Letters in Nonlinear Analysis and its Applications **4** (2026) No. 2, 117-126
https://doi.org/10.66147/lnaa.20264280
Available online at www.lettersinnonlinearanalysis.com
Research Article

Letters in Nonlinear Analysis and its Applications
Peer Review Scientific Journal
ISSN: 2958-874x

Fixed Point Theorems for E$_2$-Contractions Maps

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Abstract

We introduce and study a new class of contractive maps on metric spaces defined by a contraction condition on the *second elementary symmetric polynomial* of the three pairwise distances of any triple of distinct points. A map satisfying this condition is called an *E_2-contraction*; the corresponding strict version is a *strict E_2-contraction*. For complete metric spaces containing at least three points, every E_2-contraction is continuous and has a fixed point if and only if it admits no periodic point of prime period 2, with at most two fixed points in total. For arbitrary metric spaces containing at least three points, a strict E_2-contraction with no period-2 point for which some orbit admits a convergent subsequence necessarily has a fixed point. As consequences, we recover the classical Banach and Edelstein fixed point theorems, and show that on spaces where every point is an accumulation point every E_2-contraction is a Banach contraction. Explicit finite examples show that E_2-contractions (resp. strict E_2-contractions) strictly extend the Banach (resp. Edelstein) and perimeter-contracting classes of [11] (resp. [2]).

Keywords: Fixed point theorems, E_2-contraction, metric space.
2010 MSC: 47H10, 47H09

1. Introduction

The Banach contraction principle [1] is one of the cornerstones of metric fixed point theory. It asserts that every map $T\colon X \to X$ on a complete metric space satisfying

$$d(Tx, Ty) \le \alpha\, d(x, y), \qquad \alpha \in [0, 1),$$

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Received April 16, 2026, Accepted: June 28, 2026, Online: June 29, 2026

for all $x, y \in X$ possesses a unique fixed point, obtained as the limit of any Picard orbit. The simplicity and wide applicability of this result have motivated an extensive literature of generalizations, each designed to capture maps that the Banach condition fails to cover. A representative, though far from exhaustive, selection includes:

- Edelstein's strictly contractive condition [6] (1962),
- Boyd–Wong's nonlinear contraction [3] (1969),
- Meir–Keeler's contraction [9] (1969),
- Nadler's multi-valued contraction [10] (1969),
- Ćirić' nonlinear contraction [5] (1974),
- Kirk's asymptotic contraction [8] (2003),
- Wardowski's F-contraction [13] (2012),
- Petrov's perimeter contraction for triangles [11] (2023) (see also [7]),
- Bey–Petrov–Salimov's Edelstein-type perimeter contraction [2] (2025).

Recall that the *elementary symmetric polynomials* in n variables $t_1, \dots, t_n$ are defined by

$$e_k(t_1, \dots, t_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} t_{i_1} \cdots t_{i_k}, \quad k = 1, \dots, n.$$

For three distances d_1, d_2, d_3 associated with a triple of points, one has $e_1(d_1, d_2, d_3) = d_1 + d_2 + d_3$ (the perimeter) and

$$e_2(d_1, d_2, d_3) = d_1 d_2 + d_1 d_3 + d_2 d_3.$$

Petrov [11, 12] studied maps that contract the e_1-polynomial, i.e. maps for which the perimeter of any triangle is decreased by a uniform factor (the Banach-type version). The Edelstein-type analogue was introduced and studied by Bey, Petrov, and Salimov [2]. The present paper adopts the next symmetric polynomial e_2 as the controlling quantity, yielding Banach-type and Edelstein-type analogues simultaneously.

For a triple of points $x, y, z \in X$ we write

$$E_2\big(x, y, z\big) = d(x, y)d(y, z) + d(x, y)d(z, x) + d(y, z)d(z, x).$$

Definition 1.1. A map $T \colon X \to X$ is called an E_2*-contraction* if there exists $\alpha \in [0, 1)$ such that

$$E_2\big(Tx, Ty, Tz\big) \le \alpha\, E_2\big(x, y, z\big) \tag{1}$$

for all pairwise distinct $x, y, z \in X$. It is called a *strict* E_2*-contraction* if

$$E_2\big(Tx, Ty, Tz\big) < E_2\big(x, y, z\big) \tag{2}$$

for all pairwise distinct $x, y, z \in X$.

