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Proximal point algorithm for convex minimisation on Banach spheres

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Abstract

In this paper, we study a proximal point algorithm for convex minimisation on Banach spheres. In the spherical framework, we consider resolvent operators associated with proper lower semicontinuous spherically convex functions and investigate the asymptotic behaviour of iterative sequences generated by these resolvents. Under suitable geometric assumptions on the underlying Banach sphere and a sequential delta-continuity assumption on the duality mapping, we prove that the generated sequence delta-converges to a minimiser of the objective function. This result provides a delta-convergence principle for proximal point algorithms in the setting of Banach spheres.

Keywords: Banach sphere, convex minimisation, proximal point algorithm, spherically convex function
2010 MSC: 47H10, 58C30

1. Introduction

Let A be a set-valued mapping from a Hilbert space H into H. To find a point $x \in H$ such that $0_H \in Ax$ is called a zero point problem for A. It is well known that the class of zero point problems includes many nonlinear problems, such as convex minimisation problems, equilibrium problems, and fixed point problems, among others.

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Given a maximal monotone operator A on a Hilbert space H and a positive real number r, we can define a mapping J_{rA} by

$$J_{rA}x = (I + rA)^{-1}x$$

for $x \in H$. Note that $(I + rA)^{-1}$ is single-valued even if A is set-valued. We call this mapping J_{rA} the resolvent operator for rA. One of the fundamental properties of the resolvent operator is that the set of all fixed points of J_{rA} coincides with the set of all zero points of A. On the other hand, the proximal point algorithm is a typical zero point approximation method. Rockafellar [11] proved the following approximation theorem for finding a zero point of a maximal monotone operator:

Theorem 1.1 (Rockafellar [11]). *Let H be a Hilbert space and A a maximal monotone operator on H having a zero point. Let $\{r_n\}$ be a sequence of real numbers such that $\inf_{k\in\mathbb{N}} r_k > 0$. For a given initial point $x_1 \in H$, generate a sequence $\{x_n\}$ of H by*

$$x_{n+1} = J_{r_nA}x_n = (I + r_nA)^{-1}x_n$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges weakly to some zero point of A.

Recently, approximation theorems by the proximal point algorithm for convex functions and equilibrium problems have been established in geodesic spaces with curvature bounded above, called CAT(κ) spaces. See, for instance, [1, 3, 5]. In the setting of geodesic spaces, since there is no natural notion of weak convergence, we often use Δ-convergence [9, 10] instead. This notion coincides with weak convergence when the underlying space is a Hilbert space. Nevertheless, for instance, in the setting of spheres and hyperboloids, Δ-convergence does not reduce to weak convergence since they are not linear spaces. The following theorem concerns the proximal point algorithm for convex functions on CAT(1) spaces:

Theorem 1.2 (Kajimura–Kimura [4], Sudo [12]). *Let f be a lower semicontinuous proper convex function on an admissible complete* CAT(1) *space X, and assume that f has a minimiser. Let $\{r_n\}$ be a real sequence such that $\inf_{k\in\mathbb{N}} r_k > 0$. For an initial point $x_1 \in X$, generate an iterative sequence $\{x_n\}$ by*

$$x_{n+1} = \operatorname*{Argmin}_{y\in X}\left(f(y) - \frac{\cos d(x_n, y)}{r_n}\right)$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ Δ-converges to a minimiser of f, which equals

$$\lim_{n\to\infty} P_{\operatorname{Min} f}x_n,$$

where $P_{\operatorname{Min} f}$ is the metric projection onto the set $\operatorname{Min} f$ of all minimisers of f.

These developments suggest that proximal point algorithms can be studied beyond linear spaces by replacing linear convexity with an appropriate geometric convexity. Motivated by these developments, Kimura and Sudo [6] initiated the study of fixed point theory on unit spheres of Banach spaces. They introduced the notion of delta-convergence in the setting of Banach spheres and obtained some fundamental properties of closedness and compactness with respect to the delta-convergence; see [7].

