x_{n+1} \leq \cdots$$

and

$$y_0 \geq F(y_0, x_0) = y_1 \geq F^2(y_0, x_0) = y_2 \geq \cdots \geq F^{n+1}(y_0, x_0) = y_{n+1} \geq \cdots$$

If $(x_{n+1}, y_{n+1}) = (x_n, y_n)$, then

$$\begin{aligned}(x_{n+1}, y_{n+1}) &= (F^{n+1}(x_0, y_0), F^{n+1}(y_0, x_0)) \\ (x_n, y_n) &= F(F^n(x_0, y_0), F^n(y_0, x_0)) \\ &= F(x_n, y_n).\end{aligned}$$

Hence F has a coupled fixed point. Suppose that $(x_{n+1}, y_{n+1}) \neq (x_n, y_n)$ for all $n \geq 0$, that is we assume either $x_{n+1} = F(x_n, y_n) \neq x_n$ or $y_{n+1} = F(y_n, x_n) \neq y_n$. We claim that, for all $n \geq 0$

$$\begin{aligned}G_b(F^n(x_0, y_0), F^{n+1}(x_0, y_0), F^{n+1}(x_0, y_0)) \\ \leq \frac{k^n}{2} [G_b(x_0, F(x_0, y_0), F(x_0, y_0)) + G_b(y_0, F(y_0, x_0), F(y_0, x_0))]\end{aligned}\quad (3)$$

and

$$\begin{aligned}G_b(F^n(y_0, x_0), F^{n+1}(y_0, x_0), F^{n+1}(y_0, x_0)) \\ \leq \frac{k^n}{2} [G_b(y_0, F(y_0, x_0), F(y_0, x_0)) + G_b(x_0, F(x_0, y_0), F(x_0, y_0))]\end{aligned}\quad (4)$$

We prove this by using induction on n . For $n = 1$, we have

$$\begin{aligned}G_b(F(x_0, y_0), F^2(x_0, y_0), F^2(x_0, y_0)) \\ = G_b(F(x_0, y_0), F(F(x_0, y_0), F(y_0, x_0)), F(F(x_0, y_0), F(y_0, x_0))) \\ \leq \frac{k}{2} [G_b(x_0, F(x_0, y_0), F(x_0, y_0)) + G_b(y_0, F(y_0, x_0), F(y_0, x_0))].\end{aligned}$$

Since $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, therefore

$$G_b(x_1, x_2, x_2) \leq \frac{k}{2} [G_b(x_0, x_1, x_1) + G_b(y_0, y_1, y_1)].$$

Similarly, we can prove that

$$G_b(y_1, y_2, y_2) \leq \frac{k}{2} [G_b(y_0, y_1, y_1) + G_b(x_0, x_1, x_1)].$$

Thus inequalities (3) and (4) true for $n = 1$.

Now assume that inequalities (3) and (4) true for $n = m$. By using $F^{m+1}(x_0, y_0) \geq F^m(x_0, y_0)$ and $F^{m+1}(y_0, x_0) \leq F^m(y_0, x_0)$, we get

$$\begin{aligned}G_b(F^{m+1}(x_0, y_0), F^{m+2}(x_0, y_0), F^{m+2}(x_0, y_0)) \\ = G_b(F(F^m(x_0, y_0), F^m(y_0, x_0)), F(F^{m+1}(x_0, y_0), F^{m+1}(y_0, x_0)), F(F^{m+1}(x_0, y_0), F^{m+1}(y_0, x_0))) \\ \leq \frac{k}{2} [G_b(F^m(x_0, y_0), F^{m+1}(x_0, y_0), F^{m+1}(x_0, y_0)) + G_b(F^m(y_0, x_0), F^{m+1}(y_0, x_0), F^{m+1}(y_0, x_0))].\end{aligned}$$

Since $F^m(x_0, y_0) = F(x_m, y_m) \geq x_m$ and $F^m(y_0, x_0) = F(y_m, x_m) \leq y_m$. Also since either $x_{m+1} = F(x_m, y_m) \neq x_m$ or $y_{m+1} = F(y_m, x_m) \neq y_m$. Therefore

