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Caputo fractional systems with variable coefficients: Existence and stability results via Peano–Baker series

V. Kavitha[a], R. Sowmiya[b], D. Baleanu[c], M. Mallika Arjunan[d,*]

[a]*Division of Mathematics and Robotics Engineering, Karunya Institute of Technology and Sciences, Coimbatore-641114, Tamil Nadu, India.*

[b]*Department of Mathematics, Coimbatore Institute of Engineering and Technology, Coimbatore–641109, Tamil Nadu, India.*

[c]*Department of Computer Science and Mathematics, Lebanese American University, Beirut, Lebanon.*

[d]*Department of Mathematics, School of Arts, Sciences, Humanities and Education, SASTRA Deemed to be University, Thanjavur-613401, Tamil Nadu, India.*

Abstract

This paper studies linear fractional differential systems with variable coefficients involving the Caputo derivative $(0 < \alpha < 1)$. The solution framework is formulated in the Banach space $C(\mathcal{J}, \mathbb{R}^n)$, reflecting the regularity of Caputo-type solutions. A state-transition matrix is constructed via the generalized Peano–Baker series and shown to converge uniformly with a Mittag–Leffler bound. Existence and uniqueness results are established for both homogeneous and inhomogeneous problems, with the latter yielding a mixed-kernel representation. Stability of the trivial solution is characterized through Mittag–Leffler estimates and spectral conditions, alongside bounded-input bounded-output stability. Illustrative examples validate the theoretical findings and highlight the analytical framework.

Keywords: Caputo fractional derivative, variable-coefficient systems, generalised Peano–Baker series, state-transition matrix, existence and uniqueness, Mittag–Leffler stability, bounded-input bounded-output stability (BIBO)
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[*] Corresponding author

Email addresses: `kavi_velubagyam@yahoo.co.in` (V. Kavitha), `sowmivishnu07@gmail.com` (R. Sowmiya), `dumitru.baleanu@lau.edu.lb` (D. Baleanu), `arjunphd07@yahoo.co.in` (M. Mallika Arjunan)

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1. Introduction

Fractional differential equations (FDEs) have become an indispensable modelling tool across science and engineering because they encode non-local memory effects and hereditary properties that classical integer-order models cannot capture. For additional details, we suggest the reader to refer the monographs of Podlubny [23], Kilbas et al. [12], Samko et al. [26], and Diethelm [8], while a broad survey of contemporary engineering applications, spanning biomedical imaging, control, material science, and signal processing, is given in [3, 13, 17, 25, 27].

Among the various fractional operators in the literature, the Caputo derivative ${}^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}$ occupies a privileged position for initial-value problems. In contrast to the Riemann–Liouville derivative, whose solutions carry the integrable singularity $(\xi - \xi_0)^{\alpha-1}$ at the base point and require initial data in a weighted space, the Caputo operator admits the standard pointwise initial condition $\mathbf{x}(\xi_0) = \widetilde{\mathbf{x}}_0$, its solutions belong to the classical space $C(\mathcal{J}, \mathbb{R}^n)$, and the Caputo derivative of a constant vanishes identically, properties that make it the natural choice for physically motivated initial-value problems [8, 12, 23, 24]. The two-parameter Mittag–Leffler function [10], which plays the role of the matrix exponential in the fractional setting, arises naturally in the solution theory for both operators.

Caputo-type FDE models appear concretely in several areas: anomalous diffusion and maximum principles for time-fractional equations [16], complex spatio-temporal patterns in fractional reaction-diffusion systems near bifurcation points [6], and diffusion-wave equations with mass absorption under harmonic excitation [7]. These applications all give rise to linear or linearised systems whose analytical treatment requires a rigorous solution theory.

The analytical study of linear FDE systems with constant coefficients is now well established. Explicit state-transition matrices expressed through matrix Mittag–Leffler functions were derived by Chikrii and Eidelman [4] and Chikrii and Matichin [5]. These representations were subsequently applied to time-optimal control, optimal control, and differential game problems for fractional-order linear systems in a series of papers by Matychyn and Onyshchenko [18, 19, 20, 22]. The constant-coefficient setting is, however, too restrictive for many real-world models: linearised aircraft dynamics, restricted-growth population models, and distributed-parameter battery models all lead to linear FDEs with time-varying coefficient matrices [21].

The analytical treatment of variable-coefficient linear FDE systems requires a fundamentally different approach. The classical tool for this task is the Peano–Baker series, whose fractional generalisation was introduced in the context of initialized fractional calculus by Lorenzo and Hartley [15] and further developed in [2]. Eckert et al. [9] constructed a generalised Peano–Baker (GPB) series for time-variant FDEs and obtained state-transition matrices for initialized systems. Building on this framework, Matychyn [21] derived explicit solutions for linear variable-coefficient systems governed by both the Riemann–Liouville and Caputo derivatives, establishing the form of the state-transition matrices and solution formulae under an assumed uniform convergence of the GPB series. That work, however, left three essential questions unanswered: *(i)* no self-contained proof of GPB convergence was given; *(ii)* no existence or uniqueness theory was established in the correct function space; *(iii)* no stability analysis of any kind was developed.

The Mittag–Leffler stability of fractional-order systems was pioneered by Li et al. [14] through a fractional Lyapunov direct method, and the key fractional Leibniz inequality for Lyapunov function differentiation was established by Aguila-Camacho et al. [1]. A generalised Gronwall inequality tailored to Caputo FDEs, which underpins uniqueness proofs and comparison principles, was obtained by Ye et al. [28] and in its singular form by Henry [11]. Despite this body of stability theory for constant-coefficient systems, a corresponding treatment for variable-coefficient Caputo systems driven by the GPB series has not been developed.

The stability estimates developed below are also motivated by applications in control theory and physics. In fractional control systems, time-varying coefficient matrices naturally occur after linearisation along a reference trajectory or after gain scheduling. In physical models, variable coefficients describe non-homogeneous viscoelastic media, anomalous diffusion in non-uniform environments, and systems whose memory response changes with the operating regime. These settings require state-transition matrices and stability estimates that remain valid beyond the constant-coefficient Mittag–Leffler formula.

The present paper fills precisely this gap. The main contributions are as follows.

- (i) The solution space $C(\mathcal{I}, \mathbb{R}^n)$ is identified as the correct phase space for the Caputo initial-value problem, and a Volterra integral equivalence is established (Proposition 3.2).
- (ii) The GPB series for the Caputo state-transition matrix is proved to converge absolutely and uniformly with the sharp Mittag–Leffler bound $\|\boldsymbol{\Psi}(\xi,\xi_0)\| \le \mathbb{E}_\alpha(\mathfrak{M}(\xi-\xi_0)^\alpha)$, derived by a recursive Beta-function induction (Theorem 3.4).
- (iii) Existence and uniqueness are proved for both the homogeneous and inhomogeneous Cauchy problems (Theorems 3.7–3.10). The inhomogeneous solution reveals a structurally novel mixed-kernel representation combining the Caputo matrix $\boldsymbol{\Psi}$ with the Riemann–Liouville matrix $\boldsymbol{\Phi}$; all required properties of $\boldsymbol{\Phi}$ are derived self-containedly within the proof (Lemma 3.5).
- (iv) Mittag–Leffler stability is characterised via a Lyapunov-function criterion (Theorem 3.18) and a spectral negativity condition (Theorem 3.20), and bounded-input bounded-output stability is established for the inhomogeneous system with an explicit convolution bound (Theorem 3.23).

The paper is organised as follows. Section 2 collects the necessary operator definitions, function spaces, and the Mittag–Leffler function. Section 3 presents the existence and uniqueness theory together with the GPB convergence analysis. The stability results and three illustrative examples appear in the subsequent subsections of Section 3. Section 4 states the conclusion and future directions.

2. Preliminaries

Throughout this article, we fix the compact interval $\mathcal{I} = [\xi_0, \xi_1]$ with $0 \le \xi_0 < \xi_1 < \infty$, and set $\mathring{\mathcal{I}} = (\xi_0, \xi_1)$. The compact interval is used for the local well-posedness and GPB convergence results. In the stability subsection, the notation $\mathcal{I}_T := [\xi_0, T]$ and $\mathcal{I}_\infty := [\xi_0, \infty)$ is used explicitly, since asymptotic and BIBO stability are global notions. We write $n \in \mathbb{N}$ for the state dimension, $\mathbb{R}^{n\times n}$ for the algebra of real $n \times n$ matrices, $\mathfrak{I}_n$ for the $n\times n$ identity matrix, and $\|\cdot\|$ for the induced matrix (or Euclidean vector) norm. The Euler Gamma function is denoted by $\Gamma(\,\cdot\,)$ and the Beta function by $\mathrm{B}(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$.

2.1. Riemann–Liouville and Caputo Operators

Definition 2.1. [23][Riemann–Liouville fractional integral and derivative] Let $0 < \alpha < 1$ and let $\mathbf{f} : \mathcal{I} \to \mathbb{R}^n$ be absolutely continuous. The Riemann–Liouville fractional integral of order α with base point ξ_0 is defined by

$$ {}^{\mathcal{RL}}_{\xi_0}\mathfrak{I}^{\alpha}_{\xi}\,\mathbf{f}(\xi) \;:=\; \frac{1}{\Gamma(\alpha)} \int_{\xi_0}^{\xi} (\xi-\varsigma)^{\alpha-1}\,\mathbf{f}(\varsigma)\,d\varsigma, \qquad \xi \in \mathcal{I}. \tag{1} $$

The Riemann–Liouville fractional derivative of order α is defined by

$$ {}^{\mathcal{RL}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\,\mathbf{f}(\xi) \;:=\; \frac{d}{d\xi}\Big({}^{\mathcal{RL}}_{\xi_0}\mathfrak{I}^{1-\alpha}_{\xi}\,\mathbf{f}(\xi)\Big), \qquad \xi \in \mathring{\mathcal{I}}. \tag{2} $$

Remark 2.2. In contrast to classical differentiation, ${}^{\mathcal{RL}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\,c \neq 0$ for a nonzero constant c; in fact, ${}^{\mathcal{RL}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\,c = c\,(\xi-\xi_0)^{-\alpha}/\Gamma(1-\alpha)$, which blows up as $\xi \to \xi_0^+$. This motivates the regularised Caputo operator below.

Definition 2.3. [23][Caputo fractional derivative] Under the hypotheses of Definition 2.1, the Caputo fractional derivative of order α is

$$ {}^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\,\mathbf{f}(\xi) \;:=\; {}^{\mathcal{RL}}_{\xi_0}\mathfrak{I}^{1-\alpha}_{\xi}\left(\frac{d}{d\xi}\mathbf{f}(\xi)\right), \quad \xi \in \mathring{\mathcal{I}}. $$

The two operators are linked through the following identity, which is fundamental to the entire analysis.

Proposition 2.4 (Interrelation of operators)**.** *For* $\mathbf{f} \in C^1(\mathcal{I}, \mathbb{R}^n)$ *and* $0 < \alpha < 1$*,*

$$ {}^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\,\mathbf{f}(\xi) \;=\; {}^{\mathcal{RL}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\,\mathbf{f}(\xi) \;-\; \frac{\mathbf{f}(\xi_0)}{\Gamma(1-\alpha)}\,(\xi-\xi_0)^{-\alpha}. $$

In particular, the two operators coincide whenever $\mathbf{f}(\xi_0) = \mathbf{0}$*.*

2.2. Fundamental Operational Identities

Lemma 2.5. *[12][Semigroup property and left-inversion] Let $\alpha, \beta > 0$. For every $\mathbf{f} \in C(\mathcal{J}, \mathbb{R}^n)$ for which the expressions below are well-defined, the following hold:*

- (i) ${}^{\mathcal{RL}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\left({}^{\mathcal{RL}}_{\xi_0}\mathfrak{I}^{\alpha}_{\xi}\,\mathbf{f}(\xi)\right) = \mathbf{f}(\xi)$;
- (ii) ${}^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\left({}^{\mathcal{RL}}_{\xi_0}\mathfrak{I}^{\alpha}_{\xi}\,\mathbf{f}(\xi)\right) = \mathbf{f}(\xi)$;
- (iii) ${}^{\mathcal{RL}}_{\xi_0}\mathfrak{I}^{\alpha}_{\xi}\left({}^{\mathcal{RL}}_{\xi_0}\mathfrak{I}^{\beta}_{\xi}\,\mathbf{f}(\xi)\right) = {}^{\mathcal{RL}}_{\xi_0}\mathfrak{I}^{\alpha+\beta}_{\xi}\,\mathbf{f}(\xi)$.