Motivated by the resolvent formulation in CAT(1) spaces, we study a proximal point algorithm for convex minimisation problems on Banach spheres. More precisely, we introduce resolvent operators associated with proper lower semicontinuous spherically convex functions and investigate the asymptotic behaviour of iterative sequences generated by these resolvents. Under suitable assumptions on the underlying Banach space and the duality mapping, we prove that the generated sequence delta-converges to a minimiser of the objective function. Thus, our result may be regarded as a Banach-spherical analogue of proximal point approximation theory in Hilbert spaces and CAT(1) spaces.

2. Preliminaries

Let E be a Banach space. We denote the value of $f \in E^*$ at $x \in E$ by $\langle x, f\rangle$. We denote the unit sphere of E by

$$S_E = \{x \in E \mid \|x\| = 1\}.$$

We define the duality mapping J on E by

$$Jx = \left\{ f \in E^* \;\middle|\; \langle x, f\rangle = \|x\|^2 = \|f\|^2 \right\}$$

for $x \in E$. We say that E is strictly convex if for $x, y \in S_E$, $x = y$ whenever

$$\|x + y\| = 2.$$

We say that E is uniformly convex if for sequences $\{x_n\}, \{y_n\}$ in S_E,

$$\lim_{n\to\infty} \|x_n - y_n\| = 0$$

whenever

$$\lim_{n\to\infty} \|x_n + y_n\| = 2.$$

It is clear that the uniform convexity of E implies its strict convexity. We say that E is smooth if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S_E$. We say that E is uniformly smooth if the above limit is attained uniformly in $x, y \in S_E$. We have the following facts:

- If E is uniformly convex or uniformly smooth, then E is reflexive.
- If E is reflexive, then E is smooth if and only if E^* is strictly convex.
- If E is reflexive, then E is strictly convex if and only if E^* is smooth.
- E is smooth if and only if the duality mapping J becomes single-valued.
- If E is smooth and reflexive, then J is surjective.
- If E is smooth and strictly convex, then J is injective.
- If E is smooth, strictly convex and reflexive, then the duality mapping $J^* = J^{-1}$, where J^* is the duality mapping on E^*.
- E is uniformly convex if and only if E^* is uniformly smooth.
- E is uniformly smooth if and only if E^* is uniformly convex.
- If E is uniformly smooth, then J is norm-to-norm continuous.

For more details on Banach spaces and their geometry, see, for instance, [2, 14].

In what follows, we introduce some notions on the unit spheres of Banach spaces. Let E be a Banach space. It is well known that

$$|\langle x, f\rangle| \le \|x\|\|f\| = 1$$

for each $x \in S_E$ and $f \in S_{E^*}$. Using this fact, we define a function ρ on $S_E \times S_{E^*}$ by

$$\rho(x, f) = \arccos \langle x, f\rangle$$

for $x \in S_E$ and $f \in S_{E^*}$. Then, the following hold:

- For any $x \in S_E$ and $f \in S_{E^*}$, $\rho(x, f) \geq 0$;
- if E is smooth and strictly convex, then for $x, y \in S_E$, $x = y$ if and only if $\rho(x, Jy) = 0$.

Let S_E be the unit sphere of a Banach space E and $x, y \in S_E$. If $y \neq -x$, then

$$tx + (1-t)y \neq 0$$

for all $t \in [0, 1]$. Indeed, if there exists $t_0 \in [0, 1]$ such that

$$t_0 x + (1-t_0)y = 0,$$

then $t_0 \neq 1$ since we get a contradiction if $t_0 = 1$. Furthermore, we have

$$y = \frac{-t_0}{1-t_0}x,$$

and hence

$$\frac{t_0}{1-t_0} = \left\| \frac{-t_0}{1-t_0}x \right\| = \|y\| = 1.$$

This implies that $t_0 = 1/2$. Therefore, $y = -x$. Taking the contrapositive, we have the desired statement. Using this fact, for $x, y \in S_E$ with $y \neq -x$, we have

$$\frac{tx + (1-t)y}{\|tx + (1-t)y\|} \in S_E.$$

We define the spherical convex combination of x and y with a ratio $t \in [0, 1]$ by

$$tx \overset{\mathrm{s}}{\oplus} (1-t)y = \frac{tx + (1-t)y}{\|tx + (1-t)y\|}.$$