$$\begin{aligned}G_b(F^{m+1}(x_0, y_0), F^{m+2}(x_0, y_0), F^{m+2}(x_0, y_0)) \\ \leq \frac{k}{2} \left[\frac{k^m}{2} [G_b(x_0, F(x_0, y_0), F(x_0, y_0)) + G_b(y_0, F(y_0, x_0), F(y_0, x_0))] \right. \\ \left. + \frac{k^m}{2} [G_b(y_0, F(y_0, x_0), F(y_0, x_0)) + G_b(x_0, F(x_0, y_0), F(x_0, y_0))] \right] \\ \leq \frac{k^{m+1}}{2} [G_b(x_0, F(x_0, y_0), F(x_0, y_0)) + G_b(y_0, F(y_0, x_0), F(y_0, x_0))].\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned} & G_b(F^{m+1}(y_0, x_0), F^{m+2}(y_0, x_0), F^{m+2}(y_0, x_0)) \\ & \leq \frac{k^{m+1}}{2} [G_b(y_0, F(y_0, x_0), F(y_0, x_0)) + G_b(x_0, F(x_0, y_0), F(x_0, y_0))]. \end{aligned}$$

Hence equations (3) and (4) are true for $n = m + 1$. Thus by principle of mathematical induction equations (3) and (4) are true for all $n \geq 0$.

Now claim that $\{x_n\} = \{F^n(x_0, y_0)\}$ and $\{y_n\} = \{F^n(y_0, x_0)\}$ are Cauchy sequences. For all $n, m, n < m$ and by (v) -th property of G_b -metric space, we have

$$\begin{aligned} & G_b(F^n(x_0, y_0), F^m(x_0, y_0), F^m(x_0, y_0)) \\ & \leq s \left[G_b(F^n(x_0, y_0), F^{n+1}(x_0, y_0), F^{n+1}(x_0, y_0)) + G_b(F^{n+1}(x_0, y_0), F^m(x_0, y_0), F^m(x_0, y_0)) \right] \\ & \leq \frac{sk^n}{2} \left[G_b(x_0, F(x_0, y_0), F(x_0, y_0)) + G_b(y_0, F(y_0, x_0), F(y_0, x_0)) \right] \\ & \quad + s^2 \left[G_b(F^{n+1}(x_0, y_0), F^{n+2}(x_0, y_0), F^{n+2}(x_0, y_0)) + G_b(F^{n+2}(x_0, y_0), F^m(x_0, y_0), F^m(x_0, y_0)) \right] \\ & \leq \left(\frac{sk^n}{2} + \frac{s^2k^{n+1}}{2} + \frac{s^3k^{n+2}}{2} + \cdots + \frac{s^{m-1}k^{n+m-2}}{2} + \frac{s^{m-1}k^{n+m-1}}{2} \right) \left[G_b(x_0, F(x_0, y_0), F(x_0, y_0)) \right. \\ & \quad \left. + G_b(y_0, F(y_0, x_0), F(y_0, x_0)) \right] \\ & \leq \frac{sk^n}{2} \left(1 + sk + (sk)^2 + \cdots + (sk)^{m-2} + (sk)^{m-2}k \right) \left[G_b(x_0, F(x_0, y_0), F(x_0, y_0)) \right. \\ & \quad \left. + G_b(y_0, F(y_0, x_0), F(y_0, x_0)) \right] \\ & \leq sk^n \left[\frac{1 - (sk)^{n-(m-2)}}{(1 - sk)} + k(sk)^{m-2} \right] \left[G_b(x_0, F(x_0, y_0), F(x_0, y_0)) + G_b(y_0, F(y_0, x_0), F(y_0, x_0)) \right], \end{aligned}$$

that is

$$G_b(x_n, x_m, x_m) \leq sk^n \left[\frac{1 - (sk)^{n-(m-2)}}{(1 - sk)} + k(sk)^{m-2} \right] \left[G_b(x_0, x_1, x_1) + G_b(y_0, y_1, y_1) \right].$$

As $n, m \rightarrow \infty$, we get

$$\lim_{n, m \rightarrow \infty} G_b(x_n, x_m, x_m) = 0.$$

Thus $\{x_n\} = \{F^n(x_0, y_0)\}$ is a G_b -Cauchy sequence. Since (X, G_b) is complete G_b -metric space, hence $\{x_n\}$ is a G_b -convergent sequence. That is as $n \rightarrow \infty, x_n \rightarrow x$.

