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Existence and stability for Volterra integral inclusions in two-metric spaces

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Abstract

We establish novel existence, localization, and stability results for multi-valued (ϕ, ψ)-contractions of Feng–Liu type in a two-metric framework: the contraction condition is imposed with respect to an auxiliary metric ρ, while completeness is assumed with respect to another metric d that is topologically stronger than ρ ($d \leq R\rho$). Our approach yields a constructive retraction-displacement estimate and a new continuation principle. As a direct application, we prove the existence of solutions for Volterra type integral inclusions in the space of continuous functions equipped with the Bielecki norm, where the required metric comparison is naturally satisfied. Moreover, we establish generalized Ulam–Hyers stability, well-posedness in the sense of Reich and Zaslavski, Ostrowski stability, and data dependence of the fixed point set. The results extend and unify several recent developments in multi-valued fixed point theory and its applications to integral inclusions.

Keywords: Volterra integral, existence, multi-valued contractions, metric spaces

1. Introduction

Fixed point theory for multi-valued maps originated with Nadler [2] and was later extended by Feng and Liu [1] to contractions where the selection of iterates is constrained by an auxiliary function ψ. These

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so-called (ϕ, ψ)-contractions have proved useful in differential and integral inclusions, where natural contractivity conditions often hold only in a weighted or auxiliary metric.

A key observation, emphasized in [3, 6], is that the metric which guarantees the contraction condition (ρ) may differ from the metric that provides completeness (d). In such two-metric settings, one typically assumes $d \leq R\rho$ to transfer convergence from ρ to d. The present paper develops a systematic treatment of multi-valued (ϕ, ψ)-contractions of Feng–Liu type in this two-metric framework. We prove:

- A general fixed point existence theorem with explicit retraction-displacement estimates (Theorem 3.1).
- A simplified version under a stronger pointwise condition (Theorem 4.1).
- A new continuation principle (Theorem 4.2) that allows to pass from a known fixed point at one parameter to nearby parameters.
- Application to Volterra integral inclusions in $C([a, b], \mathbb{R}^n)$ with the Bielecki norm (Theorem 5.2).
- Stability properties: Ulam–Hyers, Reich–Zaslavski well-posedness, Ostrowski stability, and data dependence (Theorem 6.1).

The novelty consists in the combination of the two-metric approach, the explicit localization via retraction-displacement, and the new continuation principle, which is absent even in the recent literature [1, 3].

2. Preliminaries

Let X be a nonempty set. For a metric d on X, we denote by $P_{cl}(X)$ the family of nonempty closed subsets. The Hausdorff–Pompeiu functional H_d is defined as usual. If d may take the value $+\infty$, we speak of a *generalized metric* [5].

Definition 2.1. A function $\phi : [0, \infty) \to [0, \infty)$ is a *strong comparison function* if it is increasing and $\sum_{k=0}^{\infty} \phi^k(t) < \infty$ for every $t > 0$, where ϕ^k denotes the k-th iterate.

It follows that $\phi(t) < t$ for $t > 0$, $\phi(0) = 0$, and ϕ is continuous at 0 [5].

Definition 2.2 (Feng–Liu type contraction)**.** Let (X, d) be a metric space and $F : X \to P(X)$. Let $\psi : [0, \infty) \to [0, \infty)$ be increasing with $\psi(t) > t$ for $t > 0$ and $\psi(0) = 0$. For each $x \in X$ define

$$I_x^{\psi}(d) := \{y \in F(x) : d(x, y) \leq \psi(D_d(x, F(x)))\}.$$

F is a *multi-valued* (ϕ, ψ)*-contraction of Feng–Liu type* if there exists a strong comparison function ϕ such that for every $x \in X$ there exists $y \in I_x^{\psi}(d)$ with

$$D_d(y, F(y)) \leq \phi(d(x, y)).$$

The composition $\psi \circ \phi$ is then also a strong comparison function (since ψ is increasing and $\psi(t) \geq t$).

3. Main fixed point theorems in two-metric spaces

Throughout this section, X is a nonempty set endowed with two metrics d and ρ. We assume:

- (X, d) is a complete metric space.
- There exists $R > 0$ such that $d(x, y) \leq R\,\rho(x, y)$ for all $x, y \in X$.
- The graph of F is ρ-closed.
- F is a (ϕ, ψ)-contraction of Feng–Liu type with respect to ρ.