Lemma 2.6. *[12, 21][Action on power functions] For $0 < \alpha < 1$ and $\beta > 0$, the following identities hold:*

$$ {}^{\mathcal{RL}}_{\xi_0}\mathfrak{I}^{\alpha}_{\xi}\,(\xi-\xi_0)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\xi-\xi_0)^{\beta+\alpha-1}, \tag{3} $$

$$ {}^{\mathcal{RL}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\,(\xi-\xi_0)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\xi-\xi_0)^{\beta-\alpha-1}, \quad \beta \neq \alpha, \tag{4} $$

$$ {}^{\mathcal{RL}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\left(\frac{(\xi-\xi_0)^{\alpha-1}}{\Gamma(\alpha)}\right) = 0, \tag{5} $$

$$ {}^{\mathcal{RL}}_{\xi_0}\mathfrak{I}^{1-\alpha}_{\xi}\left(\frac{(\xi-\xi_0)^{\alpha-1}}{\Gamma(\alpha)}\right)\Bigg|_{\xi=\xi_0} = 1. \tag{6} $$

2.3. Function Spaces and Weighted Norms

Definition 2.7 (Solution space–Caputo case)**.** The natural solution space for the Caputo system (10) is the Banach space

$$ C\big(\mathcal{J}, \mathbb{R}^n\big) := \Big\{\mathbf{f} : \mathcal{J} \to \mathbb{R}^n \;\Big|\; \mathbf{f} \text{ is continuous on } \mathcal{J}\Big\}, \tag{7} $$

equipped with the uniform norm

$$ \|\mathbf{f}\|_\infty := \sup_{\xi\in\mathcal{J}} \|\mathbf{f}(\xi)\|. \tag{8} $$

Definition 2.8 (Bielecki norm–Caputo case)**.** For $\delta > 0$, the Bielecki norm on $C(\mathcal{J}, \mathbb{R}^n)$ is

$$ \|\mathbf{f}\|_\delta := \sup_{\xi\in\mathcal{J}} e^{-\delta(\xi-\xi_0)}\|\mathbf{f}(\xi)\|. \tag{9} $$

The norms $\|\cdot\|_\infty$ and $\|\cdot\|_\delta$ are equivalent for each fixed $\delta > 0$, since $e^{-\delta(\xi_1-\xi_0)}\|\mathbf{f}\|_\infty \le \|\mathbf{f}\|_\delta \le \|\mathbf{f}\|_\infty$. The Bielecki norm is used to render the Volterra operator strictly contractive in fixed-point arguments.

2.4. The Mittag–Leffler Function

Definition 2.9. [12][Two-parameter Mittag–Leffler function] For $\alpha, \beta > 0$ and $z \in \mathbb{C}$, the two-parameter Mittag–Leffler function is the entire function

$$ \mathbb{E}_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}. $$

The one-parameter variant is $\mathbb{E}_\alpha(z) := \mathbb{E}_{\alpha,1}(z)$.

Lemma 2.10. *[12][Asymptotic decay and positivity] Let $0 < \alpha < 1$ and $\lambda > 0$.*

- (i) *$\mathbb{E}_\alpha(-\lambda\xi^\alpha) > 0$ for all $\xi \ge 0$.*
- (ii) *There exists a constant $C_\alpha > 0$, depending only on α, such that $\mathbb{E}_\alpha(-\lambda\xi^\alpha) \le \frac{C_\alpha}{1+\lambda\xi^\alpha}$, $\xi \ge 0$.*
- (iii) *$\mathbb{E}_\alpha(-\lambda\xi^\alpha) \to 0$ as $\xi \to +\infty$.*

3. Caputo System: Existence, Uniqueness, and the Generalised Peano–Baker Series

3.1. Problem Formulation and Equivalent Integral Equation

We now turn to linear fractional differential systems governed by the Caputo operator. The homogeneous Caputo Cauchy problem reads

$$\begin{cases} {}^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\,\mathbf{x}(\xi) = \mathcal{A}(\xi)\,\mathbf{x}(\xi), \quad \xi \in \mathring{\mathcal{J}}, \\ \mathbf{x}(\xi_0) = \widetilde{\mathbf{x}}_0, \end{cases} \tag{10}$$

and the inhomogeneous Caputo Cauchy problem reads

$$\begin{cases} {}^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\,\mathbf{x}(\xi) = \mathcal{A}(\xi)\,\mathbf{x}(\xi) + \mathbf{u}(\xi), \quad \xi \in \mathring{\mathcal{J}}, \\ \mathbf{x}(\xi_0) = \widetilde{\mathbf{x}}_0, \end{cases} \tag{11}$$

where $0 < \alpha < 1$, $\widetilde{\mathbf{x}}_0 \in \mathbb{R}^n$, $\mathcal{A} \in C\big(\mathcal{J}, \mathbb{R}^{n\times n}\big)$, and $\mathbf{u} \in C\big(\mathcal{J}, \mathbb{R}^n\big)$.

We first distinguish the mild and classical meanings of solution. This distinction is important because the fixed-point and GPB constructions are naturally carried out in the Banach space $C(\mathcal{J}, \mathbb{R}^n)$, whereas the pointwise Caputo equation requires additional fractional regularity.

Definition 3.1 (Mild and classical Caputo solutions)**.** A function $\mathbf{x} \in C(\mathcal{J}, \mathbb{R}^n)$ is called a mild solution of the homogeneous problem (10) if it satisfies the Volterra integral equation (12) for every $\xi \in \mathcal{J}$. It is called a mild solution of the inhomogeneous problem (11) if it satisfies (13).

A mild solution is called a classical Caputo solution if, in addition, $\mathbf{x} - \widetilde{\mathbf{x}}_0$ belongs to the range of ${}^{\mathcal{RL}}_{\xi_0}\mathfrak{I}^{\alpha}_{\xi}$ acting on $C(\mathcal{J}, \mathbb{R}^n)$, the Caputo derivative ${}^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\mathbf{x}(\xi)$ exists on $\mathring{\mathcal{J}}$, and the corresponding differential equation in (10) or (11) holds pointwise.

Proposition 3.2 (Volterra integral equivalence–Caputo case)**.** *Let $\mathcal{A} \in C(\mathcal{J}, \mathbb{R}^{n\times n})$ and $\mathbf{u} \in C(\mathcal{J}, \mathbb{R}^n)$. Every classical Caputo solution of* (10) *satisfies*

$$\mathbf{x}(\xi) \;=\; \widetilde{\mathbf{x}}_0 \;+\; \frac{1}{\Gamma(\alpha)} \int_{\xi_0}^{\xi} (\xi - \varsigma)^{\alpha-1} \mathcal{A}(\varsigma)\,\mathbf{x}(\varsigma)\,d\varsigma, \qquad \xi \in \mathcal{J}. \tag{12}$$

Conversely, any $\mathbf{x} \in C(\mathcal{J}, \mathbb{R}^n)$ satisfying (12) *is a mild solution of* (10)*; if the additional fractional regularity stated in Definition 3.1 holds, then it is a classical Caputo solution. Similarly, every classical Caputo solution of* (11) *satisfies*

$$\mathbf{x}(\xi) \;=\; \widetilde{\mathbf{x}}_0 \;+\; \frac{1}{\Gamma(\alpha)} \int_{\xi_0}^{\xi} (\xi - \varsigma)^{\alpha-1} \big[\mathcal{A}(\varsigma)\,\mathbf{x}(\varsigma) + \mathbf{u}(\varsigma)\big]\,d\varsigma, \quad \xi \in \mathcal{J}, \tag{13}$$

and conversely (13) *defines the corresponding mild solution.*

Proof. Assume first that $\mathbf{x}$ is a classical Caputo solution of (10). Applying ${}^{\mathcal{RL}}_{\xi_0}\mathfrak{I}^{\alpha}_{\xi}$ to the differential equation and using the standard Caputo identity

$${}^{\mathcal{RL}}_{\xi_0}\mathfrak{I}^{\alpha}_{\xi}\big({}^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\mathbf{x}(\xi)\big) = \mathbf{x}(\xi) - \mathbf{x}(\xi_0), \tag{14}$$

we obtain (12) after substituting $\mathbf{x}(\xi_0) = \widetilde{\mathbf{x}}_0$. The same argument gives (13) for the inhomogeneous problem.

Conversely, if (12) or (13) holds, then $\mathbf{x} \in C(\mathcal{J}, \mathbb{R}^n)$ satisfies the corresponding integral equation and is therefore a mild solution by Definition 3.1. If $\mathbf{x} - \widetilde{\mathbf{x}}_0$ lies in the range of the Riemann–Liouville fractional integral and the pointwise Caputo derivative exists, applying ${}^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}$ to the Volterra equation and using Lemma 2.5 recovers the differential equation on $\mathring{\mathcal{J}}$. The integral term vanishes at $\xi = \xi_0$, so the initial condition also holds. □

3.2. The Generalised Peano–Baker Series for the Caputo System

Definition 3.3 (Caputo Peano–Baker sequence and state-transition matrix)**.** The Caputo Peano–Baker matrix sequence $\{\boldsymbol{\Psi}_k(\xi,\xi_0)\}_{k=0}^{\infty}$ is defined recursively by

$$\boldsymbol{\Psi}_0(\xi,\xi_0) := \mathfrak{I}_n, \tag{15}$$

$$\begin{aligned}\boldsymbol{\Psi}_{k+1}(\xi,\xi_0) &:= {}^{\mathcal{RL}}_{\xi_0}\mathfrak{I}^{\alpha}_{\xi}\big(\mathcal{A}(\xi)\,\boldsymbol{\Psi}_k(\xi,\xi_0)\big)\\ &= \frac{1}{\Gamma(\alpha)}\int_{\xi_0}^{\xi}(\xi-\varsigma)^{\alpha-1}\,\mathcal{A}(\varsigma)\,\boldsymbol{\Psi}_k(\varsigma,\xi_0)\,d\varsigma, \qquad k\ge 0.\end{aligned} \tag{16}$$

The Caputo generalised Peano–Baker (GPB) series is

$$\boldsymbol{\Psi}(\xi,\xi_0) := \sum_{k=0}^{\infty}\boldsymbol{\Psi}_k(\xi,\xi_0), \tag{17}$$

and $\boldsymbol{\Psi}(\xi,\xi_0)$ is called the Caputo state-transition matrix of system (10).

Assumption 1. The matrix function $\mathcal{A}$ is continuous on $\mathfrak{I}$ and there exists a finite constant $\mathfrak{M}>0$ such that

$$\|\mathcal{A}(\xi)\| \le \mathfrak{M} \qquad \text{for all } \xi\in\mathfrak{I}. \tag{18}$$

Since $\mathcal{A}\in C(\mathfrak{I},\mathbb{R}^{n\times n})$ and $\mathfrak{I}$ is compact, one may always choose $\mathfrak{M}:=\max_{\xi\in\mathfrak{I}}\|\mathcal{A}(\xi)\|$.

Theorem 3.4 (Uniform convergence of the Caputo GPB series)**.** *Under Assumption 1, the k-th Caputo Peano–Baker kernel satisfies the pointwise bound*

$$\big\|\boldsymbol{\Psi}_k(\xi,\xi_0)\big\| \le \frac{\mathfrak{M}^k\,(\xi-\xi_0)^{k\alpha}}{\Gamma\big(k\alpha+1\big)}, \qquad k\ge 0,\ \xi\in\mathfrak{I}. \tag{19}$$

Consequently, the series (17) converges absolutely and uniformly on $\mathfrak{I}$, and the Caputo state-transition matrix satisfies

$$\big\|\boldsymbol{\Psi}(\xi,\xi_0)\big\| \le \mathbb{E}_{\alpha}\big(\mathfrak{M}(\xi-\xi_0)^{\alpha}\big), \qquad \xi\in\mathfrak{I}_{\infty}, \tag{20}$$

where $\mathbb{E}_{\alpha}(z):=\mathbb{E}_{\alpha,1}(z)$ is the one-parameter Mittag–Leffler function.