Similarly $\{y_n\} = \{F^n(y_0, x_0)\}$ is a G_b -Cauchy sequence. Since (X, G_b) is complete G_b -metric space, hence $\{y_n\}$ is a G_b -convergent sequence. That is as $n \rightarrow \infty, y_n \rightarrow y$. Now we show that F has a coupled fixed point in X . By using equation (3) we have

$$G_b(F^n(x_0, y_0), F^{n+1}(x_0, y_0), F^{n+1}(x_0, y_0)) \leq \frac{k^n}{2} [G_b(x_0, F(x_0, y_0), F(x_0, y_0)) + G_b(y_0, F(y_0, x_0), F(y_0, x_0))].$$

That is

$$G_b(x_n, F(x_n, y_n), F(x_n, y_n)) \leq \frac{k^n}{2} [G_b(x_0, x_1, x_1) + G_b(y_0, y_1, y_1)].$$

Letting $n \rightarrow \infty$ and since F is continuous, we have

$$G_b(x, F(x, y), F(x, y)) \leq 0 \Rightarrow F(x, y) = x.$$

Similarly, we have $F(y, x) = y$. Hence (x, y) is a coupled fixed point of X .

Now, we extend the theorem 2.16 in view of the existence and uniqueness of a common fixed point. It is given as follows.

Theorem 3.2. Let (X, G_b) be a G_b -metric space. Let $F : X \rightarrow X$ and $g : X \rightarrow X$ be two mappings such that

$$G_b(Fx, Fy, Fz) \leq kG_b(gx, gy, gz) \tag{5}$$

for all $x, y, z \in X$. Assume that F and g satisfy the following conditions:

- (i) $F(X) \subseteq g(X)$
- (ii) $g(X)$ is G_b -complete and
- (iii) g is G_b -continuous and commutes with F

If $k \in [0, 1)$, then there is a unique x in X such that $gx = Fx = x$.

Proof: Let $x_0 \in X$ and given that $F(X) \subseteq g(X)$, we choose $x_1 \in X$ such that $gx_1 = Fx_0$. Also, we choose $x_2 \in X$ such that $gx_2 = Fx_1$. Continuing in this way, we can construct the sequence $\{x_n\}$ in X such that $gx_{n+1} = Fx_n$. By inequality (5) and for $n \in N$, we have

$$\begin{aligned} G_b(gx_n, gx_{n+1}, gx_{n+1}) &= G_b(Fx_{n-1}, Fx_n, Fx_n) \\ &\leq kG_b(gx_{n-1}, gx_n, gx_n) \\ &\vdots \\ &\leq k^n G_b(gx_0, gx_1, gx_1). \end{aligned} \tag{6}$$

For all $n, m \in N, n < m$ and by (v)-th property of G_b - metric space, we have

$$\begin{aligned} G_b(gx_n, gx_m, gx_m) &\leq s [G_b(gx_n, gx_{n+1}, gx_{n+1}) + G_b(gx_{n+1}, gx_m, gx_m)] \\ &\leq s [G_b(gx_n, gx_{n+1}, gx_{n+1}) + s [G_b(gx_{n+1}, gx_{n+2}, gx_{n+2}) + G_b(gx_{n+2}, gx_m, gx_m)]] \\ &\leq sG_b(gx_n, gx_{n+1}, gx_{n+1}) + s^2G_b(gx_{n+1}, gx_{n+2}, gx_{n+2}) + \dots \\ &\quad + s^{m-1} [G_b(gx_{m-2}, gx_{m-1}, gx_{m-1}) + G_b(gx_{m-1}, gx_m, gx_m)] \\ &\leq sk^n G_b(gx_0, gx_1, gx_1) + s^2k^{n+1}G_b(gx_0, gx_1, gx_1) + \dots \\ &\quad + s^{m-1}k^{n+m-2}G_b(gx_0, gx_1, gx_1) + s^{m-1}k^{n+m-1}G_b(gx_0, gx_1, gx_1) \\ &\leq sk^n [1 + sk + (sk)^2 + \dots + (sk)^{m-2}] G_b(gx_0, gx_1, gx_1) \\ &\quad + (sk)^{m-2}kG_b(gx_0, gx_1, gx_1) \\ &\leq sk^n \left[\frac{1 - (sk)^{n-(m-2)}}{(1 - sk)} + (sk)^{m-2}k \right] G_b(gx_0, gx_1, gx_1). \end{aligned}$$

That is

$$G_b(gx_n, gx_m, gx_m) \leq sk^n \left[\frac{1 - (sk)^{n-(m-2)}}{(1 - sk)} + k(sk)^{m-2} \right] G_b(gx_0, gx_1, gx_1)$$

As $n, m \rightarrow \infty$, we get

$$\lim_{n, m \rightarrow \infty} G_b(gx_n, gx_m, gx_m) = 0.$$

Thus $\{gx_n\}$ is a G_b -Cauchy sequence in $g(X)$. Since $g(X)$ is G_b -complete, hence $\{gx_n\}$ is a G_b -convergent sequence. That is as $n \rightarrow \infty, gx_n \rightarrow x$.