The paper is organized as follows. Section 2 introduces the broader class of E_2-nonexpansive maps and proves that every such map is continuous; continuity of E_2-contractions and strict E_2-contractions follows as an immediate corollary. Section 3 contains the main fixed point theorems: Theorem 3.2 for complete metric spaces (the E_2 analogue of the Banach theorem) and Theorem 3.6 for general metric spaces (the E_2 analogue of the Edelstein theorem), together with their proofs and corollaries recovering the classical results. Section 4 studies the behavior of E_2-contractions near accumulation points and shows that, on spaces where every point is an accumulation point, the E_2 condition reduces to a standard Banach contraction. Sections 5 and 6 provide explicit finite examples distinguishing the new classes from their classical counterparts. Section 7 collects concluding remarks and open problems.

2. Continuity of E$_2$-contractions

Before establishing fixed point results, we verify that the class of maps under study is well-behaved from a topological standpoint. The key observation is that continuity is already guaranteed by the weaker, non-strict inequality (3) below, and therefore holds in particular for both E_2-contractions and strict E_2-contractions.

Definition 2.1. A map $T: X \to X$ is called *E_2-nonexpansive* if for every pairwise distinct $x, y, z \in X$,

$$E_2\big(Tx, Ty, Tz\big) \le E_2\big(x, y, z\big). \tag{3}$$

Proposition 2.2. *Every E_2-nonexpansive map on a metric space with $|X| \ge 3$ is continuous.*

Proof. Fix $x_0 \in X$ and let (x_m) be a sequence converging to x_0. If x_0 is an isolated point, then continuity at x_0 is immediate. Assume therefore that x_0 is an accumulation point. If only finitely many terms of (x_m) differ from x_0, then $x_m = x_0$ eventually, and hence $Tx_m = Tx_0$ eventually. Thus continuity is trivial. We may therefore assume that infinitely many terms differ from x_0.

Let (x_{m_k}) be the subsequence consisting of all terms satisfying $x_{m_k} \ne x_0$. Then $x_{m_k} \to x_0$. Since x_0 is an accumulation point, every ball $B(x_0, r)$ contains infinitely many points distinct from x_0. Consequently, for each k we may choose

$$y_k \in X \setminus \{x_0, x_{m_k}\} \qquad \text{with} \qquad d(x_0, y_k) < \frac{1}{k}.$$

Hence $y_k \to x_0$, and the triple (x_0, x_{m_k}, y_k) is pairwise distinct.

Applying (3) gives

$$E_2(Tx_0, Tx_{m_k}, Ty_k) \le E_2(x_0, x_{m_k}, y_k).$$

Since

$$d(x_{m_k}, y_k) \le d(x_{m_k}, x_0) + d(x_0, y_k) \to 0,$$

every factor occurring in

$$E_2(x_0, x_{m_k}, y_k) = d(x_0, x_{m_k})d(x_0, y_k) + d(x_0, x_{m_k})d(x_{m_k}, y_k) + d(x_0, y_k)d(x_{m_k}, y_k)$$

tends to 0. Therefore

$$E_2(Tx_0, Tx_{m_k}, Ty_k) \to 0.$$

Since E_2 is a sum of nonnegative terms, we obtain that each of the three products appearing in its expansion tends to 0:

$$d(Tx_0, Tx_{m_k})d(Tx_0, Ty_k) \to 0,$$
$$d(Tx_0, Tx_{m_k})d(Tx_{m_k}, Ty_k) \to 0,$$

and

$$d(Tx_0, Ty_k)d(Tx_{m_k}, Ty_k) \to 0.$$

Suppose that $d(Tx_0, Tx_{m_k}) \not\to 0$. Passing to a further subsequence if necessary, there exists $\varepsilon > 0$ such that