Proof. We proceed by mathematical induction on k.

Base case ($k=0$). From (15) and $\|\mathfrak{I}_n\|=1$,

$$\big\|\boldsymbol{\Psi}_0(\xi,\xi_0)\big\| = 1 = \frac{\mathfrak{M}^0\,(\xi-\xi_0)^{0\cdot\alpha}}{\Gamma(0\cdot\alpha+1)} = \frac{1}{\Gamma(1)},$$

confirming (19) for $k=0$.

Inductive step. Assume (19) holds for some fixed $k\ge 0$. Applying (16) and Assumption 1,

$$\begin{aligned}\big\|\boldsymbol{\Psi}_{k+1}(\xi,\xi_0)\big\| &\le \frac{1}{\Gamma(\alpha)}\int_{\xi_0}^{\xi}(\xi-\varsigma)^{\alpha-1}\,\|\mathcal{A}(\varsigma)\|\,\|\boldsymbol{\Psi}_k(\varsigma,\xi_0)\|\,d\varsigma\\ &\le \frac{\mathfrak{M}^{k+1}}{\Gamma(\alpha)\,\Gamma(k\alpha+1)}\int_{\xi_0}^{\xi}(\xi-\varsigma)^{\alpha-1}\,(\varsigma-\xi_0)^{k\alpha}\,d\varsigma.\end{aligned}$$

Evaluating the convolution integral by the Beta-function identity with $\beta=k\alpha+1$, one has

$$\begin{aligned}\int_{\xi_0}^{\xi}(\xi-\varsigma)^{\alpha-1}(\varsigma-\xi_0)^{k\alpha}\,d\varsigma &= \mathrm{B}(\alpha,\,k\alpha+1)\,(\xi-\xi_0)^{(k+1)\alpha}\\ &= \frac{\Gamma(\alpha)\,\Gamma(k\alpha+1)}{\Gamma((k+1)\alpha+1)}\,(\xi-\xi_0)^{(k+1)\alpha},\end{aligned}$$

we obtain

$$\big\|\mathbf{\Psi}_{k+1}(\xi,\xi_0)\big\| \;\le\; \frac{\mathfrak{M}^{k+1}\,(\xi-\xi_0)^{(k+1)\alpha}}{\Gamma\big((k+1)\alpha+1\big)},$$

completing the induction. Summing over all $k \geq 0$ and identifying the series as $\mathbb{E}_\alpha$ (Definition 2.9), we have

$$\sum_{k=0}^{\infty}\big\|\mathbf{\Psi}_k(\xi,\xi_0)\big\| \;\le\; \sum_{k=0}^{\infty}\frac{\big[\mathfrak{M}(\xi-\xi_0)^{\alpha}\big]^k}{\Gamma(k\alpha+1)} \;=\; \mathbb{E}_\alpha(\mathfrak{M}(\xi-\xi_0)^{\alpha})\,,$$

which is continuous and bounded on the compact set $\mathfrak{I}$. The Weierstrass M-test therefore guarantees absolute and uniform convergence of (17). □

3.3. Properties of the state-transition matrix

Lemma 3.5. *[Properties of the RL state-transition matrix] For each fixed $\varsigma \in \mathring{\mathfrak{I}}$, define the matrix-valued function $\mathbf{\Phi}(\cdot,\varsigma) : \mathfrak{I} \to \mathbb{R}^{n\times n}$ by the Peano–Baker series*

$$\mathbf{\Phi}(\xi,\varsigma) \;:=\; \sum_{k=0}^{\infty}\mathbf{\Phi}_k(\xi,\varsigma), \tag{21}$$

where

$$\mathbf{\Phi}_0(\xi,\varsigma) \;:=\; \frac{(\xi-\varsigma)^{\alpha-1}}{\Gamma(\alpha)}\,\mathfrak{I}_n, \tag{22}$$

$$\mathbf{\Phi}_{k+1}(\xi,\varsigma) \;:=\; \frac{1}{\Gamma(\alpha)}\int_{\varsigma}^{\xi}(\xi-\rho)^{\alpha-1}\,\mathcal{A}(\rho)\,\mathbf{\Phi}_k(\rho,\varsigma)\,d\rho, \qquad k \geq 0. \tag{23}$$

Under Assumption 1, the weighted series

$$(\xi-\varsigma)^{1-\alpha}\mathbf{\Phi}(\xi,\varsigma) = \sum_{k=0}^{\infty}(\xi-\varsigma)^{1-\alpha}\mathbf{\Phi}_k(\xi,\varsigma)$$

converges absolutely and uniformly on the triangular set $\{(\xi,\varsigma) : \xi_0 \leq \varsigma < \xi \leq \xi_1\}$. Moreover, the unweighted series converges locally uniformly on every strip $\xi \geq \varsigma + \varepsilon$, $\varepsilon > 0$. The following two identities hold for every $\xi \in (\varsigma, \xi_1)$:

(P1) *Fractional differential equation:*

$${}^{\mathcal{RL}}_{\varsigma}\mathfrak{D}^{\alpha}_{\xi}\,\mathbf{\Phi}(\xi,\varsigma) \;=\; \mathcal{A}(\xi)\,\mathbf{\Phi}(\xi,\varsigma). \tag{24}$$

(P2) *Initial condition:*

$${}^{\mathcal{RL}}_{\varsigma}\mathfrak{I}^{1-\alpha}_{\xi}\,\mathbf{\Phi}(\xi,\varsigma)\Big|_{\xi=\varsigma} \;=\; \mathfrak{I}_n. \tag{25}$$

Proof. Convergence. The kernel bound holds:

$$\big\|\mathbf{\Phi}_k(\xi,\varsigma)\big\| \;\le\; \frac{\mathfrak{M}^k\,(\xi-\varsigma)^{(k+1)\alpha-1}}{\Gamma\big((k+1)\alpha\big)}, \qquad k \geq 0, \tag{26}$$

proved by induction on k via the same Beta-function argument used in Theorem 3.4 (replacing ξ_0 with ς throughout). Since the factor $(\xi-\varsigma)^{\alpha-1}$ is singular on the diagonal, the appropriate uniform object is the weighted kernel. Multiplying (26) by $(\xi-\varsigma)^{1-\alpha}$ gives

$$\big\|(\xi-\varsigma)^{1-\alpha}\mathbf{\Phi}_k(\xi,\varsigma)\big\| \le \frac{\mathfrak{M}^k(\xi-\varsigma)^{k\alpha}}{\Gamma((k+1)\alpha)} \le \frac{\mathfrak{M}^k(\xi_1-\xi_0)^{k\alpha}}{\Gamma((k+1)\alpha)}. \tag{27}$$

Moreover,

$$\sum_{k=0}^{\infty} \frac{\mathfrak{M}^k (\xi_1-\xi_0)^{k\alpha}}{\Gamma((k+1)\alpha)} < \infty,$$

because it is a shifted Mittag–Leffler type series. Hence the Weierstrass M-test proves absolute and uniform convergence of the weighted series. On every strip $\xi \geq \varsigma+\varepsilon$, the singular factor is bounded by $\varepsilon^{\alpha-1}$, and the same estimate yields local uniform convergence of the unweighted series.

Proof of (P1). Local uniform convergence away from the diagonal, together with the weighted estimate (27), allows term-wise Riemann–Liouville differentiation on every strip $\xi \geq \varsigma+\varepsilon$ and then passage to the limit $\varepsilon \to 0^+$:

$${}^{\mathcal{RL}}_{\varsigma}\mathfrak{D}^{\alpha}_{\xi}\,\mathbf{\Phi}(\xi,\varsigma) = \sum_{k=0}^{\infty} {}^{\mathcal{RL}}_{\varsigma}\mathfrak{D}^{\alpha}_{\xi}\,\mathbf{\Phi}_k(\xi,\varsigma).$$

For $k=0$: applying identity (5) (with base point ς in place of ξ_0),

$${}^{\mathcal{RL}}_{\varsigma}\mathfrak{D}^{\alpha}_{\xi}\left(\frac{(\xi-\varsigma)^{\alpha-1}}{\Gamma(\alpha)}\,\mathfrak{I}_n\right) = \mathbf{0}. \tag{28}$$

For $k \geq 1$: since $\mathbf{\Phi}_k(\xi,\varsigma) = {}^{\mathcal{RL}}_{\varsigma}\mathfrak{J}^{\alpha}_{\xi}(\mathcal{A}(\xi)\mathbf{\Phi}_{k-1}(\xi,\varsigma))$ by (23), the left-inversion identity Lemma 2.5(i) (with base ς) gives

$${}^{\mathcal{RL}}_{\varsigma}\mathfrak{D}^{\alpha}_{\xi}\,\mathbf{\Phi}_k(\xi,\varsigma) = {}^{\mathcal{RL}}_{\varsigma}\mathfrak{D}^{\alpha}_{\xi}\Big({}^{\mathcal{RL}}_{\varsigma}\mathfrak{J}^{\alpha}_{\xi}\big(\mathcal{A}(\xi)\mathbf{\Phi}_{k-1}(\xi,\varsigma)\big)\Big) = \mathcal{A}(\xi)\,\mathbf{\Phi}_{k-1}(\xi,\varsigma). \tag{29}$$

Combining (28) and (29) and re-indexing:

$${}^{\mathcal{RL}}_{\varsigma}\mathfrak{D}^{\alpha}_{\xi}\,\mathbf{\Phi}(\xi,\varsigma) = \sum_{k=1}^{\infty} \mathcal{A}(\xi)\,\mathbf{\Phi}_{k-1}(\xi,\varsigma) = \mathcal{A}(\xi)\sum_{k=0}^{\infty} \mathbf{\Phi}_k(\xi,\varsigma) = \mathcal{A}(\xi)\,\mathbf{\Phi}(\xi,\varsigma),$$

establishing (24).

Proof of (P2). Applying ${}^{\mathcal{RL}}_{\varsigma}\mathfrak{J}^{1-\alpha}_{\xi}$ to the series term-by-term, justified by the uniform convergence of the weighted series and the estimate (31):

Term $k=0$: Using identity (6) with base point ς,

$${}^{\mathcal{RL}}_{\varsigma}\mathfrak{J}^{1-\alpha}_{\xi}\left(\frac{(\xi-\varsigma)^{\alpha-1}}{\Gamma(\alpha)}\right)\Bigg|_{\xi=\varsigma} = 1, \tag{30}$$

so ${}^{\mathcal{RL}}_{\varsigma}\mathfrak{J}^{1-\alpha}_{\xi}\,\mathbf{\Phi}_0(\xi,\varsigma)\big|_{\xi=\varsigma} = \mathfrak{I}_n$.