Since g is G_b -continuous, therefore $\{ggx_n\}$ is a G_b -convergent to gx . Also g is commutes with F , therefore $ggx_{n+1} = gFx_n = Fgx_n$. Hence

$$\begin{aligned} G_b(ggx_{n+1}, Fx, Fx) &= G_b(Fgx_n, Fx, Fx) \\ &\leq kG_b(ggx_n, gx, gx). \end{aligned}$$

As $n \rightarrow \infty$ and G_b is continuous, then we have

$$G_b(gx, Fx, Fx) \leq kG_b(gx, gx, gx) = 0.$$

Thus $gx = Fx$. Hence (x, y) is a coupled coincidence point of F and g .

Since $\{gx_{n+1}\}$ is subsequence of $\{gx_n\}$, therefore $\{gx_{n+1}\}$ is a G_b -convergent to x . Thus

$$\begin{aligned} G_b(gx_{n+1}, gx, gx) &= G_b(gx_{n+1}, Fx, Fx) \\ &= G_b(Fx_n, Fx, Fx) \\ &\leq kG_b(gx_n, gx, gx). \end{aligned}$$

As $n \rightarrow \infty$ and G_b is continuous, then we have

$$G_b(x, gx, gx) \leq kG_b(x, gx, gx) \Rightarrow (1 - k)G_b(x, gx, gx) \leq 0.$$

Since $k \in [0, 1)$, therefore

$$G_b(x, gx, gx) = 0.$$

Hence $x = gx$. This implies that $gx = Fx = x$.

For uniqueness suppose $x \neq u$ and $u \in X$ is another fixed point of F and g , i.e. $gu = Fu = u$. Then

$$G_b(x, u, u) = G_b(Fx, Fu, Fu) \leq kG_b(gx, gu, gu) \leq kG_b(x, u, u) \Rightarrow (1 - k)G_b(x, u, u) \leq 0.$$

Since $k \in [0, 1)$, we have $G_b(x, u, u) = 0$, that is $x = u$. Hence F and g have unique common fixed point.

Corollary 3.3. *Let (X, G_b) be a G_b -metric space. Let $F : X \rightarrow X$ and $g : X \rightarrow X$ be two mappings such that*

$$G_b(Fx, Fy, Fy) \leq kG_b(gx, gy, gy)$$

for all $x, y \in X$. Assume that F and g satisfy the following conditions:

- (i) $F(X) \subseteq g(X)$
- (ii) $g(X)$ is G_b -complete and
- (iii) g is G_b -continuous and commutes with F

If $k \in [0, 1)$ then there is a unique x in X such that $gx = Fx = x$.

Proof: By taking $z = y$ in Theorem (2.2), we get proof of this corollary.

Theorem 3.4. *Let (X, G_b) be a complete G_b -metric space. Let $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ be two mappings such that*

$$G_b(F(x, y), F(u, v), F(w, z)) \leq \frac{k}{2} [G_b(gx, gu, gw) + G_b(gy, gv, gz)] \quad (7)$$

Assume that F and g satisfy the following conditions:

- (i) $F(X \times X) \subseteq g(X)$
- (ii) $g(X)$ is G_b -complete and
- (iii) g is G_b -continuous and commutes with F .

If $k \in [0, 1)$ and for $x, u, w \in X$ with $gx \geq gu \geq gw$ and for $y, v, z \in X$ with $gy \leq gv \leq gz$, where either $u \neq w$ or $v \neq z$. If there exists $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$, then there exists $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$, that is F and g have coupled coincidence points. Also, there is a unique common fixed point x in X such that $gx = F(x, x) = x$.

Proof: Let $x_0, y_0 \in X$ and given that $F(X \times X) \subseteq g(X)$, we choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Also, we choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continuing in this way, we can construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$. By equation (7) and for $n \in N$, we have

$$\begin{aligned} G_b(gx_n, gx_{n+1}, gx_{n+1}) &= G_b(F(x_{n-1}, y_{n-1}), F(x_n, y_n), F(x_n, y_n)) \\ &\leq \frac{k}{2} [G_b(gx_{n-1}, gx_n, gx_n) + G_b(gy_{n-1}, gy_n, gy_n)]. \end{aligned} \quad (8)$$