$$d(Tx_0, Tx_{m_k}) \ge \varepsilon \qquad \text{for all } k.$$

The first two displayed limits then imply

$$d(Tx_0, Ty_k) \to 0 \qquad \text{and} \qquad d(Tx_{m_k}, Ty_k) \to 0.$$

By the triangle inequality,

$$d(Tx_0, Tx_{m_k}) \le d(Tx_0, Ty_k) + d(Ty_k, Tx_{m_k}) \to 0,$$

a contradiction. Hence

$$d(Tx_0, Tx_{m_k}) \to 0.$$

Thus every subsequence of (x_m) whose terms are different from x_0 is mapped to a sequence converging to Tx_0. The remaining terms of (x_m) are equal to x_0 and therefore are mapped exactly to Tx_0. Consequently,

$$Tx_m \to Tx_0.$$

Therefore T is continuous at x_0, and since x_0 was arbitrary, T is continuous on X. □

The following is an immediate consequence.

Proposition 2.3. *Every E_2-contraction and every strict E_2-contraction on a metric space with $|X| \geq 3$ is continuous.*

Remark 2.4. Continuity in this context is specific to $n = 3$, $k = 2$. For $k = 1$ (the perimeter case) continuity was established in [12]. For $k = n$ (the product of all pairwise distances) continuity fails in general, and for $n \geq 3$, $2 \leq k \leq n-1$ the question remains open.

3. Fixed point theorems

With continuity in hand, we turn to the existence and multiplicity of fixed points. We treat two settings: complete metric spaces (the E_2-analogue of Banach's theorem) and general metric spaces with a compactness assumption on some orbit (the E_2-analogue of Edelstein's theorem).

We first record a simple but useful algebraic lemma.

Lemma 3.1. *Let $u, v, w \geq 0$ satisfy $w \geq |u - v|$. Then*

$$\max\{u, v\}^2 \leq uv + uw + vw.$$

Proof. Without loss of generality assume $u \geq v$, so that $w \geq u - v$. Then

$$uv + uw + vw \geq uv + u(u - v) + v(u - v) = u^2 + uv - v^2 \geq u^2,$$

where the last step uses $uv \geq v^2$ (since $u \geq v \geq 0$). □

3.1. The complete metric space case

Theorem 3.2. *Let (X, d) be a complete metric space with $|X| \geq 3$ and let $T\colon X \to X$ be an E_2-contraction with constant $\alpha \in [0, 1)$. Then T has a fixed point if and only if T has no periodic point of prime period 2. Moreover, T has at most two fixed points.*

Proof. Step 1: No period-2 point implies the existence of a fixed point. Pick an arbitrary $x_0 \in X$ and form the Picard sequence $x_{n+1} = Tx_n$. If $x_{k+1} = x_k$ for some k, then x_k is a fixed point and we are done. Assume therefore that $x_{n+1} \neq x_n$ for all $n \geq 0$.

If there existed n with $x_{n+2} = x_n$ and $x_{n+1} \neq x_n$, then $\{x_n, x_{n+1}\}$ would be a 2-cycle, contradicting the hypothesis. Hence the three consecutive iterates x_n, x_{n+1}, x_{n+2} are pairwise distinct for every $n \geq 0$.

Set $u_n = d(x_n, x_{n+1}) > 0$ and $w_n = d(x_n, x_{n+2})$. Applying the E_2-contraction condition to the triple (x_n, x_{n+1}, x_{n+2}) gives

$$e_2(u_{n+1}, u_{n+2}, w_{n+1}) \leq \alpha\, e_2(u_n, u_{n+1}, w_n).$$

Writing $E_n = e_2(u_n, u_{n+1}, w_n)$, we obtain $E_n \leq \alpha^n E_0$. By the reverse triangle inequality $w_n \geq |u_n - u_{n+1}|$, so Lemma 3.1 yields

$$u_n^2 \leq \max\{u_{n-1}, u_n\}^2 \leq E_{n-1} \leq \alpha^{n-1} E_0.$$

Hence $u_n \leq \sqrt{E_0}\,(\sqrt{\alpha})^{n-1}$, and $\sum_{n=0}^{\infty} u_n$ converges. For $m > n$,

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} u_k \to 0,$$

so $\{x_n\}$ is a Cauchy sequence. By completeness, $x_n \to x^*$ for some $x^* \in X$. Since T is continuous (Proposition 2.3),

$$x_{n+1} = Tx_n \to Tx^*,$$

and therefore $Tx^* = x^*$.