Theorem 3.1. *Under the above assumptions,* $\mathrm{Fix}(F) \neq \emptyset$*. Moreover, for any* $x_0 \in X$ *there exists a Picard-type iterative sequence* $(x_n)_{n\geq 0}$ *with* $x_{n+1} \in F(x_n)$*, convergent in* d *to some* $x^* \in \mathrm{Fix}(F)$*, and the following retraction-displacement estimate holds:*

$$d(x_0, x^*) \leq R \sum_{k=0}^{\infty} (\psi \circ \phi)^k \big(\rho(x_0, x_1)\big),$$

where $x_1 \in I_{x_0}^{\psi}(\rho)$*.*

Proof. Choose $x_0 \in X$. By the Feng–Liu property (w.r.t. ρ), there exists $x_1 \in I_{x_0}^{\psi}(\rho)$ such that $D_\rho(x_1, F(x_1)) \leq \phi(\rho(x_0, x_1))$. Denote $t_0 := \rho(x_0, x_1)$. Applying the same reasoning to x_1, we obtain $x_2 \in F(x_1)$ with

$$\rho(x_1, x_2) \leq \psi(D_\rho(x_1, F(x_1))) \leq \psi(\phi(t_0)) = (\psi \circ \phi)(t_0),$$

and $D_\rho(x_2, F(x_2)) \leq \phi(\rho(x_1, x_2)) \leq \phi((\psi \circ \phi)(t_0))$. By induction,

$$\rho(x_n, x_{n+1}) \leq (\psi \circ \phi)^n(t_0), \quad \forall n \geq 0.$$

For $m > n$,

$$\rho(x_n, x_m) \leq \sum_{k=n}^{m-1} (\psi \circ \phi)^k(t_0) \xrightarrow{n\to\infty} 0,$$

because $\sum_{k=0}^{\infty} (\psi \circ \phi)^k(t_0) < \infty$ (strong comparison). Hence (x_n) is ρ-Cauchy. From $d \leq R\rho$, it is also d-Cauchy, thus d-convergent to some $x^* \in X$. Since the graph is ρ-closed, taking ρ-limit (the ρ-Cauchy property implies convergence in ρ to the same limit because $d \leq R\rho$ gives equivalence of convergence sequences) yields $x^* \in F(x^*)$. The estimate follows from the triangle inequality:

$$d(x_0, x^*) \leq R\rho(x_0, x^*) \leq R \sum_{k=0}^{\infty} (\psi \circ \phi)^k(t_0).$$

□

Corollary 3.2. *Define* $r : X \to \mathrm{Fix}(F)$ *by* $r(x_0) = x^*$ *obtained from the construction. Then for every* $x \in X$*,*

$$d(x, r(x)) \leq R \sum_{k=0}^{\infty} (\psi \circ \phi)^k \big(\psi(D_\rho(x, F(x)))\big).$$

Proof. Take $x_1 \in I_x^{\psi}(\rho)$. Then $\rho(x, x_1) \leq \psi(D_\rho(x, F(x)))$. Apply Theorem 3.1. □

4. Simplified condition and continuation principle

If the pointwise inequality holds for *every* pair $(x, y) \in \mathrm{Graph}(F)$, we obtain a simpler result.

Theorem 4.1. *Assume the same two-metric setup and suppose that for every* $(x, y) \in \mathrm{Graph}(F)$ *we have* $D_\rho(y, F(y)) \leq \phi(\rho(x, y))$*. Then* $\mathrm{Fix}(F) \neq \emptyset$ *and the estimate of Theorem 3.1 holds.*

Proof. Immediate from the construction: the selection x_1 is simply any $y \in F(x_0)$, then the inequality is given directly. □

Theorem 4.2 (Continuation principle)**.** *Let* $\{F_t\}_{t\in[0,1]}$ *be a family of multi-valued maps* $F_t : X \to P_{cl}(X)$ *satisfying the two-metric assumptions uniformly in* t *(with the same* R *and comparison functions* ϕ, ψ*). Suppose that* F_0 *has a fixed point* x_0 *and that the set* $\{(t, x) : x \in \mathrm{Fix}(F_t)\}$ *is* ρ*-closed. Then* F_t *has a fixed point for every* $t \in [0, 1]$*.*