Now

$$\begin{aligned} G_b(gx_{n-1}, gx_n, gx_n) &= G_b(F(x_{n-2}, y_{n-2}), F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1})) \\ &\leq \frac{k}{2} [G_b(gx_{n-2}, gx_{n-1}, gx_{n-1}) + G_b(gy_{n-2}, gy_{n-1}, gy_{n-1})] \end{aligned}$$

and

$$\begin{aligned} G_b(gy_{n-1}, gy_n, gy_n) &= G_b(F(y_{n-2}, x_{n-2}), F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1})) \\ &\leq \frac{k}{2} [G_b(gy_{n-2}, gy_{n-1}, gy_{n-1}) + G_b(gx_{n-2}, gx_{n-1}, gx_{n-1})] \end{aligned}$$

Therefore

$$\begin{aligned} G_b(gx_{n-1}, gx_n, gx_n) + G_b(gy_{n-1}, gy_n, gy_n) \\ \leq k [G_b(gx_{n-2}, gx_{n-1}, gx_{n-1}) + G_b(gy_{n-2}, gy_{n-1}, gy_{n-1})] \end{aligned}$$

which is holds for all $n \in N$. Hence the equation (8) becomes

$$\begin{aligned} G_b(gx_n, gx_{n+1}, gx_{n+1}) &\leq \frac{k^2}{2} [G_b(gx_{n-2}, gx_{n-1}, gx_{n-1}) + G_b(gy_{n-2}, gy_{n-1}, gy_{n-1})] \\ &\leq \frac{k^3}{2} [G_b(gx_{n-3}, gx_{n-2}, gx_{n-2}) + G_b(gy_{n-3}, gy_{n-2}, gy_{n-2})] \cdots \\ &\leq \frac{k^n}{2} [G_b(gx_0, gx_1, gx_1) + G_b(gy_0, gy_1, gy_1)]. \end{aligned}$$

Thus for all $n \in N$, we have

$$G_b(gx_n, gx_{n+1}, gx_{n+1}) \leq \frac{k^n}{2} [G_b(gx_0, gx_1, gx_1) + G_b(gy_0, gy_1, gy_1)]. \quad (9)$$

For all $n, m \in N, n < m$ and by (v)-th property of G_b -metric space, we have

$$\begin{aligned}
 G_b(gx_n, gx_m, gx_m) &\leq s [G_b(gx_n, gx_{n+1}, gx_{n+1}) + G_b(gx_{n+1}, gx_m, gx_m)] \\
 &\leq s [G_b(gx_n, gx_{n+1}, gx_{n+1}) + s [G_b(gx_{n+1}, gx_{n+2}, gx_{n+2}) + G_b(gx_{n+2}, gx_m, gx_m)]] \\
 &\leq sG_b(gx_n, gx_{n+1}, gx_{n+1}) + s^2G_b(gx_{n+1}, gx_{n+2}, gx_{n+2}) + \dots \\
 &\quad + s^{m-1} [G_b(gx_{m-2}, gx_{m-1}, gx_{m-1}) + G_b(gx_{m-1}, gx_m, gx_m)] \\
 &\leq \frac{k^n}{2} [G_b(gx_0, gx_1, gx_1) + G_b(gy_0, gy_1, gy_1)] \\
 &\quad + s^2 \frac{k^{n+1}}{2} [G_b(gx_0, gx_1, gx_1) + G_b(gy_0, gy_1, gy_1)] \\
 &\quad + \dots + s^{m-1} \frac{k^{n+m-2}}{2} [G_b(gx_0, gx_1, gx_1) + G_b(gy_0, gy_1, gy_1)] \\
 &\quad + s^{m-1} \frac{k^{n+m-1}}{2} [G_b(gx_0, gx_1, gx_1) + G_b(gy_0, gy_1, gy_1)] \\
 &\leq \frac{sk^n}{2} [[1 + sk + (sk)^2 + \dots + (sk)^{m-2}] [G_b(gx_0, gx_1, gx_1) + G_b(gy_0, gy_1, gy_1)] \\
 &\quad + (sk)^{m-2}k] [G_b(gx_0, gx_1, gx_1) + G_b(gy_0, gy_1, gy_1)] \\
 &\leq \frac{sk^n}{2} \left[\frac{1 - (sk)^{n-(m-2)}}{(1 - sk)} + (sk)^{m-2}k \right] [G_b(gx_0, gx_1, gx_1) + G_b(gy_0, gy_1, gy_1)].
 \end{aligned}$$

That is

$$G_b(gx_n, gx_m, gx_m) \leq \frac{sk^n}{2} \left[\frac{1 - (sk)^{n-(m-2)}}{(1 - sk)} + k(sk)^{m-2} \right] [G_b(gx_0, gx_1, gx_1) + G_b(gy_0, gy_1, gy_1)].$$

As $n, m \rightarrow \infty$, we get

$$\lim_{n,m \rightarrow \infty} G_b(gx_n, gx_m, gx_m) = 0.$$

Thus $\{gx_n\}$ is a G_b -Cauchy sequence in $g(X)$. Since $g(X)$ is G_b -complete, hence $\{gx_n\}$ is a G_b -convergent sequence. That is as $n \rightarrow \infty, gx_n \rightarrow x$.