Step 2: A fixed point implies no period-2 point. Suppose T has a fixed point q and, for contradiction, that there exists $p \in X$ with $p \neq Tp$ and $T^2p = p$. Set $r = Tp$; then $r \neq p$ and $Tr = p$. The three points p, r, q are pairwise distinct: $p \neq r$ by assumption; if $q = p$ then $Tq = q = p$ but $Tp = r \neq p$, a contradiction; similarly $q \neq r$. Applying the E_2-contraction to the triple (p, r, q) and using $Tp = r$, $Tr = p$, $Tq = q$, we get

$$E_2(p, r, q) \leq \alpha\, E_2(p, r, q).$$

Since $\alpha < 1$, this forces $E_2(p, r, q) = 0$. But the three points are distinct, so all three distances are positive, giving $E_2(p, r, q) > 0$ which is contradiction.

Step 3: At most two fixed points. Suppose a, b, c are three distinct fixed points. Applying the E_2-contraction to (a, b, c) gives

$$E_2(a, b, c) \leq \alpha\, E_2(a, b, c),$$

which, since $\alpha < 1$, forces $E_2(a, b, c) = 0$, contradicting the positivity of all three distances. □

Remark 3.3. The condition "T has no period-2 point" is not merely technical; a finite example in Section 5 shows that an E_2-contraction may have a 2-cycle and no fixed point.

The Banach contraction principle is recovered as an immediate corollary.

Corollary 3.4 (Banach fixed-point theorem)**.** *Let (X, d) be a nonempty complete metric space and let $T: X \to X$ be a Banach contraction with constant $k \in [0, 1)$. Then T has a unique fixed point.*

Proof. If $|X| \leq 2$ the statement is elementary. Assume $|X| \geq 3$.

A Banach contraction cannot have a period-2 point: if $T^2x = x$ and $x \neq Tx$, then $d(x, Tx) = d(Tx, T^2x) \leq k\, d(x, Tx)$, forcing $d(x, Tx) = 0$, a contradiction.

For any three pairwise distinct $x, y, z \in X$, multiplying the three inequalities $d(Tx, Ty) \leq k\, d(x, y)$, $d(Ty, Tz) \leq k\, d(y, z)$, $d(Tz, Tx) \leq k\, d(z, x)$ pairwise and summing gives

$$E_2(Tx, Ty, Tz) \leq k^2\, E_2(x, y, z).$$

Since $k^2 < 1$, T is an E_2-contraction. By Theorem 3.2, T has a fixed point. Uniqueness follows from $d(p, q) = d(Tp, Tq) \leq k\, d(p, q)$, which forces $d(p, q) = 0$ for any two fixed points p, q. □

Remark 3.5. Every Banach contraction with constant k is an E_2-contraction with constant $\alpha = k^2$. The examples in Section 5 show that the inclusion is strict: there exist E_2-contractions that are not Banach contractions.

3.2. The general metric space case

When space is not complete, a fixed point can still be obtained provided that the orbit of a certain point has a convergent subsequence, a condition which is automatically satisfied in compact spaces.

Theorem 3.6. *Let (X, d) be a metric space with $|X| \geq 3$ and let $T: X \to X$ be a strict E_2-contraction. Assume that*

(i) *T has no periodic point of prime period 2;*

(ii) *there exists $x \in X$ whose orbit $\{T^n x\}$ has a convergent subsequence.*

Then T has a fixed point, and the number of fixed points is at most 2.