We say that a subset X of S_E is spherically convex if

$$tx \overset{\mathrm{s}}{\oplus} (1-t)y \in X$$

for all $x, y \in X$ with $y \neq -x$ and $t \in [0, 1]$. Suppose further that E is smooth. Then, we say that a subset X of S_E is admissible [5, 6] if

$$\rho(x, Jy) < \frac{\pi}{2}$$

for all $x, y \in X$. If a subset X of S_E is admissible, then $y \neq -x$ for all $x, y \in X$. In fact, if there exist $x, y \in X$ such that $y = -x$, then

$$0 < \langle x, Jy \rangle = -\|x\|^2 = -1,$$

which is a contradiction.

Let S_E be the unit sphere of a smooth and uniformly convex Banach space E and X a nonempty, admissible, closed and spherically convex subset of S_E. Let C be a nonempty closed spherically convex subset of X. Then, for $x \in X$, there exists a unique point $y_x \in C$ such that

$$\rho(y_x, Jx) = \inf_{y \in C} \rho(y, Jx).$$

We define the spherical projection Π_C onto C by $\Pi_C x = y_x$ for each $x \in X$. Then,

$$\rho(y, J\Pi_C x) \leq \rho(y, Jx)$$

for all $x \in X$ and $y \in C$. For more details, see [6, 7].

We next define delta-convergence of sequences in the unit spheres of Banach spaces. Let S_E be the unit sphere of a smooth Banach space E and $\{x_n\}$ a sequence in S_E. We call $x \in S_E$ an asymptotic centre of $\{x_n\}$ if

$$\limsup_{n\to\infty} \rho(x, Jx_n) = \inf_{y\in S_E} \limsup_{n\to\infty} \rho(y, Jx_n).$$

If a sequence $\{x_n\}$ in S_E satisfies

$$\inf_{y\in S_E} \limsup_{n\to\infty} \rho(y, Jx_n) < \frac{\pi}{2},$$

then we say that $\{x_n\}$ is spherically bounded. We say that a sequence $\{x_n\}$ in S_E delta-converges [6, 10] to $x \in S_E$ if x is the unique asymptotic centre of every subsequence of $\{x_n\}$. We have the following:

Theorem 2.1 (Kimura–Sudo [7])**.** *Let S_E be the unit sphere of a smooth and uniformly convex Banach space E. If a sequence $\{x_n\}$ in S_E is spherically bounded, then it has a unique asymptotic centre and a delta-convergent subsequence.*

Theorem 2.2 (Kimura–Sudo [7])**.** *Let S_E be the unit sphere of a uniformly smooth and uniformly convex Banach space E, and let X be a nonempty admissible closed subset of S_E such that JX is spherically convex. Then, the unique asymptotic centre of a spherically bounded sequence in X is included in X.*

If a Banach space E is smooth, strictly convex and reflexive, then for a sequence $\{x_n\}$ in S_E, $f \in S_{E^*}$ is an asymptotic centre of $\{Jx_n\}$ if and only if

$$\limsup_{n\to\infty} \rho(x_n, f) = \inf_{g\in S_{E^*}} \limsup_{n\to\infty} \rho(x_n, g).$$

Similarly, for a sequence $\{x_n\}$ in S_E, $\{Jx_n\}$ is spherically bounded if and only if

$$\inf_{g\in S_{E^*}} \limsup_{n\to\infty} \rho(x_n, g) < \frac{\pi}{2}.$$

Let S_E be the unit sphere of a smooth Banach space. We say that the duality mapping J on E is sequentially delta-continuous [7] on a subset X of S_E if the following statement holds: For $x \in X$ and a sequence $\{x_n\}$ in X, if $\{x_n\}$ delta-converges to x, then $\{Jx_n\}$ delta-converges to Jx.