Similarly $\{gy_n\}$ is a G_b -Cauchy sequence in $g(X)$. Since $g(X)$ is G_b -complete, hence $\{gy_n\}$ is a G_b -convergent sequence. That is as $n \rightarrow \infty, gy_n \rightarrow y$.

Since g is G_b -continuous, therefore $\{ggx_n\}$ is a G_b -convergent to gx and $\{ggy_n\}$ is a G_b -convergent to gy . Also g commutes with F , therefore $ggx_{n+1} = g(F(x_n, y_n)) = F(gx_n, gy_n)$ and $ggy_{n+1} = g(F(y_n, x_n)) = F(gy_n, gx_n)$. Hence

$$\begin{aligned}
 G_b(ggx_{n+1}, F(x, y), F(x, y)) &= G_b(F(gx_n, gy_n), F(x, y), F(x, y)) \\
 &\leq \frac{k}{2} [G_b(ggx_n, gx, gx) + G_b(ggy_n, gy, gy)].
 \end{aligned}$$

As $n \rightarrow \infty$ and G_b is continuous, then we have

$$G_b(gx, F(x, y), F(x, y)) \leq \frac{k}{2} [G_b(gx, gx, gx) + G_b(gy, gy, gy)] = 0.$$

Thus $gx = F(x, y)$. Similarly, we can show that $gy = F(y, x)$. Hence (x, y) is a coupled coincidence point of F and g .

Since $\{gx_{n+1}\}$ is subsequence of $\{gx_n\}$, therefore $\{gx_{n+1}\}$ is a G_b -convergent to x . Thus

$$\begin{aligned}
 G_b(gx_{n+1}, gx, gx) &= G_b(gx_{n+1}, F(x, y), F(x, y)) \\
 &= G_b(F(x_n, y_n), F(x, y), F(x, y)) \\
 &\leq \frac{k}{2} [G_b(gx_n, gx, gx) + G_b(gy_n, gy, gy)].
 \end{aligned}$$

As $n \rightarrow \infty$ and G_b is continuous, then we have

$$G_b(x, gx, gx) \leq \frac{k}{2} [G_b(x, gx, gx) + G_b(y, gy, gy)].$$

Similarly, we can show that

$$G_b(y, gy, gy) \leq \frac{k}{2} [G_b(x, gx, gx) + G_b(y, gy, gy)].$$

Hence

$$\begin{aligned} G_b(x, gx, gx) + G_b(y, gy, gy) &\leq k [G_b(x, gx, gx) + G_b(y, gy, gy)] \\ \Rightarrow (1 - k) [G_b(x, gx, gx) + G_b(y, gy, gy)] &\leq 0. \end{aligned}$$

Since $k \in [0, 1)$, therefore

$$G_b(x, gx, gx) + G_b(y, gy, gy) = 0.$$

It implies that $G_b(x, gx, gx) = 0$ and $G_b(y, gy, gy) = 0$. Hence $x = gx$ and $y = gy$, so $gx = F(x, x) = x$.

For uniqueness suppose $x \neq u$ and $u \in X$ is another fixed point of F and g , i.e. $gu = F(u, u) = u$. Then

$$\begin{aligned} G_b(x, u, u) &= G_b(F(x, x), F(u, u), F(u, u)) \\ &\leq kG_b(gx, gu, gu) = kG_b(x, u, u) \\ \Rightarrow (1 - k)G_b(x, u, u) &\leq 0. \end{aligned}$$

Since $k \in [0, 1)$, we have $G_b(x, u, u) = 0$, that is $x = u$. Hence F and g have unique common fixed point.

Corollary 3.5. *Let (X, G_b) be a complete G_b -metric space. Let $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ be two mappings such that*

$$G_b(F(x, y), F(u, v), F(u, v)) \leq \frac{k}{2} [G_b(gx, gu, gu) + G_b(gy, gv, gv)]$$

for all $x, y, u, v \in X$. Assume that F and g satisfy the following conditions:

- (i) $F(X \times X) \subseteq g(X)$
- (ii) $g(X)$ is G_b -complete and
- (iii) g is G_b -continuous and commutes with F

If $k \in [0, 1)$ then there is a unique x in X such that $gx = F(x, x) = x$.