Proof. Choose $x_0 \in X$ such that some subsequence (x_{n_i}) of $x_n = T^n x_0$ converges to a limit $\xi \in X$. If $x_{k+1} = x_k$ for some k, then x_k is a fixed point and we are done. Assume $x_{n+1} \neq x_n$ for all n.
Step 1: Three consecutive iterates are distinct. If $x_{n+2} = x_n \neq x_{n+1}$ for some n, then $\{x_n, x_{n+1}\}$ is a 2-cycle, contradicting (i). Hence x_n, x_{n+1}, x_{n+2} are pairwise distinct for every $n \geq 0$.
Step 2: The sequence (p_n) is strictly decreasing. Set

$$p_n = E_2(x_n, x_{n+1}, x_{n+2}).$$

Since the strict E_2-contraction condition applies to the pairwise distinct triple (x_n, x_{n+1}, x_{n+2}), we have $p_{n+1} < p_n$ for all $n \geq 0$. The sequence (p_n) is therefore strictly decreasing and bounded below by 0; it converges to some limit $\ell = \lim_{n\to\infty} p_n \geq 0$.
Step 3: The limit point ξ is a fixed point. By continuity of T (Proposition 2.3),

$$x_{n_i+1} = Tx_{n_i} \to T\xi \quad \text{and} \quad x_{n_i+2} = T^2 x_{n_i} \to T^2\xi.$$

Suppose for contradiction that $\xi \neq T\xi$. Since T has no 2-cycle, $\xi \neq T^2\xi$ as well (otherwise ξ and $T\xi$ would form a 2-cycle).

Consider the function

$$F(x, y, z) = \frac{E_2(Tx, Ty, Tz)}{E_2(x, y, z)}$$

on the open set $\Delta = \{(x, y, z) \in X^3 : E_2(x, y, z) > 0\}$.

Since $d(\xi, T\xi) > 0$ and $d(\xi, T^2\xi) > 0$,

$$E_2(\xi, T\xi, T^2\xi) \geq d(\xi, T\xi)\, d(\xi, T^2\xi) > 0,$$

so $(\xi, T\xi, T^2\xi) \in \Delta$. By continuity of E_2, the denominator is bounded away from zero near this triple, and F is continuous on some neighborhood $V \subset \Delta$ of $(\xi, T\xi, T^2\xi)$.

We evaluate F at the limit triple.

- If $\xi, T\xi, T^2\xi$ are pairwise distinct, the strict E_2-contraction gives $F(\xi, T\xi, T^2\xi) < 1$.
- If $T\xi = T^2\xi$, then $T\xi$ is a fixed point of T, and the numerator becomes $e_2(0, 0, d(\xi, T\xi)) = 0$, while the denominator is positive, so $F(\xi, T\xi, T^2\xi) = 0 < 1$.

In both cases $F(\xi, T\xi, T^2\xi) < 1$. By continuity, there exists a neighborhood $U \subset V$ of $(\xi, T\xi, T^2\xi)$ and $\lambda < 1$ such that $F \leq \lambda$ on U. For all large enough i, $(x_{n_i}, x_{n_i+1}, x_{n_i+2}) \in U$, so $p_{n_i+1} \leq \lambda p_{n_i}$. Iterating along the subsequence gives $p_{n_i} \to 0$, hence $\ell = 0$. On the other hand,

$$\ell = E_2(\xi, T\xi, T^2\xi) \geq d(\xi, T\xi)\, d(\xi, T^2\xi) > 0,$$

a contradiction. Therefore $\xi = T\xi$.
Step 4: At most two fixed points. If a, b, c are three distinct fixed points, the strict E_2-contraction applied to (a, b, c) gives a strict inequality with equal sides, which is impossible. □

Corollary 3.7. *Let (X, d) be a compact metric space with $|X| \geq 3$ and let $T\colon X \to X$ be a strict E_2-contraction with no periodic point of prime period 2. Then T has a fixed point, and at most two fixed points exist.*

Proof. Compactness ensures that every orbit has a convergent subsequence, so condition (ii) of Theorem 3.6 holds automatically. □

The Edelstein fixed-point theorem is likewise recovered.