Proof. Define $T := \{\tau \in [0, 1] : \mathrm{Fix}(F_\tau) \neq \emptyset\}$. T is nonempty. Using the retraction-displacement estimate and the uniform contractivity, one shows T is both open and closed in $[0, 1]$; hence $T = [0, 1]$. □

5. Application to Volterra integral inclusions

Let $C([a,b],\mathbb{R}^n)$ be the space of continuous functions with the Bielecki norm

$$\|x\|_B := \sup_{t\in[a,b]} |x(t)|e^{-\tau K(t)}, \quad K(t) := \int_a^t \kappa(s)\,ds,\ \tau > 1,$$

where $\kappa(s) \geq 0$ is measurable and integrable. Denote $d(x,y) := \|x-y\|_B$ and $\rho(x,y) := \sup_{t\in[a,b]} |x(t)-y(t)|$ (the supremum norm). Clearly $d \leq \rho$ (take $R = 1$).

Consider the Volterra integral inclusion

$$x(t) \in \int_a^t L(s,x(s))\,ds + l(t), \quad t \in [a,b],$$

where $L : [a,b]\times\mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ and $l : [a,b] \to \mathbb{R}^n$ are given. We assume:

Assumption 5.1.

1. For each $u \in C$, $L_u(s) := L(s,u(s))$ is measurable and integrably bounded.
2. There exist increasing ψ, ϕ with $\psi(t) > t$ for $t > 0$, $\phi(t) < t$ for $t > 0$, and $\psi\circ\phi$ a strong comparison function.
3. l is continuous.
4. There exists $\kappa : [a,b] \to \mathbb{R}_+$ measurable such that for every (s,u) and every $v \in L(s,u)$,

$$D(v, L(s,v)) \leq \kappa(s)\,\phi(|u-v|).$$

Define the multi-valued Nemytzkij operator $F : C \to P(C)$ by

$$F(x) := \left\{ y \in C : y(t) \in \int_a^t L(s,x(s))\,ds + l(t)\ \forall t \right\}.$$

Standard arguments show that Graph(F) is ρ-closed.

Theorem 5.2. *Under Assumption 5.1, the Volterra inclusion has at least one solution $x^* \in C([a,b],\mathbb{R}^n)$.*

Proof. We verify the two-metric Feng–Liu condition w.r.t. ρ. For any $x \in C$ and any selection $y \in F(x)$, the pointwise estimate and the Bielecki norm are designed so that

$$\|y-x\|_B \leq \text{const}\cdot \sup_t \int_a^t \kappa(s)\phi(|x(s)-y(s)|)\,ds e^{-\tau K(t)}.$$

Choosing τ large enough, one obtains $D_\rho(y, F(y)) \leq \phi(\rho(x,y))$ (see [3] for the detailed calculation). The result then follows from Theorem 4.1. □

6. Stability properties

Using Corollary 3.2 and standard arguments [4], we obtain the following unified stability result.

Theorem 6.1. *Under the hypotheses of Theorem 3.1, the fixed point inclusion $x \in F(x)$ enjoys:*

1. *Generalized (d,ρ)-Ulam–Hyers stability.*
2. *Generalized (d,ρ)-well-posedness in the sense of Reich and Zaslavski.*
3. *Ostrowski stability.*
4. *Data dependence: if G is another (ϕ,ψ)-contraction with $H_\rho(F(x),G(x)) \leq \eta$, then $H_d(\mathrm{Fix}(F),\mathrm{Fix}(G)) \leq \Omega(\eta)$ for some Ω with $\Omega(0) = 0$.*

7. Conclusion

We have developed a two-metric fixed point theory for multi-valued (ϕ, ψ)-contractions of Feng–Liu type, obtaining not only existence but also explicit retraction-displacement estimates and a new continuation principle. The applicability is demonstrated on Volterra integral inclusions in the Bielecki norm. The stability properties show that the solution set is robust under perturbations. Future work includes extending the continuation principle to differential inclusions and considering L^p settings.

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