Terms $k \geq 1$: From (26) and the power-function identity (3) (with $\beta=(k+1)\alpha$ and base ς),

$$\begin{aligned} \left\|{}^{\mathcal{RL}}_{\varsigma}\mathfrak{J}^{1-\alpha}_{\xi}\,\mathbf{\Phi}_k(\xi,\varsigma)\right\| &\leq \frac{\mathfrak{M}^k}{\Gamma\big((k+1)\alpha\big)} \cdot {}^{\mathcal{RL}}_{\varsigma}\mathfrak{J}^{1-\alpha}_{\xi}\big[(\xi-\varsigma)^{(k+1)\alpha-1}\big] \\ &= \frac{\mathfrak{M}^k}{\Gamma\big((k+1)\alpha\big)} \cdot \frac{\Gamma\big((k+1)\alpha\big)}{\Gamma\big(k\alpha+1\big)}\,(\xi-\varsigma)^{k\alpha} \\ &= \frac{\mathfrak{M}^k\,(\xi-\varsigma)^{k\alpha}}{\Gamma(k\alpha+1)}. \end{aligned} \tag{31}$$

Evaluating (31) at $\xi=\varsigma$: since $k\alpha>0$ for $k \geq 1$,

$$\left\|{}^{\mathcal{RL}}_{\varsigma}\mathfrak{J}^{1-\alpha}_{\xi}\,\mathbf{\Phi}_k(\xi,\varsigma)\big|_{\xi=\varsigma}\right\| \leq \frac{\mathfrak{M}^k\,0^{k\alpha}}{\Gamma(k\alpha+1)} = 0, \qquad k \geq 1.$$

Summing over all k, we have

$${}^{\mathcal{RL}}_{\varsigma}\mathfrak{J}^{1-\alpha}_{\xi}\,\mathbf{\Phi}(\xi,\varsigma)\Big|_{\xi=\varsigma} = \mathfrak{I}_n + \mathbf{0} = \mathfrak{I}_n,$$

establishing (25). □

Lemma 3.6 (Kernel differentiation for the mixed representation)**.** *Let* $0 < \alpha < 1$*,* $\mathcal{A} \in C(\mathfrak{I}, \mathbb{R}^{n\times n})$*, and* $\mathbf{u} \in C(\mathfrak{I}, \mathbb{R}^n)$*. Define*

$$\mathbf{z}(\xi) := \int_{\xi_0}^{\xi} \mathbf{\Phi}(\xi, \varsigma)\mathbf{u}(\varsigma)\, d\varsigma, \qquad \xi \in \mathfrak{I}.$$

Then $\mathbf{z} \in C(\mathfrak{I}, \mathbb{R}^n)$*,* $\mathbf{z}(\xi_0) = \mathbf{0}$*,* $\mathbf{z}$ *belongs to the fractional domain of the Caputo operator, and*

$$^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\mathbf{z}(\xi) = \mathcal{A}(\xi)\mathbf{z}(\xi) + \mathbf{u}(\xi), \qquad \xi \in \mathring{\mathfrak{I}}. \tag{32}$$

Proof. The continuity of $\mathbf{z}$ follows from the integrability of the singular kernel $(\xi - \varsigma)^{\alpha-1}$ and the weighted uniform convergence proved in Lemma 3.5. Using the series representation of $\mathbf{\Phi}$, one has

$$\mathbf{z}(\xi) = \sum_{k=0}^{\infty} \int_{\xi_0}^{\xi} \mathbf{\Phi}_k(\xi, \varsigma)\mathbf{u}(\varsigma)\, d\varsigma,$$

where the interchange of summation and integration follows from the Weierstrass majorant (27) and $\mathbf{u} \in C(\mathfrak{I}, \mathbb{R}^n)$. Applying $^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}$ term by term is justified by dominated convergence on $[\xi_0 + \varepsilon, \xi]$ and then by letting $\varepsilon \to 0^+$. The leading kernel $\mathbf{\Phi}_0(\xi, \varsigma) = (\xi - \varsigma)^{\alpha-1}\mathfrak{I}_n/\Gamma(\alpha)$ gives the boundary contribution $\mathbf{u}(\xi)$, because $^{\mathcal{RL}}_{\varsigma}\mathfrak{J}^{1-\alpha}_{\xi}\mathbf{\Phi}_0(\xi, \varsigma)|_{\xi=\varsigma} = \mathfrak{I}_n$. For the remaining kernels, Lemma 3.5(P1) yields

$$^{\mathcal{RL}}_{\varsigma}\mathfrak{D}^{\alpha}_{\xi}\mathbf{\Phi}(\xi, \varsigma) = \mathcal{A}(\xi)\mathbf{\Phi}(\xi, \varsigma).$$

Consequently,

$$^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\mathbf{z}(\xi) = \int_{\xi_0}^{\xi} \mathcal{A}(\xi)\mathbf{\Phi}(\xi, \varsigma)\mathbf{u}(\varsigma)\, d\varsigma + \mathbf{u}(\xi) = \mathcal{A}(\xi)\mathbf{z}(\xi) + \mathbf{u}(\xi).$$

This proves (32) without invoking any unsupported fractional Leibniz rule. ☐

3.4. Existence of Solutions

Theorem 3.7 (Existence–homogeneous Caputo system)**.** *Under Assumption 1, the function*

$$\mathbf{x}(\xi) \ := \ \mathbf{\Psi}(\xi, \xi_0)\, \widetilde{\mathbf{x}}_0 \tag{33}$$

belongs to $C(\mathfrak{I}, \mathbb{R}^n)$ *and is a mild solution of the homogeneous problem* (10)*. When the additional fractional regularity in Definition 3.1 is satisfied, it is also a classical Caputo solution.*

Proof. Step 1: The Caputo GPB series satisfies the fractional equation. Uniform convergence (Theorem 3.4) justifies term-wise application of $^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}$ to (17). For the zeroth term, using $^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\, c = \mathbf{0}$ for any constant c (cf. identity (5) and Proposition 2.4):

$$^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\, \mathbf{\Psi}_0(\xi, \xi_0) \ = \ ^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\, \mathfrak{I}_n \ = \ \mathbf{0}.$$

For each $k \geq 1$, applying Lemma 2.5(ii) to (16), one has

$$^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\, \mathbf{\Psi}_k(\xi, \xi_0) \ = \ ^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\big(^{\mathcal{RL}}_{\xi_0}\mathfrak{J}^{\alpha}_{\xi}\big(\mathcal{A}(\xi)\, \mathbf{\Psi}_{k-1}(\xi, \xi_0)\big)\big) \ = \ \mathcal{A}(\xi)\, \mathbf{\Psi}_{k-1}(\xi, \xi_0).$$

Re-indexing the sum, we get

$$^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\, \mathbf{\Psi}(\xi, \xi_0) = \sum_{k=1}^{\infty} \mathcal{A}(\xi)\, \mathbf{\Psi}_{k-1}(\xi, \xi_0) = \mathcal{A}(\xi) \sum_{k=0}^{\infty} \mathbf{\Psi}_k(\xi, \xi_0) = \mathcal{A}(\xi)\, \mathbf{\Psi}(\xi, \xi_0).$$

Step 2: Verification of the initial condition. Evaluating (17) at $\xi = \xi_0$: $\mathbf{\Psi}_0(\xi_0, \xi_0) = \mathfrak{I}_n$, and for $k \geq 1$, from (19), we have

$$\|\mathbf{\Psi}_k(\xi_0, \xi_0)\| \leq \mathfrak{M}^k(\xi_0 - \xi_0)^{k\alpha}/\Gamma(k\alpha + 1) = 0.$$

Hence $\mathbf{\Psi}(\xi_0, \xi_0) = \mathfrak{I}_n$, and consequently $\mathbf{x}(\xi_0) = \mathbf{\Psi}(\xi_0, \xi_0)\, \widetilde{\mathbf{x}}_0 = \widetilde{\mathbf{x}}_0$.

Step 3: Membership in $C(\mathfrak{I}, \mathbb{R}^n)$*.* From (20) and (33), one has

$$\|\mathbf{x}(\xi)\| \ \leq \ \mathbb{E}_{\alpha}(\mathfrak{M}(\xi - \xi_0)^{\alpha})\, \|\widetilde{\mathbf{x}}_0\|,$$

which is bounded and continuous on $\mathfrak{I}$ (since $\mathbb{E}_{\alpha}$ is an entire function), so $\mathbf{x} \in C(\mathfrak{I}, \mathbb{R}^n)$. ☐

Theorem 3.8 (Existence–inhomogeneous Caputo system). *Under Assumption 1 and* $\mathbf{u} \in C(\mathfrak{I}, \mathbb{R}^n)$*, the function*

$$\mathbf{x}(\xi) \;:=\; \mathbf{\Psi}(\xi,\xi_0)\,\widetilde{\mathbf{x}}_0 \;+\; \int_{\xi_0}^{\xi} \mathbf{\Phi}(\xi,\varsigma)\,\mathbf{u}(\varsigma)\,d\varsigma, \qquad \xi \in \mathfrak{I}, \tag{34}$$

where $\mathbf{\Psi}(\xi,\xi_0)$ *is the Caputo state-transition matrix (17) and* $\mathbf{\Phi}(\xi,\varsigma)$ *is the Riemann–Liouville state-transition matrix (21), belongs to* $C(\mathfrak{I},\mathbb{R}^n)$ *and is a mild solution of the inhomogeneous Caputo problem (11). Under the additional fractional regularity stated in Definition 3.1, it is a classical Caputo solution.*

Proof. The proof proceeds as follows.

Step 1: Boundedness and membership in $C(\mathfrak{I},\mathbb{R}^n)$*.*

From the Caputo GPB bound (20) and the RL GPB bound (26),

$$\begin{aligned}\|\mathbf{x}(\xi)\| \;&\le\; \big\|\mathbf{\Psi}(\xi,\xi_0)\big\|\,\|\widetilde{\mathbf{x}}_0\| + \int_{\xi_0}^{\xi} \big\|\mathbf{\Phi}(\xi,\varsigma)\big\|\,\|\mathbf{u}(\varsigma)\|\,d\varsigma \\ &\le\; \mathbb{E}_\alpha(\mathfrak{M}(\xi-\xi_0)^\alpha)\,\|\widetilde{\mathbf{x}}_0\| + \|\mathbf{u}\|_{C(\mathfrak{I})}\int_{\xi_0}^{\xi}(\xi-\varsigma)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\mathfrak{M}(\xi-\varsigma)^\alpha)\,d\varsigma.\end{aligned}$$

Since $\mathbb{E}_\alpha$ and $\mathbb{E}_{\alpha,\alpha}$ are entire functions, both terms are continuous and bounded on $\mathfrak{I}$, giving $\mathbf{x} \in C(\mathfrak{I},\mathbb{R}^n)$.

Step 2: Direct differentiation of the mixed kernel.

Apply ${}^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}$ to the homogeneous part and to the mixed-kernel part separately. For the homogeneous part, Step 1 of Theorem 3.7 gives

$${}^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\big[\mathbf{\Psi}(\xi,\xi_0)\,\widetilde{\mathbf{x}}_0\big] = \mathcal{A}(\xi)\,\mathbf{\Psi}(\xi,\xi_0)\,\widetilde{\mathbf{x}}_0. \tag{35}$$

For the inhomogeneous part, put

$$\mathbf{z}(\xi) = \int_{\xi_0}^{\xi} \mathbf{\Phi}(\xi,\varsigma)\mathbf{u}(\varsigma)\,d\varsigma.$$

By Lemma 3.6,

$${}^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\mathbf{z}(\xi) = \mathcal{A}(\xi)\int_{\xi_0}^{\xi} \mathbf{\Phi}(\xi,\varsigma)\mathbf{u}(\varsigma)\,d\varsigma + \mathbf{u}(\xi). \tag{36}$$

Thus the proof uses the kernel-differentiation identity (32), which was derived from the uniformly convergent GPB series and the Riemann–Liouville initial relation, rather than a formal Caputo Leibniz rule.

Combining (35) and (36), we have

$$\begin{aligned}{}^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\,\mathbf{x}(\xi) \;&=\; \mathcal{A}(\xi)\,\mathbf{\Psi}(\xi,\xi_0)\,\widetilde{\mathbf{x}}_0 \;+\; \mathcal{A}(\xi)\int_{\xi_0}^{\xi}\mathbf{\Phi}(\xi,\varsigma)\,\mathbf{u}(\varsigma)\,d\varsigma \;+\; \mathbf{u}(\xi) \\ &=\; \mathcal{A}(\xi)\,\mathbf{x}(\xi) + \mathbf{u}(\xi),\end{aligned}$$

which is precisely the differential equation in (11).