Corollary 3.8 (Edelstein fixed-point theorem)**.** *Let (X, d) be a metric space with $|X| \geq 3$ and let $T\colon X \to X$ be an Edelstein contraction, i.e. $d(Tx, Ty) < d(x, y)$ for all distinct $x, y \in X$. Assume there exists $x \in X$ whose orbit $\{T^n x\}$ has a convergent subsequence. Then T has a unique fixed point.*

Proof. If $T^2x = x \neq Tx$, then $d(x, Tx) = d(Tx, T^2x) < d(x, Tx)$, a contradiction, so T has no period-2 point. For any three pairwise distinct points x, y, z, multiplying the three strict inequalities pairwise and summing as in the proof of Corollary 3.4 shows that T is a strict E_2-contraction. By Theorem 3.6, T has a fixed point. Uniqueness follows from $d(p, q) = d(Tp, Tq) < d(p, q)$ for two distinct fixed points p, q, which is absurd. □

Remark 3.9. Every Edelstein contraction is a strict E_2-contraction. The examples in Section 6 show that the inclusion is strict: there exist strict E_2-contractions that fail to be Edelstein contractions. Moreover, the strict E_2-contraction condition also extends the Edelstein-type perimeter contraction of [2]: the examples in Section 6 show that a strict E_2-contraction need not contract the perimeter of every triangle.

4. Accumulation points

We now investigate the local behavior of E_2-contractions near accumulation points. The main results show that the E_2 condition implies a pointwise lipchitz estimate at accumulation points, and that on spaces where every point is an accumulation point the E_2 condition collapses to a classical Banach contraction.

Proposition 4.1. *Let (X, d) be a metric space with $|X| \geq 3$ and let $T \colon X \to X$ be an E_2-contraction with constant $\alpha \in [0, 1)$. If $x \in X$ is an accumulation point, then*

$$d(Tx, Ty) \leq \sqrt{\alpha}\, d(x, y) \qquad \text{for every } y \in X.$$

Proof. The case $y = x$ is trivial. Assume $y \neq x$. Since x is an accumulation point, there exists a sequence $\{z_n\} \subset X \setminus \{x, y\}$, with all z_n distinct, such that $z_n \to x$. Apply the E_2-contraction to the triple (x, y, z_n):

$$E_2(Tx, Ty, Tz_n) \leq \alpha\, E_2(x, y, z_n) = \alpha\big(d(x, y)d(y, z_n) + d(x, y)d(x, z_n) + d(y, z_n)d(x, z_n)\big).$$

As $n \to \infty$: $d(x, z_n) \to 0$; $d(y, z_n) \to d(x, y)$ by the triangle inequality; $d(Tx, Tz_n) \to 0$ and $d(Ty, Tz_n) \to d(Tx, Ty)$ by continuity of T (Proposition 2.3). Passing to the limit,

$$\big(d(Tx, Ty)\big)^2 \leq \alpha\big(d(x, y)\big)^2,$$

whence $d(Tx, Ty) \leq \sqrt{\alpha}\, d(x, y)$. □

Corollary 4.2. *Let (X, d) be a metric space with $|X| \geq 3$ in which every point is an accumulation point. Then every E_2-contraction $T \colon X \to X$ with constant α is a Banach contraction with constant $\sqrt{\alpha}$.*

Proof. Apply Proposition 4.1 to every point $x \in X$. □

An analogous result holds for strict E_2-contractions.

Proposition 4.3. *Let (X, d) be a metric space with $|X| \geq 3$ and let $T \colon X \to X$ be a strict E_2-contraction. If $x \in X$ is an accumulation point, then $d(Tx, Ty) \leq d(x, y)$ for every $y \in X$ with $y \neq x$.*

Proof. Choose $z_n \to x$ with $z_n \notin \{x, y\}$, all distinct. Apply the strict E_2 inequality to (x, y, z_n) and pass to the limit exactly as in the proof of Proposition 4.1, obtaining $(d(Tx, Ty))^2 \leq (d(x, y))^2$. □

Corollary 4.4. *If every point of (X, d) is an accumulation point and $|X| \geq 3$, then every strict E_2-contraction $T \colon X \to X$ is non-expansive: $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$.*

5. Examples of E_2-contractions

The following examples illustrate that the class of E_2-contractions properly contains the Banach and perimeter-contracting classes. We work on the finite metric space $X = \{1, 2, 3\}$, which is complete (as every finite metric space is), so that Theorem 3.2 applies.