Proof: By taking $z = u$ and $w = v$ in Theorem (2.4), we get proof of this corollary. Enumerations:

Example 3.6. Let $X = \mathbb{R}$. Let us define $G : X \times X \times X \rightarrow [0, +\infty)$ by

$$G_b(x, y, z) = |2x - y| + |2y - z| + |2x - z|$$

for all $x, y, z \in X$. Then (X, G_b) be a complete G_b -metric space. Define a map $F : X \times X \rightarrow X$ by

$$F(x, y) = \frac{x}{3} - \frac{y}{4}.$$

Then for all $x, y, z, u, v, w \in X$, we have

$$\begin{aligned} G_b(F(x, y), F(u, v), F(w, z)) &= \left| \left(\frac{2x}{3} - \frac{y}{2} \right) - \left(\frac{u}{3} - \frac{v}{4} \right) \right| + \left| \left(\frac{2u}{3} - \frac{v}{2} \right) - \left(\frac{w}{3} - \frac{z}{4} \right) \right| \\ &\quad + \left| \left(\frac{2x}{3} - \frac{y}{2} \right) - \left(\frac{w}{3} - \frac{z}{4} \right) \right| \\ &\leq \frac{1}{3} (|2x - u| + |2u - w| + |2x - w|) \\ &\quad + \frac{1}{4} (|2y - v| + |2v - z| + |2y - z|) \\ &\leq \frac{k}{2} [G_b(x, u, w) + G_b(y, v, z)]. \end{aligned}$$

where $k \in [0, 1)$. It is easy to see that 0 is the unique fixed point.

Example 3.7. Let $X = \mathbb{R}$ Let us define $G : X \times X \times X \rightarrow [0, +\infty)$ by

$$G_b(x, y, z) = |2x - y| + |2y - z| + |2x - z|$$

for all $x, y, z \in X$. Then (X, G_b) be a complete G_b -metric space. Define a map $F : X \times X \rightarrow X$ by

$$F(x, y) = \frac{x}{3} - \frac{y}{4}$$

and $g : X \rightarrow X$ by $g(x) = \frac{4x}{9}$. Then for all $x, y, z, u, v, w \in X$, we have

$$\begin{aligned} G_b(F(x, y), F(u, v), F(w, z)) &= \left| \left(\frac{2x}{3} - \frac{y}{2} \right) - \left(\frac{u}{3} - \frac{v}{4} \right) \right| + \left| \left(\frac{2u}{3} - \frac{v}{2} \right) - \left(\frac{w}{3} - \frac{z}{4} \right) \right| \\ &\quad + \left| \left(\frac{2x}{3} - \frac{y}{2} \right) - \left(\frac{w}{3} - \frac{z}{4} \right) \right| \\ &\leq \frac{1}{3} (|2x - u| + |2u - w| + |2x - w|) \\ &\quad + \frac{1}{4} (|2y - v| + |2v - z| + |2y - z|) \\ &\leq \frac{1}{3} [(|2x - u| + |2u - w| + |2x - w|) + (|2y - v| + |2v - z| + |2y - z|)]. \end{aligned}$$

$$\begin{aligned} G_b(gx, gu, gw) + G_b(gy, gv, gz) &= \left| \left(\frac{8x}{9} - \frac{4u}{9} \right) \right| + \left| \left(\frac{8u}{9} - \frac{4w}{9} \right) \right| + \left| \left(\frac{8x}{9} - \frac{4w}{9} \right) \right| \\ &\quad + \left| \left(\frac{8y}{9} - \frac{4v}{9} \right) \right| + \left| \left(\frac{8v}{9} - \frac{4z}{9} \right) \right| + \left| \left(\frac{8y}{9} - \frac{4z}{9} \right) \right| \\ &\leq \frac{4}{9} [(|2x - u| + |2u - w| + |2x - w|) \\ &\quad + (|2y - v| + |2v - z| + |2y - z|)]. \end{aligned}$$

Hence

$$G_b(F(x, y), F(u, v), F(w, z)) \leq \frac{k}{2} [G_b(gx, gu, gw) + G_b(gy, gv, gz)],$$

where $k \in [0, 1)$. It is easy to see that 0 is the unique fixed point.