For the initial condition, evaluate (34) at $\xi = \xi_0$: the integral $\int_{\xi_0}^{\xi_0}(\cdots)\,d\varsigma = \mathbf{0}$, and $\mathbf{\Psi}(\xi_0,\xi_0) = \mathfrak{I}_n$ (since $\mathbf{\Psi}_0(\xi_0,\xi_0) = \mathfrak{I}_n$ and $\mathbf{\Psi}_k(\xi_0,\xi_0) \le \mathfrak{M}^k(\xi_0-\xi_0)^{k\alpha}/\Gamma(k\alpha+1) = 0$ for $k \ge 1$ by (19)). Therefore $\mathbf{x}(\xi_0) = \mathfrak{I}_n\,\widetilde{\mathbf{x}}_0 = \widetilde{\mathbf{x}}_0$, confirming the initial condition of (11). ☐

3.5. Uniqueness of Solutions

Theorem 3.9 (Uniqueness–homogeneous Caputo system). *Under Assumption 1, the solution (33) of problem (10) is the unique solution in* $C(\mathfrak{I},\mathbb{R}^n)$*.*

Proof. Let $\mathbf{x}_1, \mathbf{x}_2 \in C(\mathfrak{I}, \mathbb{R}^n)$ be two solutions of (10). By Proposition 3.2, both satisfy (12); subtracting,

$$\|\mathbf{x}_1(\xi) - \mathbf{x}_2(\xi)\| \;\le\; \frac{\mathfrak{M}}{\Gamma(\alpha)} \int_{\xi_0}^{\xi} (\xi - \varsigma)^{\alpha-1} \|\mathbf{x}_1(\varsigma) - \mathbf{x}_2(\varsigma)\| \, d\varsigma.$$

Introduce the sup-norm residual $\boldsymbol{\varepsilon} := \sup_{\varsigma \in \mathfrak{I}} \|\mathbf{x}_1(\varsigma) - \mathbf{x}_2(\varsigma)\|$, which is finite since $\mathbf{x}_1, \mathbf{x}_2 \in C(\mathfrak{I}, \mathbb{R}^n)$. Using Lemma 2.6, identity (3) with $\beta = 1$:

$$\|\mathbf{x}_1(\xi) - \mathbf{x}_2(\xi)\| \;\le\; \frac{\mathfrak{M}\,(\xi - \xi_0)^{\alpha}}{\Gamma(\alpha + 1)}\, \boldsymbol{\varepsilon}.$$

Iterating N times yields

$$\boldsymbol{\varepsilon} \;\le\; \frac{\mathfrak{M}^N (\xi_1 - \xi_0)^{N\alpha}}{\Gamma(N\alpha + 1)}\, \boldsymbol{\varepsilon} \;\xrightarrow{N \to \infty}\; 0,$$

since $\Gamma(N\alpha + 1)$ grows super-exponentially by Stirling's formula. Hence $\boldsymbol{\varepsilon} = 0$, which gives $\mathbf{x}_1 \equiv \mathbf{x}_2$ on $\mathfrak{I}$. ☐

Theorem 3.10 (Uniqueness–inhomogeneous Caputo system)**.** *Under Assumption 1 and* $\mathbf{u} \in C(\mathfrak{I}, \mathbb{R}^n)$*, the solution* (34) *of problem* (11) *is the unique solution in* $C(\mathfrak{I}, \mathbb{R}^n)$.

Proof. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions of (11) in $C(\mathfrak{I}, \mathbb{R}^n)$, setting $\mathbf{w} := \mathbf{x}_1 - \mathbf{x}_2$ shows that $\mathbf{w}$ satisfies the homogeneous problem (10) with zero initial datum $\widetilde{\mathbf{x}}_0 = \mathbf{0}$. Theorem 3.9 then gives $\mathbf{w} \equiv \mathbf{0}$, i.e., $\mathbf{x}_1 \equiv \mathbf{x}_2$ on $\mathfrak{I}$. ☐

Remark 3.11 (Constant-coefficient reduction)**.** When $\mathcal{A}(\xi) \equiv \mathcal{A}$ is constant, Lemma 2.6, identity (3), gives

$$\boldsymbol{\Psi}_k(\xi, \xi_0) \;=\; \frac{(\xi - \xi_0)^{k\alpha}}{\Gamma(k\alpha + 1)}\, \mathcal{A}^k,$$

so that

$$\boldsymbol{\Psi}(\xi, \xi_0) \;=\; \sum_{k=0}^{\infty} \frac{\mathcal{A}^k\, (\xi - \xi_0)^{k\alpha}}{\Gamma(k\alpha + 1)} \;=\; \mathbb{E}_{\alpha}(\mathcal{A}(\xi - \xi_0)^{\alpha}),$$

the matrix Mittag–Leffler function $\mathbb{E}_{\alpha,1}$. Equation (33) then reduces to

$$\mathbf{x}(\xi) \;=\; \mathbb{E}_{\alpha}(\mathcal{A}(\xi - \xi_0)^{\alpha})\, \widetilde{\mathbf{x}}_0,$$

in full agreement with the constant-coefficient Caputo theory of [12, 21]. Note that the RL formula involves $\mathbb{E}_{\alpha,\alpha}$ (two-parameter), while the Caputo formula involves $\mathbb{E}_{\alpha} = \mathbb{E}_{\alpha,1}$ (one-parameter), directly reflecting the difference in the zeroth GPB terms.

Remark 3.12 (Reduction, truncation, and computational cost)**.** The variable-coefficient construction reduces to the classical constant-coefficient formula when $\mathcal{A}(\xi) \equiv \mathcal{A}$. The following comparison summarizes the reduction.

Quantity	Variable coefficient	Constant coefficient
Caputo transition matrix	$\boldsymbol{\Psi}(\xi, \xi_0) = \sum_{k=0}^{\infty} \boldsymbol{\Psi}_k(\xi, \xi_0)$	$\mathbb{E}_{\alpha}(\mathcal{A}(\xi - \xi_0)^{\alpha})$
RL transition kernel	$\boldsymbol{\Phi}(\xi, \varsigma) = \sum_{k=0}^{\infty} \boldsymbol{\Phi}_k(\xi, \varsigma)$	$(\xi - \varsigma)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\mathcal{A}(\xi - \varsigma)^{\alpha})$

For numerical computation, the GPB series may be truncated. If $\boldsymbol{\Psi}_N := \sum_{k=0}^{N} \boldsymbol{\Psi}_k$ and $z = \mathfrak{M}(\xi - \xi_0)^{\alpha}$, then Theorem 3.4 yields the computable a priori error estimate

$$\|\boldsymbol{\Psi}(\xi, \xi_0) - \boldsymbol{\Psi}_N(\xi, \xi_0)\| \le \sum_{k=N+1}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)} = \mathbb{E}_{\alpha}(z) - \sum_{k=0}^{N} \frac{z^k}{\Gamma(k\alpha + 1)}.$$

Similarly, for the weighted Riemann–Liouville kernel,

$$\left\|(\xi-\varsigma)^{1-\alpha}\big(\mathbf{\Phi}(\xi,\varsigma)-\mathbf{\Phi}_N(\xi,\varsigma)\big)\right\| \le \sum_{k=N+1}^{\infty} \frac{\mathfrak{M}^k(\xi-\varsigma)^{k\alpha}}{\Gamma((k+1)\alpha)}.$$

The computational cost of the N-term approximation is governed by the number of nested Volterra integrations and matrix multiplications. In practice, the Mittag–Leffler majorants above provide a stopping criterion: terms are added until the displayed tail estimate is below the prescribed tolerance.

3.6. Stability Concepts for the Caputo System

The preceding well-posedness results are formulated on a compact interval $\mathcal{I} = [\xi_0, \xi_1]$. In contrast, stability, asymptotic stability, and BIBO stability are global concepts. Hence, throughout the stability subsection, we consider the system on

$$\mathcal{I}_\infty := [\xi_0, \infty),$$

and write $\mathcal{I}_T := [\xi_0, T]$ for arbitrary finite subintervals. The estimates obtained on $\mathcal{I}_T$ are valid for each $T > \xi_0$; the asymptotic conclusions are stated only on $\mathcal{I}_\infty$. In this subsection, Assumption 1 is understood in its global form, namely $\sup_{\xi\ge\xi_0} \|\mathcal{A}(\xi)\| \le \mathfrak{M} < \infty$.

Since Caputo solutions belong to $C(\mathcal{I}_\infty, \mathbb{R}^n)$ and satisfy the pointwise initial condition $\mathbf{x}(\xi_0) = \widetilde{\mathbf{x}}_0$, the classical Lyapunov definitions carry over without any weighted modification.

Definition 3.13 (Stability–Caputo case)**.** The trivial solution $\mathbf{x} \equiv \mathbf{0}$ of the homogeneous system (10) (with $\widetilde{\mathbf{x}}_0 = \mathbf{0}$) is said to be:

(i) stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\widetilde{\mathbf{x}}_0\| < \delta$ implies $\|\mathbf{x}(\xi)\| < \varepsilon$ for all $\xi \in \mathcal{I}_\infty$;
(ii) asymptotically stable if it is stable and, moreover, $\|\mathbf{x}(\xi)\| \to 0$ as $\xi \to +\infty$;
(iii) Mittag–Leffler stable if there exist constants $\nu, \lambda > 0$ such that

$$\|\mathbf{x}(\xi)\| \;\le\; \Big[\nu\|\widetilde{\mathbf{x}}_0\|^2\,\mathbb{E}_\alpha(-\lambda(\xi-\xi_0)^\alpha)\Big]^{1/2}, \qquad \xi \in \mathcal{I}_\infty. \tag{37}$$

Remark 3.14. Mittag–Leffler stability implies asymptotic stability. Indeed, from (37) and Lemma 2.10(iii), $\mathbb{E}_\alpha(-\lambda(\xi-\xi_0)^\alpha) \to 0$ as $\xi \to +\infty$, so $\|\mathbf{x}(\xi)\| \to 0$.

Now, we recall the following auxiliary lemmas for the stability analysis.

Lemma 3.15 (Caputo comparison principle)**.** *Let $v \in C(\mathcal{I}_\infty, [0,\infty))$ and suppose that, for every $T > \xi_0$, v satisfies on $\mathcal{I}_T$ the fractional integral inequality equivalent to*

$$^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi} v(\xi) \le -\mu v(\xi), \qquad \mu > 0,$$

in the mild sense. Then

$$v(\xi) \;\le\; v(\xi_0)\,\mathbb{E}_\alpha(-\mu(\xi-\xi_0)^\alpha)\,, \qquad \xi \in \mathcal{I}_\infty. \tag{38}$$

Proof. For C^1 functions this is the standard Caputo comparison principle. For a mild solution, one first regularizes v on $[\xi_0+\varepsilon, T]$, applies the classical comparison result there, and then lets $\varepsilon \to 0^+$. Continuity of v and dominated convergence for the weakly singular kernel $(\xi-\varsigma)^{\alpha-1}$ yield the same estimate on $\mathcal{I}_T$. Since T is arbitrary, the estimate holds on $\mathcal{I}_\infty$. □

Lemma 3.16 (Fractional Leibniz inequality [1])**.** *Let $\mathbf{x} \in C^1(\mathcal{I}_\infty, \mathbb{R}^n)$ and $0 < \alpha < 1$. Then*

$$^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\big(\mathbf{x}^\top(\xi)\,\mathbf{x}(\xi)\big) \;\le\; 2\,\mathbf{x}^\top(\xi)\,^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\,\mathbf{x}(\xi). \tag{39}$$

For mild solutions used in the present stability proofs, this inequality is applied through the same regularization-on-$\mathcal{I}_T$ argument as in Lemma 3.15.

Lemma 3.17 (Mittag–Leffler convolution identity)**.** *For* $0 < \alpha < 1$ *and* $\lambda > 0$*,*

$$\int_{\xi_0}^{\xi} (\xi - \varsigma)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(-\lambda(\xi - \varsigma)^{\alpha})\, d\varsigma \;=\; \frac{1}{\lambda}\Big[1 - \mathbb{E}_{\alpha}(-\lambda(\xi - \xi_0)^{\alpha})\Big], \quad \xi \in \mathfrak{I}_{\infty}. \tag{40}$$

Proof. Differentiating the series Definition 2.9 term-by-term with respect to ξ:

$$\frac{d}{d\xi}\,\mathbb{E}_{\alpha}(-\lambda(\xi - \xi_0)^{\alpha}) = -\lambda(\xi - \xi_0)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(-\lambda(\xi - \xi_0)^{\alpha})\,.$$

Integrating both sides from ξ_0 to ξ and using $\mathbb{E}_{\alpha}(0) = 1$, we obtain

$$\mathbb{E}_{\alpha}(-\lambda(\xi - \xi_0)^{\alpha}) - 1 = -\lambda \int_{\xi_0}^{\xi} (\xi - \varsigma)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(-\lambda(\xi - \varsigma)^{\alpha})\, d\varsigma,$$

from which (40) follows upon rearranging. □

3.7. Mittag–Leffler Stability via Lyapunov Functions

Assumption 2. There exists a continuously differentiable function $V : \mathfrak{I}_{\infty} \times \mathbb{R}^n \to [0, \infty)$ and constants $c_1, c_2, c_3 > 0$ such that, for all $(\xi, \mathbf{x}) \in \mathfrak{I}_{\infty} \times \mathbb{R}^n$,

$$c_1\|\mathbf{x}\|^2 \;\le\; V(\xi, \mathbf{x}) \;\le\; c_2\|\mathbf{x}\|^2, \tag{41}$$

$${}^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\, V\big(\xi, \mathbf{x}(\xi)\big) \;\le\; -c_3\, V\big(\xi, \mathbf{x}(\xi)\big), \tag{42}$$

where the Caputo derivative in (42) is evaluated along every solution $\mathbf{x}$ of (10).