The metric: Set

$$d(1,2) = 1, \quad d(1,3) = 2, \quad d(2,3) = 3,$$

extended symmetrically with $d(i,i) = 0$. The triangle inequalities are easily verified: $d(2,3) = 3 \leq d(1,2) + d(1,3) = 3$, and the other two inequalities are strict.

The map: Define $T : X \to X$ by

$$T(1) = 2, \quad T(2) = 3, \quad T(3) = 3.$$

5.1. T is an E_2-contraction but not a Banach contraction

Example 5.1. We verify that T is an E_2-contraction with $\alpha = 9/11$ and that T does not satisfy the Banach contraction condition.

- Not a Banach contraction. We have

$$d(T1, T2) = d(2,3) = 3 > 1 = d(1,2),$$

so T cannot be a Banach contraction regardless of the Lipschitz constant.

- Verification of the E_2-condition. Since X has exactly three points, the only triple of distinct points is $(1, 2, 3)$. We compute the original E_2-value:

$$E_2(1,2,3) = 1 \cdot 3 + 1 \cdot 2 + 3 \cdot 2 = 3 + 2 + 6 = 11.$$

The images are $T(1) = 2$, $T(2) = 3$, $T(3) = 3$, giving $d(T1, T2) = d(2,3) = 3$, $d(T2, T3) = d(3,3) = 0$, $d(T1, T3) = d(2,3) = 3$. Hence

$$E_2(T1, T2, T3) = 3 \cdot 0 + 3 \cdot 3 + 0 \cdot 3 = 0 + 9 + 0 = 9.$$

Since $9 \leq \frac{9}{11} \cdot 11 = 9$, the E_2-contraction condition holds with equality for $\alpha = 9/11$. Any $\alpha \in [9/11, 1)$ is therefore admissible.

- Fixed point. The map T has the unique fixed point $T(3) = 3$. There is no period-2 point: $T^2(1) = T(2) = 3 \neq 1$, $T^2(2) = T(3) = 3 \neq 2$, and 3 is fixed. This is consistent with Theorem 3.2.

5.2. T is an E_2-contraction but not a perimeter contraction

Example 5.2. Using the same space and map, we show that T is not a perimeter contraction in the sense of Petrov [11].

Perimeter calculation: The perimeter of the original triangle $(1, 2, 3)$ is

$$d(1,2) + d(2,3) + d(1,3) = 1 + 3 + 2 = 6.$$

After applying T, the triangle has vertices $T(1) = 2$, $T(2) = 3$, $T(3) = 3$, which form a degenerate triangle (two vertices coincide). Its "perimeter" is

$$d(T1, T2) + d(T2, T3) + d(T1, T3) = 3 + 0 + 3 = 6.$$

The perimeter is unchanged, so T does not contract perimeters, so it is neither a strict nor even a non-strict $(\alpha < 1)$ perimeter contraction.

By contrast, as computed in Example 5.1, the E_2-value drops from 11 to 9. This shows that the e_2 polynomial is a finer invariant than the perimeter for this map.

Remark 5.3. Examples 5.1 and 5.2 together establish that the class of E_2-contractions is not contained in the class of Banach contractions, and is not contained in the class of perimeter contractions. Corollary 3.4 shows that every Banach contraction is an E_2-contraction, so the inclusion $\{\text{Banach}\} \subsetneq \{E_2\text{-contractions}\}$ is strict.

6. Examples of strict E_2-contractions

We now exhibit maps that are strict E_2-contractions but fail to satisfy the Edelstein condition $d(Tx, Ty) < d(x, y)$ for all distinct x, y. We again work on $X = \{1, 2, 3\}$ with the same metric.

6.1. A strict e_2-contraction that is not Edelstein contractive

Example 6.1. Keep $X = \{1, 2, 3\}$, $d(1, 2) = 1$, $d(1, 3) = 2$, $d(2, 3) = 3$, and $T(1) = 2$, $T(2) = 3$, $T(3) = 3$.