Example 2.3 (Kimura–Sudo [7], Sudo [13])**.** Since the duality mapping of a Hilbert space coincides with the identity mapping, it is clearly sequentially delta-continuous duality mappings on the whole unit sphere.

For $p > 1$, Let $\mathbb{R}^m$ be the finite-dimensional space with the norm

$$\|x\| = \left(\sum_{k=1}^{m} |x_k|^p\right)^{1/p}.$$

Then, the delta-convergence on the unit sphere of $\mathbb{R}^m$ coincides with the norm convergence. Thus, the duality mapping is sequentially delta-continuous on the whole unit sphere.

For $p > 1$, let l^p be the real sequence space equipped with its usual norm. Let $r \in \,]0, 1[$. We define a subset of the unit sphere S_{l^p} of l^p by

$$X = \{(x_k) \in S_{l^p} \mid x_1 \geq r \text{ and } \forall i \in \mathbb{N} \setminus \{1\},\, x_i \geq 0\}.$$

In this setting, the duality mapping J on l^p is sequentially delta-continuous on X.

We shall use the following consequence:

Theorem 2.4. *Let S_E be the unit sphere of a uniformly smooth and uniformly convex Banach space E, and let X be a nonempty admissible closed spherically convex subset of S_E. Assume that the duality mapping J is sequentially delta-continuous on X. Let C be a nonempty, closed and spherically convex subset of X. If a sequence $\{x_n\}$ in C delta-converges to $x \in X$, then $x \in C$.*

Proof. Let $\{x_n\}$ be a delta-convergent sequence in C and let $x \in X$ be its delta-limit. Then, since J is sequentially delta-continuous, $\{Jx_n\}$ delta-converges to Jx. Put $C^* = JC$. To apply Theorem 2.2 to the dual space E^* and C^*, we show that E^* and C^* satisfy the required conditions as follows:

- E^* is uniformly smooth and uniformly convex;
- C^* is nonempty admissible closed;
- $J^*C^* = C$ is spherically convex;
- $\{Jx_n\}$ is in C^* and spherically bounded.

By the assumptions, E^* is uniformly smooth and uniformly convex, C^* is nonempty and J^*C^* is spherically convex. Since C is closed, $J^* = J^{-1}$ is norm-to-norm continuous and

$$C^* = JC = (J^*)^{-1}C,$$

we have C^* is closed. We show that C^* is admissible. Let $f, g \in C^*$. Then, there exist $p, q \in C$ such that $f = Jp$ and $g = Jq$. Moreover, since C is a subset of X and X is admissible, we have

$$\rho(f, J^*g) = \arccos \langle Jp, J^*Jq\rangle = \arccos \langle q, Jp\rangle = \rho(q, Jp) < \frac{\pi}{2}.$$

This implies that C^* is admissible. We prove that $\{Jx_n\}$ is in $JC = C^*$ and spherically bounded. Since $\{x_n\}$ is a sequence in C, $\{Jx_n\}$ is in JC. We show that $\{Jx_n\}$ is spherically bounded. If C consists of exactly one point, then $\{Jx_n\}$ is clearly spherically bounded. Thus, we suppose that C contains more than one point, and then so is C^*. Suppose that $\{Jx_n\}$ is not spherically bounded. Then, since C^* is admissible, for any $f \in C^*$, we have

$$\frac{\pi}{2} \le \inf_{g\in S_{E^*}} \limsup_{n\to\infty} \rho(x_n, g) \le \limsup_{n\to\infty} \rho(x_n, f) \le \frac{\pi}{2},$$

and hence

$$\limsup_{n\to\infty} \rho(x_n, f) = \inf_{g\in S_{E^*}} \limsup_{n\to\infty} \rho(x_n, g).$$