Theorem 3.18 (Mittag–Leffler stability via Lyapunov function)**.** *Under Assumptions 1 and 2, the trivial solution of* (10) *is Mittag–Leffler stable. Specifically, every solution satisfies*

$$\|\mathbf{x}(\xi)\| \;\le\; \left[\frac{c_2}{c_1}\,\|\widetilde{\mathbf{x}}_0\|^2\,\mathbb{E}_{\alpha}(-c_3(\xi - \xi_0)^{\alpha})\right]^{1/2}, \qquad \xi \in \mathfrak{I}_{\infty}. \tag{43}$$

Proof. Set $v(\xi) := V(\xi, \mathbf{x}(\xi))$. From (42), ${}^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\, v(\xi) \le -c_3\, v(\xi)$. Since v is non-negative and $c_3 > 0$, Lemma 3.15 gives

$$v(\xi) \;\le\; v(\xi_0)\,\mathbb{E}_{\alpha}(-c_3(\xi - \xi_0)^{\alpha})\,. \tag{44}$$

From the left inequality in (41), $c_1\|\mathbf{x}(\xi)\|^2 \le v(\xi)$; from the right inequality at $\xi = \xi_0$, $v(\xi_0) \le c_2\|\widetilde{\mathbf{x}}_0\|^2$. Substituting into (44) and dividing by c_1:

$$\|\mathbf{x}(\xi)\|^2 \;\le\; \frac{c_2}{c_1}\,\|\widetilde{\mathbf{x}}_0\|^2\,\mathbb{E}_{\alpha}(-c_3(\xi - \xi_0)^{\alpha})\,.$$

Taking square roots yields (43), which is precisely the Mittag–Leffler condition (37) with $\nu = c_2/c_1$ and $\lambda = c_3$. □

Corollary 3.19 (Asymptotic stability via Lyapunov function)**.** *Under the hypotheses of Theorem 3.18, the trivial solution of* (10) *is asymptotically stable.*

Proof. From (43) and Lemma 2.10(iii), $\mathbb{E}_{\alpha}(-c_3(\xi - \xi_0)^{\alpha}) \to 0$ as $\xi \to +\infty$, so $\|\mathbf{x}(\xi)\| \to 0$. Stability (in the sense of Definition 3.13(i)) follows from (43) by choosing $\delta = \varepsilon/\sqrt{c_2/c_1}$. □

3.8. Spectral Stability Criterion

Assumption 3. There exists a constant $\lambda > 0$ such that

$$\mathbf{x}^\top \mathcal{A}(\xi)\mathbf{x} \ \le\ -\lambda\|\mathbf{x}\|^2 \qquad \text{for all } \mathbf{x} \in \mathbb{R}^n,\ \xi \in \mathfrak{I}_\infty. \tag{45}$$

Equivalently, all eigenvalues of the symmetric part $\frac{1}{2}(\mathcal{A}(\xi) + \mathcal{A}^\top(\xi))$ are bounded above by $-\lambda$ uniformly in ξ.

Theorem 3.20 (Spectral stability criterion)**.** *Under Assumptions 1 and 3, every solution of the homogeneous system* (10) *satisfies*

$$\|\mathbf{x}(\xi)\| \ \le\ \|\widetilde{\mathbf{x}}_0\|\, \mathbb{E}_\alpha^{1/2}(-2\lambda(\xi-\xi_0)^\alpha)\,, \qquad \xi \in \mathfrak{I}_\infty, \tag{46}$$

and the trivial solution is Mittag–Leffler stable. Consequently, the Caputo state-transition matrix satisfies

$$\big\|\boldsymbol{\Psi}(\xi,\xi_0)\big\| \ \le\ \mathbb{E}_\alpha^{1/2}(-2\lambda(\xi-\xi_0)^\alpha)\,, \qquad \xi \in \mathfrak{I}_\infty. \tag{47}$$

Proof. Define the quadratic Lyapunov function $V(\xi,\mathbf{x}) := \|\mathbf{x}\|^2 = \mathbf{x}^\top\mathbf{x}$. This satisfies Assumption 2 with $c_1 = c_2 = 1$. To verify the decay condition (42), apply the fractional Leibniz inequality (Lemma 3.16) along trajectories of (10):

$$\begin{aligned} {}^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\, \|\mathbf{x}(\xi)\|^2 \ &\le\ 2\,\mathbf{x}^\top(\xi)\, {}^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\, \mathbf{x}(\xi) \\ &=\ 2\,\mathbf{x}^\top(\xi)\,\mathcal{A}(\xi)\,\mathbf{x}(\xi) \ \le\ -2\lambda\|\mathbf{x}(\xi)\|^2. \end{aligned} \tag{48}$$

Hence Assumption 2 holds with $c_3 = 2\lambda$. The comparison principle (Lemma 3.15) applied to $v(\xi) = \|\mathbf{x}(\xi)\|^2$ with $\mu = 2\lambda$ gives

$$\|\mathbf{x}(\xi)\|^2 \ \le\ \|\widetilde{\mathbf{x}}_0\|^2\, \mathbb{E}_\alpha(-2\lambda(\xi-\xi_0)^\alpha)\,.$$

Taking square roots yields (46). Bound (47) follows since $\mathbf{x}(\xi) = \boldsymbol{\Psi}(\xi,\xi_0)\widetilde{\mathbf{x}}_0$ and $\widetilde{\mathbf{x}}_0 \in \mathbb{R}^n$ is arbitrary. ☐

Corollary 3.21 (Asymptotic stability under spectral condition)**.** *Under Assumptions 1 and 3, the trivial solution of* (10) *is asymptotically stable.*

Proof. From (46) and Lemma 2.10(iii), $\mathbb{E}_\alpha(-2\lambda(\xi-\xi_0)^\alpha) \to 0$, hence $\|\mathbf{x}(\xi)\| \to 0$ as $\xi \to +\infty$. ☐

Remark 3.22 (Sufficiency but not necessity of Assumption 3)**.** Assumption 3 is a sufficient uniform dissipativity condition; it is not necessary for asymptotic stability. In the constant-coefficient case

$${}^{\mathcal{C}}_{\xi_0}\mathfrak{D}^{\alpha}_{\xi}\mathbf{x} = A\mathbf{x},$$

the standard fractional spectral condition is $|\arg(\lambda_j(A))| > \alpha\pi/2$ for every eigenvalue $\lambda_j(A)$ of A. This may hold even when the symmetric-part condition fails. For instance,

$$A = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}$$

has the single eigenvalue -1, and hence the constant-coefficient Caputo system is asymptotically stable for every $0 < \alpha < 1$. However,

$$\frac{A + A^\top}{2} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

has eigenvalues 1 and -3. Therefore, no $\lambda > 0$ can satisfy $\mathbf{x}^\top A\mathbf{x} \le -\lambda\|\mathbf{x}\|^2$ for all $\mathbf{x} \in \mathbb{R}^2$. Thus Assumption 3 is sufficient, but not necessary.

3.9. Bounded-Input Bounded-Output Stability

Theorem 3.23 (Bounded-input bounded-output stability)**.** *Let Assumptions 1 and 3 hold, and let* $\mathbf{u} \in C(\mathcal{J}_\infty, \mathbb{R}^n)$ *with* $\sup_{\xi\in\mathcal{J}_\infty} \|\mathbf{u}(\xi)\| \le \mathfrak{U} < \infty$. *Then the solution (34) of the inhomogeneous system (11) satisfies*

$$\|\mathbf{x}(\xi)\| \;\le\; \mathbb{E}_\alpha^{1/2}(-2\lambda(\xi-\xi_0)^\alpha)\,\|\widetilde{\mathbf{x}}_0\| \;+\; \frac{\mathfrak{U}}{\lambda}\left[1-\mathbb{E}_\alpha(-\lambda(\xi-\xi_0)^\alpha)\right], \quad \xi\in\mathcal{J}_\infty. \tag{49}$$

In particular, if $\mathfrak{U}<\infty$ *then* $\|\mathbf{x}(\xi)\|$ *is uniformly bounded on* $\mathcal{J}_\infty$.

Proof. From (34), Theorem 3.20, and the RL state-transition matrix bound (26) (summed to give $\|\mathbf{\Phi}(\xi,\varsigma)\| \le (\xi-\varsigma)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(-\lambda(\xi-\varsigma)^\alpha)$ under Assumption 3), one has

$$\begin{aligned}\|\mathbf{x}(\xi)\| \;&\le\; \|\mathbf{\Psi}(\xi,\xi_0)\|\,\|\widetilde{\mathbf{x}}_0\| + \int_{\xi_0}^{\xi} \|\mathbf{\Phi}(\xi,\varsigma)\|\,\|\mathbf{u}(\varsigma)\|\,d\varsigma\\ &\le\; \mathbb{E}_\alpha^{1/2}(-2\lambda(\xi-\xi_0)^\alpha)\,\|\widetilde{\mathbf{x}}_0\| + \mathfrak{U}\int_{\xi_0}^{\xi}(\xi-\varsigma)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(-\lambda(\xi-\varsigma)^\alpha)\,d\varsigma. \end{aligned}\tag{50}$$

Applying the Mittag–Leffler convolution identity (Lemma 3.17) to the remaining integral:

$$\int_{\xi_0}^{\xi}(\xi-\varsigma)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(-\lambda(\xi-\varsigma)^\alpha)\,d\varsigma = \frac{1}{\lambda}\Big[1-\mathbb{E}_\alpha(-\lambda(\xi-\xi_0)^\alpha)\Big].$$

Substituting into (50) gives (49).

For the uniform bound, note that $\mathbb{E}_\alpha^{1/2}(-2\lambda(\xi-\xi_0)^\alpha)\le 1$ and $1-\mathbb{E}_\alpha(-\lambda(\xi-\xi_0)^\alpha)\le 1$ for all $\xi\ge\xi_0$, so $\|\mathbf{x}(\xi)\|\le\|\widetilde{\mathbf{x}}_0\|+\mathfrak{U}/\lambda$. □

Remark 3.24 (Conservativeness of the BIBO bound)**.** Estimate (49) is sharper than the uniform bound $\|\mathbf{x}(\xi)\|\le\|\widetilde{\mathbf{x}}_0\|+\mathfrak{U}/\lambda$ because it separates the decaying initial-state contribution from the forced response contribution. The uniform gain $1/\lambda$ may be conservative for oscillatory or decaying inputs, since it uses only the amplitude bound $\|\mathbf{u}\|_\infty\le\mathfrak{U}$. For the scalar constant-coefficient system with constant input, however, this gain is sharp in the usual worst-case BIBO sense.

Remark 3.25 (L^p inputs)**.** Theorem 3.23 is stated for bounded inputs because BIBO stability is an L^∞-to-L^∞ property. Extensions to L^p inputs are possible under additional kernel-integrability assumptions. For the scalar stable kernel

$$g(r)=r^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(-\lambda r^\alpha),\qquad r>0,$$

Young's convolution inequality gives an L^p-to-L^∞ estimate whenever the conjugate exponent $q=p/(p-1)$ satisfies $g\in L^q(0,\infty)$. The singularity near $r=0$ imposes $q<1/(1-\alpha)$, and hence a sufficient condition is $p>1/\alpha$. A complete L^p theory requires separate hypotheses and is therefore left as a future extension.