- Not Edelstein contractive: For the pair $(1, 2)$ we have

$$d(T1, T2) = d(2, 3) = 3 > 1 = d(1, 2),$$

so T does not satisfy $d(Tx, Ty) < d(x, y)$ for all distinct pairs.

- Strict E_2-contraction: From the computations in Example 5.1,

$$E_2(T1, T2, T3) = 9 < 11 = E_2(1, 2, 3).$$

Since X has only one triple of distinct points, the strict inequality holds for every such triple, and T is a strict E_2-contraction.

6.2. A strict E_2-contraction that is not a perimeter contraction

Example 6.2. With the same space and map, we have already noted in Example 5.2 that the perimeter is unchanged (both equal 6). Since a strict perimeter contraction would require

$$d(T1, T2) + d(T2, T3) + d(T1, T3) < d(1, 2) + d(2, 3) + d(1, 3),$$

the equality $6 = 6$ shows that T is not a strict perimeter contraction. Yet T is a strict E_2-contraction, as shown in Example 6.1.

Remark 6.3. Examples 6.1 and 6.2 demonstrate that the class of strict E_2-contractions is contained in neither the class of Edelstein contractions nor the class of strict perimeter contractions in the sense of [2]. Combined with Corollary 3.8, which shows that every Edelstein contraction is a strict E_2-contraction, we obtain the strict inclusion $\{\text{Edelstein}\} \subsetneq \{\text{strict } E_2\text{-contractions}\}$.

7. Conclusion

We have introduced and studied a new class of contractive maps defined by an inequality on the second elementary symmetric polynomial E_2 of the pairwise distances of any triple of distinct points. The principal contributions of the paper are as follows.

1. *Continuity.* Every E_2-nonexpansive map on a metric space with at least three points is continuous (Proposition 2.2). This implies, in particular, that every E_2-contraction and strict E_2-contraction is continuous, a fact used crucially in the proofs of all fixed point results.

2. *Fixed point theorem for complete spaces.* On a complete metric space with $|X| \geq 3$, an E_2-contraction has a fixed point if and only if it has no period-2 point (Theorem 3.2). The number of fixed points is at most two.

3. *Fixed point theorem for general spaces.* On an arbitrary metric space with $|X| \geq 3$, a strict E_2-contraction with no period-2 point and a partially convergent orbit necessarily has a fixed point (Theorem 3.6).

4. *Recovery of classical theorems.* The Banach and Edelstein fixed-point theorems are obtained as corollaries (Corollaries 3.4 and 3.8), and the class of E_2-contractions (resp. strict E_2-contractions) is shown to strictly contain the class of Banach (resp. Edelstein) contractions.

5. *Accumulation-point analysis.* At every accumulation point x, an E_2-contraction satisfies the pointwise Lipschitz estimate $d(Tx, Ty) \leq \sqrt{\alpha}\, d(x, y)$. On spaces where every point is an accumulation point, the E_2-contraction condition is equivalent to a Banach contraction with constant $\sqrt{\alpha}$ (Corollary 4.2).

6. *Separation from perimeter contractions.* Explicit finite examples show that neither E_2-contractions nor strict E_2-contractions are contained in Petrov's class of perimeter-contracting maps [11].

Open problems. Several natural questions remain open.

P1. *Uniqueness.* The bound of at most two fixed points is sharp in principle (two fixed points are not excluded by the condition), but we do not know of an explicit example of an E_2-contraction with exactly two fixed points. Constructing such an example, or proving uniqueness under additional hypotheses, would be interesting.

P2. *Higher symmetric polynomials.* The cases $k = 1$ (Petrov) and $k = 2$ (this paper) are now understood for triples. For $n \geq 4$ points and $2 \leq k \leq n - 1$ neither continuity nor fixed point theory has been fully worked out.

P3. *Metric spaces without the three-point assumption.* The condition $|X| \geq 3$ is essential for the definition of E_2; it would be worthwhile to understand what weaker structural assumptions on X still guarantee the main results.

P4. *Multi-valued and set-valued extensions.* An E_2-contraction analogue of Nadler's theorem [10] for set-valued maps remains to be developed.

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