References

- [1] A. Aghajani, M. Abbas, J.R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered b -metric spaces*, Math. Slovaca **64** (2014), No. 4, 941–960.
- [2] A. Aghajani, M. Abbas, J.R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered G_b -metric spaces*, Filomat **28** (2014), No. 6, 1087–1101.
- [3] A. Roldán, E. Karapınar, P. Kumam, *G -metric spaces in any number of arguments related fixed point theorems*, (2014).
- [4] B. Samet, C. Vetro, F. Vetro, *Remarks on G -metric spaces*, Int. J. Anal. (2013), Article ID 917158, 6 pages.
- [5] B. Samet, E. Karapınar, H. Aydi, V.C. Rajić, *Discussion on some coupled fixed point theorems*, Fixed Point Theory Appl. **2013** (2013), 50.
- [6] B.S. Choudhury, P. Maity, *Coupled fixed point results in generalized metric spaces*, Math. Comput. Model. **54** (2011), 73–79.
- [7] D. O'Regan, A. Petruşel, *Fixed point theorems for generalized contractions in ordered metric spaces*, J. Math. Anal. Appl. **341** (2008), No. 2, 1241–1252.
- [8] E. Karapınar, B. Kaymakçalan, K. Taş, *On coupled fixed point theorems on partially ordered G -metric spaces*, (2012).
- [9] E. Karapınar, I.M. Erhan, A.Y. Ulus, *Cyclic contractions on G -metric spaces*, (2012).
- [10] E. Karapınar, I.M. Erhan, D. Türkoğlu, *Cyclic contractions and related fixed point theorems on G -metric spaces*, (2014).
- [11] E. Karapınar, R.P. Agarwal, *Further fixed point results on G -metric spaces*, (2013).
- [12] E. Karapınar, W. Sintunavarat, P. Kumam, *Coupled fixed point theorems on partially ordered G -metric spaces*, (2012).
- [13] G.V. Lakshmikantham, Lj.B. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. **70** (2009), 4341–4349.
- [14] H. Aydi, E. Karapınar, P. Salimi, *Some fixed point results in GP -metric spaces*, (2012).
- [15] H. Aydi, E. Karapınar, Z. Mustafa, *On common fixed points in G -metric spaces using $E.A.$ property*, (2012).
- [16] H.S. Ding, E. Karapınar, *Meir–Keeler type contractions in partially ordered G -metric spaces*, (2013).
- [17] M. Asadi, E. Karapınar, P. Salimi, *A new approach to G -metric and related fixed point theorems*, (2013).
- [18] M.A. Khamsi, W.A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, Wiley-Interscience Publication.
- [19] M. Jleli, B. Samet, *Remarks on G -metric spaces and fixed point theorems*, Fixed Point Theory Appl. **2012** (2012), 210.
- [20] N. Bilgili, I.M. Erhan, E. Karapınar, D. Türkoğlu, *A note on coupled fixed point theorems for mixed g -monotone mappings in partially ordered metric spaces*, (2014).
- [21] N. Bilgili, I.M. Erhan, E. Karapınar, D. Türkoğlu, *Cyclic contractions and related fixed point theorems on G -metric spaces*, (2014).
- [22] N. Hussain, E. Karapınar, P. Salimi, P. Vetro, *Fixed point results for Gm -Meir–Keeler contractive and G - ξ -Meir–Keeler contractive mappings*, (2013).
- [23] R.P. Agarwal, E. Karapınar, *Remarks on some coupled fixed point theorems in G -metric spaces*, Fixed Point Theory Appl. **2013** (2013).
- [24] R.P. Agarwal, E. Karapınar, A. Roldán, *Fixed point theorems on quasi-metric spaces and applications to multidimensional fixed point theorems on G -metric spaces*, (2015).
- [25] S. Czerwik, *Contraction mappings in b -metric spaces*, Acta Math. Inform. Univ. Ostrav. **1** (1993), 5–11.
- [26] T.G. Bhaskar, G.V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. **65** (2006), 1379–1393.
- [27] W.A. Kirk, B. Sims, *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publishers, Dordrecht, 2001.
- [28] W. Shatanawi, *Coupled fixed point theorems in generalized metric spaces*, Hacet. J. Math. Stat. **40** (2011), No. 3, 441–447.
- [29] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal. **7** (2006), No. 2, 289–297.
- [30] Z. Mustafa, *A New Structure for Generalized Metric Spaces with Applications to Fixed Point Theory*, Ph.D. Thesis, The University of Newcastle, Australia, 2005.