Corollary 3.26 (Vanishing input implies vanishing state)**.** *Under the hypotheses of Theorem 3.23, if additionally* $\|\mathbf{u}(\xi)\|\to 0$ *as* $\xi\to+\infty$, *then* $\|\mathbf{x}(\xi)\|\to 0$.

Proof. For $\varepsilon>0$, choose $\xi^*>\xi_0$ such that $\|\mathbf{u}(\xi)\|<\varepsilon\lambda/2$ for all $\xi>\xi^*$. Split the convolution integral in (34) at ξ^*, one has

$$\begin{aligned}\|\mathbf{x}(\xi)\| \;\le\; &\mathbb{E}_\alpha^{1/2}(-2\lambda(\xi-\xi_0)^\alpha)\,\|\widetilde{\mathbf{x}}_0\|\\ &+\sup_{\varsigma\le\xi^*}\|\mathbf{u}(\varsigma)\|\cdot\int_{\xi_0}^{\xi^*}(\xi-\varsigma)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\big(-\lambda(\xi-\varsigma)^\alpha\big)d\varsigma\\ &+\frac{\varepsilon\lambda}{2}\int_{\xi^*}^{\xi}(\xi-\varsigma)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\big(-\lambda(\xi-\varsigma)^\alpha\big)d\varsigma.\end{aligned}$$

The first term $\to 0$ by Lemma 2.10(iii). The second term $\to 0$ since the integrand is bounded and the upper limit ξ^* is fixed while $\xi\to+\infty$ causes $(\xi-\varsigma)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(-\lambda(\xi-\varsigma)^\alpha)\to 0$ uniformly in $\varsigma\in[\xi_0,\xi^*]$. The third term is bounded above by $\varepsilon/2$ by Lemma 3.17. Since ε is arbitrary, $\|\mathbf{x}(\xi)\|\to 0$. □

3.10. Illustrative Examples

The following three examples are chosen to validate, in turn, Theorem 3.18 (Lyapunov approach), Theorem 3.20 (spectral criterion), and Theorem 3.23 (BIBO stability). In each case every hypothesis is checked explicitly before the stability estimate is derived.

Example 3.27 (Lyapunov-based Mittag–Leffler stability: oscillatory variable coefficient)**.** Consider the scalar homogeneous Caputo system on $\mathcal{I} = [0, \infty)$:

$$\begin{cases} {}^{\mathcal{C}}_{0}\mathfrak{D}^{\alpha}_{\xi}\, x(\xi) = -(2 + \sin\xi)\, x(\xi), \quad \xi > 0, \quad 0 < \alpha < 1, \\ x(0) = \widetilde{x}_0. \end{cases} \tag{51}$$

Verification of Assumption 1. The coefficient $\mathcal{A}(\xi) = -(2 + \sin\xi)$ is continuous on $[0, \infty)$ and satisfies $|\mathcal{A}(\xi)| \le 3 =: \mathfrak{M}$ for all $\xi \ge 0$.
Construction of the Lyapunov function. Define $V(\xi, x) := x^2$. Then V is C^1, non-negative, and satisfies

$$c_1|x|^2 \le V(\xi, x) \le c_2|x|^2 \quad \text{with} \quad c_1 = c_2 = 1. \tag{52}$$

Verification of the decay condition. Applying the fractional Leibniz inequality (Lemma 3.16) along trajectories of (51):

$$\begin{aligned} {}^{\mathcal{C}}_{0}\mathfrak{D}^{\alpha}_{\xi}\, x^2(\xi) \;&\le\; 2x(\xi)\, {}^{\mathcal{C}}_{0}\mathfrak{D}^{\alpha}_{\xi}\, x(\xi) \\ &=\; 2x(\xi) \cdot \big[-(2 + \sin\xi)\big] x(\xi) \;=\; -2(2 + \sin\xi)\, x^2(\xi). \end{aligned} \tag{53}$$

Since $2 + \sin\xi \ge 1$ for all $\xi \ge 0$, we obtain

$${}^{\mathcal{C}}_{0}\mathfrak{D}^{\alpha}_{\xi}\, V(\xi, x(\xi)) \;\le\; -2\, V(\xi, x(\xi)),$$

so Assumption 2 holds with $c_3 = 2$.
Stability estimate. By Theorem 3.18 with $\nu = c_2/c_1 = 1$ and $\lambda = c_3 = 2$,

$$|x(\xi)| \;\le\; |\widetilde{x}_0|\, \mathbb{E}_{\alpha}^{1/2}(-2\xi^{\alpha}), \qquad \xi \ge 0. \tag{54}$$

By Lemma 2.10(iii), $\mathbb{E}_{\alpha}(-2\xi^{\alpha}) \to 0$ as $\xi \to +\infty$, so $|x(\xi)| \to 0$ for every $\widetilde{x}_0 \in \mathbb{R}$. Figure 1 illustrates this decay for $\alpha \in \{0.5, 0.7, 0.8, 0.9, 1.0\}$ with $\widetilde{x}_0 = 1$. Each trajectory $|x(\xi)|$ stays inside the envelope $\mathbb{E}_{\alpha}^{1/2}(-2\xi^{\alpha})$ and tends to zero, while the mild ripples track the oscillation of the coefficient $-(2 + \sin\xi)$. The decay becomes faster and smoother as $\alpha \to 1$, recovering the classical exponential behaviour at $\alpha = 1$.

Remark 3.28. The coefficient $\mathcal{A}(\xi) = -(2 + \sin\xi)$ oscillates, so the GPB kernels $\boldsymbol{\Psi}_k(\xi, 0)$ involve integrals of trigonometric-power products and do not admit a simple closed form. The Lyapunov approach bypasses this difficulty entirely and delivers the explicit decay bound (54) directly from the pointwise estimate on $\mathcal{A}(\xi)$.

Example 3.29 (Spectral stability criterion: two-dimensional constant matrix)**.** Consider the two-dimensional homogeneous Caputo system:

$$\begin{cases} {}^{\mathcal{C}}_{0}\mathfrak{D}^{\alpha}_{\xi}\, \mathbf{x}(\xi) = \mathcal{A}\,\mathbf{x}(\xi), \quad \xi > 0, \quad 0 < \alpha < 1, \\ \mathbf{x}(0) = \widetilde{\mathbf{x}}_0, \end{cases} \tag{55}$$

with the constant symmetric matrix

$$\mathcal{A} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \qquad \text{and} \qquad \widetilde{\mathbf{x}}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{56}$$

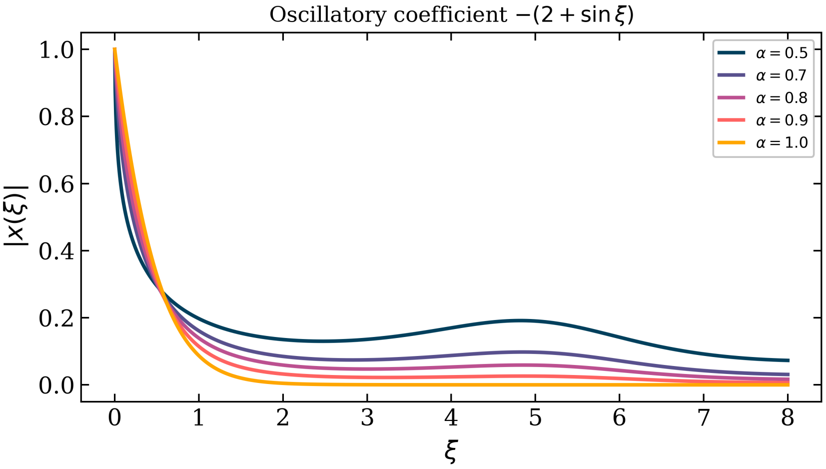


Figure 1: Lyapunov-based Mittag–Leffler stability for the scalar system (51) with oscillatory coefficient $\mathcal{A}(\xi) = -(2 + \sin\xi)$ and $x(0) = 1$, computed on $\mathcal{I} = [0, 8]$ for $\alpha \in \{0.5, 0.7, 0.8, 0.9, 1.0\}$. Every trajectory $|x(\xi)|$ decays to zero in agreement with Theorem 3.18; the mild non-monotone ripples reflect the oscillation of the coefficient, and the decay sharpens as $\alpha \to 1$.

Verification of Assumption 1. Since $\mathcal{A}$ is constant, $\|\mathcal{A}(\xi)\| \equiv \|\mathcal{A}\|$. The eigenvalues of $\mathcal{A}$ are -3 ± 1, i.e., $\mu_1 = -2$ and $\mu_2 = -4$, so the spectral norm is $\|\mathcal{A}\| = |\mu_2| = 4 =: \mathfrak{M}$.

Verification of Assumption 3. Since $\mathcal{A}$ is symmetric with eigenvalues $\mu_1 = -2 < 0$ and $\mu_2 = -4 < 0$,

$$\mathbf{x}^\top \mathcal{A}\mathbf{x} \;\le\; \mu_1\|\mathbf{x}\|^2 \;=\; -2\|\mathbf{x}\|^2, \qquad \forall\, \mathbf{x} \in \mathbb{R}^2.$$

Hence Assumption 3 holds with $\lambda = 2$.

Exact GPB series (constant-coefficient reduction). Since $\mathcal{A}$ is constant, Remark 3.11 gives

$$\mathbf{\Psi}(\xi, 0) = \mathbb{E}_\alpha(\mathcal{A}\xi^\alpha) = \sum_{k=0}^{\infty} \frac{\mathcal{A}^k\, \xi^{k\alpha}}{\Gamma(k\alpha + 1)}. \tag{57}$$

To evaluate (57) in closed form, diagonalise $\mathcal{A} = P\,\mathrm{diag}(-2, -4)\,P^{-1}$ with

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad P^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then $\mathbb{E}_\alpha(\mathcal{A}\xi^\alpha) = P\,\mathrm{diag}(\mathbb{E}_\alpha(-2\xi^\alpha), \mathbb{E}_\alpha(-4\xi^\alpha))\,P^{-1}$, giving

$$\mathbf{\Psi}(\xi, 0) = \frac{1}{2}\begin{pmatrix} \mathbb{E}_2 + \mathbb{E}_4 & \mathbb{E}_2 - \mathbb{E}_4 \\ \mathbb{E}_2 - \mathbb{E}_4 & \mathbb{E}_2 + \mathbb{E}_4 \end{pmatrix}, \tag{58}$$

where $\mathbb{E}_2 := \mathbb{E}_\alpha(-2\xi^\alpha)$ and $\mathbb{E}_4 := \mathbb{E}_\alpha(-4\xi^\alpha)$ for brevity.

Exact solution. Since $\widetilde{\mathbf{x}}_0 = (1, 1)^\top = \sqrt{2}\,\mathbf{v}_1$ where $\mathbf{v}_1 = \frac{1}{\sqrt{2}}(1, 1)^\top$ is the eigenvector for $\mu_1 = -2$,

$$\mathbf{x}(\xi) = \mathbf{\Psi}(\xi, 0)\,\widetilde{\mathbf{x}}_0 = \mathbb{E}_\alpha(-2\xi^\alpha)\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{59}$$

and thus $\|\mathbf{x}(\xi)\| = \sqrt{2}\,\mathbb{E}_\alpha(-2\xi^\alpha)$.

Verification of the spectral bound. Since $\mathcal{A}$ is symmetric, its symmetric part is itself:

$$\frac{\mathcal{A} + \mathcal{A}^\top}{2} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}.$$

The eigenvalues are -2 and -4, so the largest eigenvalue of the symmetric part is -2. Therefore, by the Rayleigh quotient,

$$\mathbf{x}^\top \mathcal{A}\mathbf{x} \le -2\|\mathbf{x}\|^2, \qquad \mathbf{x} \in \mathbb{R}^2.$$

Hence Assumption 3 is rigorously verified with $\lambda = 2$. Theorem 3.20 gives

$$\|\mathbf{x}(\xi)\| \le \|\widetilde{\mathbf{x}}_0\| \mathbb{E}_\alpha^{1/2}(-4\xi^\alpha) = \sqrt{2}\,\mathbb{E}_\alpha^{1/2}(-4\xi^\alpha)\,.$$

Together with the exact expression $\|\mathbf{x}(\xi)\| = \sqrt{2}\,\mathbb{E}_\alpha(-2\xi^\alpha)$, this confirms the spectral stability estimate for the present example. The exact and bounding trajectories both decay to zero as $\xi \to +\infty$, consistent with Corollary 3.21. Figure 2 illustrates this comparison for $\alpha = 0.8$.