Therefore, any $f \in C^*$ is an asymptotic centre of $\{Jx_n\}$. Nevertheless, since $\{Jx_n\}$ delta-converges to Jx, the point Jx is the unique asymptotic centre of $\{Jx_n\}$, and hence $Jx = f$ for all $f \in C^*$. This implies that $C^* = \{Jx\}$. This is a contradiction since C^* contains more than one point. Hence, $\{Jx_n\}$ is spherically bounded. Therefore, applying Theorem 2.2 to E^* and C^*, we have $Jx \in C^*$. Consequently, $x \in J^{-1}C^* = C$. □

3. Proximal point algorithm

In this section, we prove a convergence theorem to approximate minimisers of spherically convex functions. We first recall the resolvent operators for spherically convex functions. Let S_E be the unit sphere of a smooth and uniformly convex Banach space E, and let X be a nonempty admissible closed spherically convex subset of S_E. Let f be a function from X to $]-\infty, \infty]$. We say that f is lower semicontinuous if, for any $a \in \mathbb{R}$, the set

$$\{x \in X \mid f(x) \le a\}$$

is closed in X. We say that f is proper if there exists $x \in X$ such that

$$f(x) < \infty.$$

We say that f is spherically convex if

$$f\left(tx \overset{\mathrm{s}}{\oplus} (1-t)y\right) \le tf(x) + (1-t)f(y)$$

for all $x, y \in X$ and $t \in]0,1[$. We call $x \in X$ a minimiser of f if

$$f(x) = \inf f(X),$$

and we denote the set of all minimisers of f by $\operatorname{Min} f$. Now, we introduce the resolvent operators of spherically convex functions by Kimura and Sudo [8]. Assume that f is proper, lower semicontinuous and spherically convex. Suppose that $\operatorname{Min} f$ is nonempty. Then, for a fixed $x \in X$, there exists a unique point $R_f x \in X$ such that

$$f(R_f x) - \cos \rho(R_f x, Jx) = \inf_{y \in X} (f(y) - \cos \rho(y, Jx)),$$

that is, $R_f x$ is a unique minimiser of a function

$$f(\cdot) - \cos \rho(\cdot, Jx).$$

In this way, we can define a mapping R_f on X, and we call R_f the resolvent operator for f. For any $r > 0$, since rf is proper, lower semicontinuous and spherically convex again, and

$$\operatorname{Min}(rf) = \operatorname{Min} f \neq \emptyset,$$

we can also define R_{rf} in the same setting as R_f. That is, R_{rf} is a unique point in X such that

$$rf(R_{rf}x) - \cos \rho(R_{rf}x, Jx) = \inf_{y \in X} (rf(y) - \cos \rho(y, Jx)).$$

Theorem 3.1 (Kimura–Sudo [8])**.** *Let S_E be the unit sphere of a smooth and uniformly convex Banach space E. Let X be a nonempty admissible closed spherically convex subset of S_E, and f a proper lower semicontinuous spherically convex function from X to $]-\infty, \infty]$ having a minimiser. Then,*

$$\rho(y, JR_f x) \le \rho(y, Jx)$$

for $x \in X$ and $y \in \operatorname{Min} f$.

Kimura and Sudo [8] proved the following function-value convergence theorem for the proximal point algorithm:

Theorem 3.2 (Kimura–Sudo [8])**.** *Let E be a uniformly smooth and uniformly convex Banach space. Let X be a nonempty admissible closed and spherically convex subset of S_E such that JX is spherically convex, and let f be a proper lower semicontinuous spherically convex function having a minimiser. For an initial point $x_1 \in X$ and a sequence $\{r_n\}$ of positive real numbers, generate an iterative sequence $\{x_n\}$ in X by*

$$x_{n+1} = R_{r_n f} x_n$$

for $n \in \mathbb{N}$. Then, the following hold:

(i) *The sequence $\{\Pi_{\operatorname{Min} f} x_n\}$ converges strongly to a minimiser of f;*

(ii) *for all $n \in \mathbb{N}$,*

$$|f(x_{n+1}) - \inf f(X)| \le \frac{1 - \langle \Pi_{\operatorname{Min} f} x_1, Jx_1 \rangle}{\sum_{k=1}^{n} r_k};$$

(iii) *if $\sum_{k=1}^{\infty} r_k = \infty$, then the sequence $\{f(x_n)\}$ converges to $\inf f(X)$.*

Using these results and delta-convergence, we obtain the following approximation theorem:

Theorem 3.3. *Let S_E be the unit sphere of a uniformly smooth and uniformly convex Banach space E, and let X be a nonempty admissible closed and spherically convex subset of S_E such that JX is spherically convex. Assume that J is sequentially delta-continuous on X. Let f be a lower semicontinuous proper spherically convex function having a minimiser. For an initial point $x_1 \in X$ and a positive real sequence $\{r_n\}$ such that $\sum_{k=1}^{\infty} r_k = \infty$, generate a sequence $\{x_n\}$ in X by*

$$x_{n+1} = R_{r_n f} x_n$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ delta-converges to a minimiser of f, which equals

$$\lim_{n\to\infty} \Pi_{\operatorname{Min} f} x_n.$$

Proof. By Theorem 3.2,

- The sequence $\{\Pi_{\operatorname{Min} f} x_n\}$ converges strongly to some $x_0 \in \operatorname{Min} f$;
- the sequence $\{f(x_n)\}$ converges to $\inf f(X)$.

For a fixed $n \in \mathbb{N}$, by Theorem 3.1, we have

$$\rho(x_0, Jx_{n+1}) = \rho(x_0, JR_{r_n f} x_n) \le \rho(x_0, Jx_n),$$

and hence

$$0 \le \rho(x_0, Jx_n) \le \rho(x_0, Jx_1) < \frac{\pi}{2}$$

for all $n \in \mathbb{N}$ since $x_0, x_1 \in X$ and X is admissible. Therefore, the limit

$$\lim_{n\to\infty} \rho(x_0, Jx_n) \in \left[0, \frac{\pi}{2}\right[$$

exists, and $\{x_n\}$ is spherically bounded.

By Theorem 2.1, $\{x_n\}$ has a delta-convergent subsequence. Take an arbitrary delta-convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and let $y \in S_E$ be its delta-limit. By Theorem 2.2, we have $y \in X$. We show that y is a minimiser of f. To this end, we prove

$$f(y) \le \limsup_{i\to\infty} f(x_{n_i}).$$

For $j \in \mathbb{N}$, let

$$C_j = \left\{ x \in X \;\middle|\; f(x) \le \sup_{i \ge j} f(x_{n_i}) \right\}.$$

Then, by the lower semicontinuity and the spherical convexity of f, we obtain that C_j is closed and spherically convex for all $j \in \mathbb{N}$. Since

$$x_{n_i} \in C_j$$

for all $i \ge j$, the tail sequence of $\{x_{n_i}\}$ is contained in C_j and still delta-converges to y. Hence, by Theorem 2.4, we have $y \in C_j$. Therefore,

$$f(y) \le \sup_{i \ge j} f(x_{n_i})$$

for all $j \in \mathbb{N}$, and hence

$$f(y) \le \inf_{j\in\mathbb{N}} \sup_{i \ge j} f(x_{n_i}) = \limsup_{i\to\infty} f(x_{n_i}).$$

On the other hand, since $\{f(x_n)\}$ converges to $\inf f(X)$, we obtain that

$$\inf f(X) \le f(y) \le \limsup_{i\to\infty} f(x_{n_i}) = \lim_{n\to\infty} f(x_n) = \inf f(X).$$