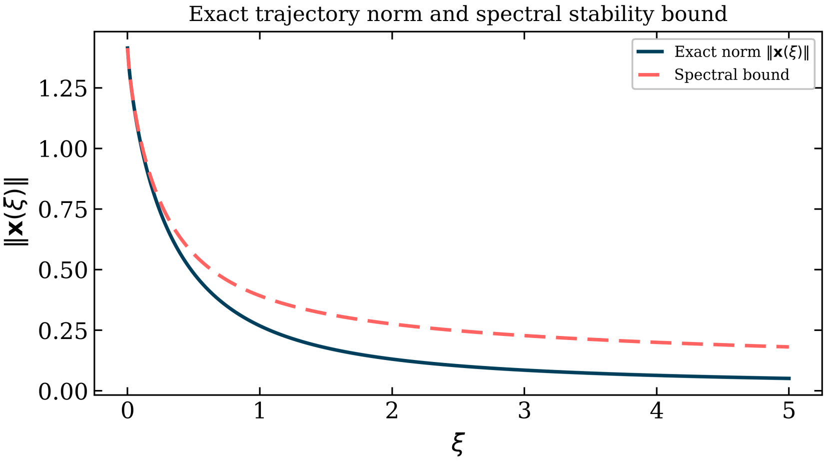


Figure 2: Spectral stability criterion for the two-dimensional system (55) with $\mathcal{A} = \left(\begin{smallmatrix} -3 & 1 \\ 1 & -3 \end{smallmatrix}\right)$, $\widetilde{\mathbf{x}}_0 = (1, 1)^\top$ and $\alpha = 0.8$. The exact norm $\|\mathbf{x}(\xi)\| = \sqrt{2}\,\mathbb{E}_\alpha(-2\xi^\alpha)$ (solid) remains below the spectral bound $\sqrt{2}\,\mathbb{E}_\alpha^{1/2}(-4\xi^\alpha)$ (dashed) for all $\xi > 0$, confirming Theorem 3.20. Both curves tend to zero as $\xi \to +\infty$, consistent with Corollary 3.21.

Example 3.30 (Bounded-input bounded-output stability: scalar inhomogeneous system with sinusoidal input)**.** Consider the scalar inhomogeneous Caputo system:

$$\begin{cases} {}^{\mathcal{C}}_{0}\mathfrak{D}^{\alpha}_{\xi}\, x(\xi) = -\lambda\, x(\xi) + \mathfrak{U}_0 \sin(\omega\xi), \quad \xi > 0, \quad 0 < \alpha < 1, \\ x(0) = \widetilde{x}_0, \end{cases} \tag{60}$$

where $\lambda, \mathfrak{U}_0, \omega > 0$ are given constants.

Verification of Assumption 1. Here $\mathcal{A}(\xi) \equiv -\lambda$ (constant), so $\|\mathcal{A}(\xi)\| = \lambda =: \mathfrak{M}$.

Verification of Assumption 3. For any $x \in \mathbb{R}$: $x \cdot \mathcal{A}(\xi) \cdot x = -\lambda x^2 \le -\lambda x^2$, so (45) holds with the same constant λ.

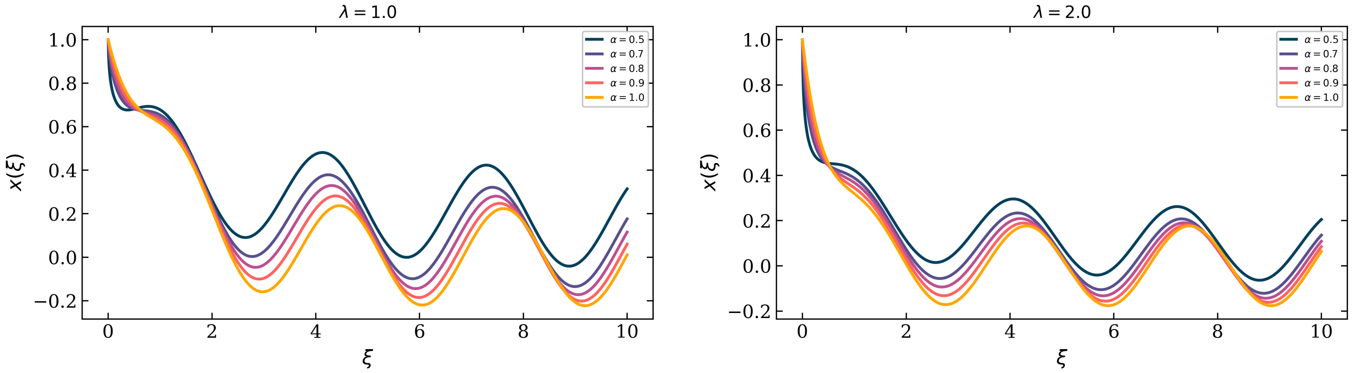
Input bound. $\sup_{\xi \ge 0} |\mathfrak{U}_0 \sin(\omega\xi)| = \mathfrak{U}_0 =: \mathfrak{U} < \infty$.

State-transition matrices. By the constant-coefficient reduction (Remark 3.11):

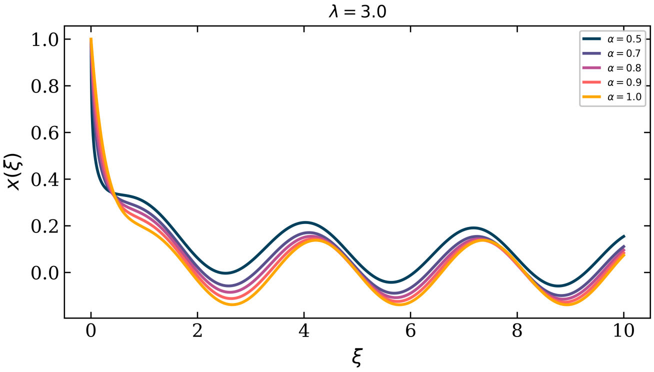
$$\mathbf{\Psi}(\xi, 0) = \mathbb{E}_\alpha(-\lambda\xi^\alpha). \tag{61}$$

The RL state-transition matrix (Lemma 3.5) with constant $\mathcal{A} = -\lambda$ reads

$$\mathbf{\Phi}(\xi, \varsigma) = (\xi - \varsigma)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(-\lambda(\xi - \varsigma)^\alpha)\,. \tag{62}$$



(a) $\lambda = 1$ (b) $\lambda = 2$



(c) $\lambda = 3$

Figure 3: Bounded-input bounded-output stability for the scalar inhomogeneous system (60) with sinusoidal forcing $\mathfrak{U}_0 \sin(\omega\xi)$, $\mathfrak{U}_0 = 0.5$, $\omega = 2$ and $x(0) = 1$, on $\mathcal{I} = [0, 10]$. Each panel fixes a damping level, (a) $\lambda = 1$, (b) $\lambda = 2$, and (c) $\lambda = 3$, and overlays $\alpha \in \{0.5, 0.7, 0.8, 0.9, 1.0\}$ with the colour assignment fixed across all panels. The response stays uniformly bounded as predicted by Theorem 3.23, and the sustained amplitude shrinks as λ grows, in line with the input-to-state gain estimate $1/\lambda$ in (66).

Closed-form solution. By Theorem 3.8:

$$x(\xi) = \mathbb{E}_\alpha(-\lambda\xi^\alpha)\,\widetilde{x}_0 + \mathfrak{U}_0 \int_0^\xi (\xi - \varsigma)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(-\lambda(\xi-\varsigma)^\alpha)\sin(\omega\varsigma)\,d\varsigma. \tag{63}$$

BIBO bound. Since $|\sin(\omega\varsigma)| \le 1$, applying Theorem 3.23 with λ as above and $\mathfrak{U} = \mathfrak{U}_0$:

$$|x(\xi)| \;\le\; \mathbb{E}_\alpha^{1/2}(-2\lambda\xi^\alpha)\,|\widetilde{x}_0| \;+\; \frac{\mathfrak{U}_0}{\lambda}\Big[1 - \mathbb{E}_\alpha(-\lambda\xi^\alpha)\Big]. \tag{64}$$

In particular, the uniform bound from Theorem 3.23 gives

$$\sup_{\xi \ge 0} |x(\xi)| \;\le\; |\widetilde{x}_0| + \frac{\mathfrak{U}_0}{\lambda}. \tag{65}$$

Zero-state response. Setting $\widetilde{x}_0 = 0$, the bound (64) reduces to

$$|x(\xi)| \;\le\; \frac{\mathfrak{U}_0}{\lambda}\Big[1 - \mathbb{E}_\alpha(-\lambda\xi^\alpha)\Big] \;\le\; \frac{\mathfrak{U}_0}{\lambda}, \qquad \xi \ge 0, \tag{66}$$

showing that the input-to-state gain is at most $1/\lambda$.

Vanishing input. If $\mathfrak{U}_0 = 0$ (unforced), (64) yields

$$|x(\xi)| \;\le\; |\widetilde{x}_0|\,\mathbb{E}_\alpha^{1/2}(-2\lambda\xi^\alpha) \;\to\; 0 \quad \text{as } \xi \to +\infty,$$

consistent with Corollary 3.26.

Numerical illustration. For $\alpha = 0.8$, $\lambda = 1$, $\mathfrak{U}_0 = 0.5$, $\omega = 2$, $\widetilde{x}_0 = 1$:

ξ	$\mathbb{E}_{0.8}(-\xi^{0.8})$	Bound (64)	Uniform bound (65)
0.5	0.6435	1.1782	1.5
1.0	0.4559	0.9266	1.5
2.0	0.2946	0.6548	1.5
5.0	0.1368	0.4376	1.5
10	0.0712	0.3286	1.5

The table confirms that the BIBO bound (64) is uniformly bounded by the constant (65), and decays as the exponential influence of $\mathbb{E}_\alpha^{1/2}$ diminishes the initial-condition contribution. Figure 3 shows the corresponding responses $x(\xi)$ on $\mathfrak{I} = [0, 10]$ for $\alpha \in \{0.5, 0.7, 0.8, 0.9, 1.0\}$ and the three damping levels (a) $\lambda = 1$, (b) $\lambda = 2$ and (c) $\lambda = 3$. In every panel the state settles into a bounded sustained oscillation driven by the input $\mathfrak{U}_0 \sin(\omega\xi)$, confirming the uniform bound of Theorem 3.23. The amplitude of the sustained response shrinks as λ increases, consistent with the input-to-state gain $1/\lambda$ in (66), and lower α produces a slower, more memory-dominated approach to the steady regime.

4. Conclusion

This paper establishes a rigorous well-posedness and stability framework for linear Caputo fractional systems with variable coefficients in the Banach space $C(\mathfrak{I}, \mathbb{R}^n)$. The generalized Peano–Baker series is constructed with a regular identity initial term and shown to converge absolutely and uniformly via a Beta-function-based Mittag–Leffler estimate. Existence and uniqueness of solutions are proved for both homogeneous and inhomogeneous problems using integral representations and Gronwall-type arguments. A notable structural result is the mixed-kernel representation of the inhomogeneous solution, involving both Caputo and Riemann–Liouville state-transition matrices. Stability is characterized through Lyapunov, spectral, and BIBO criteria, yielding explicit Mittag–Leffler bounds for system responses. The revised analysis also clarifies the distinction between mild and classical Caputo solutions, separates finite-interval well-posedness from infinite-interval stability, gives truncation estimates for numerical use of the GPB series, and explains why the uniform spectral condition is sufficient but not necessary.

Future work will focus on nonlinear extensions, numerical implementation and adaptive truncation of the generalized Peano–Baker series, optimal control applications, higher-order Caputo systems, and extensions to systems with delays, impulses, and more general input spaces such as L^p classes. These directions are natural because variable coefficients, memory effects, and nonlocal forcing terms frequently occur together in fractional control and physical models.

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