Therefore, y is a minimiser of f. Now, recall that $\{\Pi_{\operatorname{Min} f} x_n\}$ converges strongly to x_0, and

$$\cos\rho(y, J\Pi_{\operatorname{Min} f} x_{n_i}) \geq \cos\rho(y, Jx_{n_i})$$

for all $i \in \mathbb{N}$. Therefore, since the limit of $\{\cos\rho(x_0, Jx_n)\}$ exists,

$$\begin{aligned}
\liminf_{i\to\infty} \cos\rho(x_0, Jx_{n_i}) &= \lim_{i\to\infty} \cos\rho(x_0, Jx_{n_i}) = \lim_{i\to\infty} \langle x_0, Jx_{n_i}\rangle \\
&= \lim_{i\to\infty} \left(\langle \Pi_{\operatorname{Min} f} x_{n_i}, Jx_{n_i}\rangle - \langle \Pi_{\operatorname{Min} f} x_{n_i} - x_0, Jx_{n_i}\rangle\right) \\
&\geq \liminf_{i\to\infty} \left(\langle \Pi_{\operatorname{Min} f} x_{n_i}, Jx_{n_i}\rangle - \|\Pi_{\operatorname{Min} f} x_{n_i} - x_0\|\right) \\
&= \liminf_{i\to\infty} \cos\rho(\Pi_{\operatorname{Min} f} x_{n_i}, Jx_{n_i}) \\
&\geq \liminf_{i\to\infty} \cos\rho(y, Jx_{n_i}).
\end{aligned}$$

Hence, since cos is nonincreasing on $[0, \pi]$ and y is an asymptotic centre of $\{x_{n_i}\}$, we have

$$\limsup_{i\to\infty} \rho(x_0, Jx_{n_i}) \leq \limsup_{i\to\infty} \rho(y, Jx_{n_i}) = \inf_{z\in S_E} \limsup_{i\to\infty} \rho(z, Jx_{n_i})$$

This implies that x_0 is an asymptotic centre of $\{x_{n_i}\}$, and hence $y = x_0$. Therefore, $\{x_{n_i}\}$ delta-converges to x_0. Consequently, for every delta-convergent subsequence of $\{x_n\}$, its delta-limit is x_0.

We finally show that $\{x_n\}$ delta-converges to x_0. Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily and let $z \in S_E$ be the unique asymptotic centre of $\{x_{n_i}\}$. Here, we take a delta-convergent subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that

$$\lim_{j\to\infty} \rho(x_0, Jx_{n_{i_j}}) = \limsup_{i\to\infty} \rho(x_0, Jx_{n_i}).$$

The sequence $\{x_{n_{i_j}}\}$ delta-converges to x_0. Then, we obtain that

$$\limsup_{i\to\infty} \rho(x_0, Jx_{n_i}) = \lim_{j\to\infty} \rho(x_0, Jx_{n_{i_j}}) \leq \limsup_{j\to\infty} \rho(z, Jx_{n_{i_j}}) \leq \limsup_{i\to\infty} \rho(z, Jx_{n_i}).$$

Since z is the unique asymptotic centre of $\{x_{n_i}\}$, we have $z = x_0$. Consequently, x_0 is the unique asymptotic centre of every subsequence of $\{x_n\}$, and hence $\{x_n\}$ delta-converges to x_0. □

4. Conclusion

In this paper, we established a delta-convergence theorem for the proximal point algorithm on Banach spheres. More precisely, for a proper lower semicontinuous spherically convex function defined on an admissible spherically convex subset of a Banach sphere, we considered the iteration generated by the associated resolvent operators and proved that the generated sequence delta-converges to a minimiser of the function. The proof combines the function-value convergence theorem for the proximal point algorithm with the geometric theory of delta-convergence on Banach spheres.

Consequently, the obtained theorem provides a delta-convergence principle for convex minimisation problems in the framework of Banach spheres. This result may also be regarded as a spherical analogue of proximal point approximation theory in classical Banach space settings. The essential point is that delta-convergence provides an appropriate substitute for weak convergence, since Banach spheres do not possess a linear weak topology.

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