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Existence, uniqueness and Ulam-type stability for nonlocal Caputo fractional boundary value problems

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Abstract

In this paper, we study a class of nonlocal boundary value problems for Caputo fractional differential equations with integral boundary conditions and Riemann–Liouville memory effects. The equation involves lower-order terms, a nonlinear perturbation, and an internal memory term, while the boundary condition is given by an integral functional. We first derive an equivalent integral formulation and define the corresponding fixed point operator in a suitable function space. Using fractional integral estimates, Krasnoselskii's fixed point theorem, and Banach's contraction principle, we establish existence and uniqueness results for integral solutions. The resulting criteria explicitly describe the influence of the Riemann–Liouville memory term on the solvability conditions. We further prove Ulam–Hyers and Ulam–Hyers–Rassias type stability and obtain explicit stability bounds.

Keywords: Caputo derivative, Riemann–Liouville fractional integral, integral solution, nonlocal boundary condition, fixed point method, Ulam-type stability
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1. Introduction

Fractional calculus provides a useful framework for describing systems with memory and hereditary effects. It has become an important tool in the study of nonlocal dynamical models; see [8, 10, 12, 14, 16] for general background and [9, 13] for representative applications.

Fractional boundary value problems have been widely studied because of their mathematical interest and their relevance in applications. Among them, problems with integral boundary conditions are of particular interest, since such conditions describe averaged or global information on the state. Existence and qualitative properties for these problems have been investigated by several methods, especially fixed point techniques; see, for example, [2, 5].

Another important topic is Ulam-type stability. Starting from the works of Ulam, Hyers, Aoki, and Rassias [4, 11, 15, 17], this notion has been studied for many classes of differential, integral, and fractional equations; see [3, 6, 7, 18]. For recent remarks on Ulam stability in boundary value problems, we also refer to [1].

However, the simultaneous treatment of an internal Riemann–Liouville memory term and an integral boundary condition is still less developed, especially from the viewpoint of existence, uniqueness, and Ulam-type stability. In such problems, two different nonlocal mechanisms appear: one comes from the memory term in the equation, and the other comes from the integral condition at the boundary. Their combination changes the fixed point estimates and requires a careful treatment of the corresponding integral operator.

Motivated by this observation, we consider the problem

$$ {}^{C}D_a^{\alpha}x(t) + p(t)x'(t) + q(t)x(t) + \eta r(t)(I_a^{\rho}x)(t) = g(t) + \lambda f\big(t, x(t)\big), \qquad t \in [a,b], \tag{1}$$

subject to

$$x(a) = x'(a) = 0, \qquad x(b) = \int_a^b w(s)x(s)\,ds, \tag{2}$$

where $2 < \alpha < 3$, $0 < \rho < 1$, and

$$(I_a^{\rho}x)(t) := \frac{1}{\Gamma(\rho)}\int_a^t (t-s)^{\rho-1}x(s)\,ds, \qquad t \in [a,b].$$

The term $r(t)(I_a^{\rho}x)(t)$ represents a Riemann–Liouville type memory effect, while the boundary condition is nonlocal through the integral functional

$$x \mapsto \int_a^b w(s)x(s)\,ds.$$

This structure leads naturally to an equivalent integral formulation and a fixed point approach. The estimates obtained in this framework explicitly show how the memory coefficient and the nonlocal boundary term affect the solvability and stability conditions.

The main contributions of the paper are as follows:

- We derive an equivalent integral equation for (1)–(2).
- We establish existence and uniqueness results for integral solutions by using fixed point methods.
- We obtain explicit estimates showing the influence of the Riemann–Liouville memory term on the solvability conditions.
- We prove Ulam–Hyers and Ulam–Hyers–Rassias type stability results with explicit stability bounds.

The paper is organized as follows. Section 2 contains the basic definitions, estimates, and fixed point results used in the sequel. In Section 3, we derive the integral formulation and prove the existence and uniqueness results. Section 4 is devoted to Ulam-type stability.

2. Preliminaries

In this section, we recall some basic facts from fractional calculus and fixed point theory. We also introduce the notation used throughout the paper.

Let

$$L := b - a.$$

Denote by $C([a,b])$ the Banach space of continuous real-valued functions on $[a,b]$, endowed with the norm

$$\|x\|_\infty := \max_{t\in[a,b]} |x(t)|.$$

We shall mainly work in the space

$$X := C^2([a,b]),$$

with the norm

$$\|x\|_X := \|x\|_\infty + \|x'\|_\infty + \|x''\|_\infty.$$

As usual, $\Gamma(\sigma)$ denotes the Gamma function.

Definition 2.1. Let $\sigma > 0$ and $x \in C([a,b])$. The left Riemann–Liouville fractional integral of order σ is defined by

$$(I_a^\sigma x)(t) := \frac{1}{\Gamma(\sigma)} \int_a^t (t-s)^{\sigma-1} x(s)\, ds, \qquad t \in [a,b]. \tag{3}$$

In particular, for $\rho \in (0,1)$,

$$(I_a^\rho x)(t) = \frac{1}{\Gamma(\rho)} \int_a^t (t-s)^{\rho-1} x(s)\, ds, \qquad t \in [a,b].$$

Definition 2.2. Let $\alpha \in (2,3)$ and $x \in C^3([a,b])$. The left Caputo fractional derivative of order α is defined by

$$({}^C D_a^\alpha x)(t) := \frac{1}{\Gamma(3-\alpha)} \int_a^t (t-s)^{2-\alpha} x^{(3)}(s)\, ds, \qquad t \in [a,b]. \tag{4}$$

We shall use the following standard identity; see [8, 12, 14].

Proposition 2.3. *Let $\alpha \in (2,3)$ and $x \in C^3([a,b])$. Then*

$$(I_a^\alpha {}^C D_a^\alpha x)(t) = x(t) - x(a) - x'(a)(t-a) - \frac{x''(a)}{2}(t-a)^2, \qquad t \in [a,b]. \tag{5}$$

The next estimate will be used repeatedly.

Lemma 2.4. *Let $\sigma > 0$ and $x \in C([a,b])$. Then*

$$\|I_a^\sigma x\|_\infty \le \frac{L^\sigma}{\Gamma(\sigma+1)} \|x\|_\infty. \tag{6}$$

For the memory term, we shall use the following consequence.

Lemma 2.5. *Let $\rho \in (0,1)$. If $x \in X$, then*

$$\|I_a^\rho x\|_\infty \le \frac{L^\rho}{\Gamma(\rho+1)} \|x\|_\infty \le \frac{L^\rho}{\Gamma(\rho+1)} \|x\|_X. \tag{7}$$

We next introduce the weighted memory operator. For $r \in C([a,b])$, define

$$(\mathcal{R}x)(t) := r(t)(I_a^\rho x)(t), \qquad t \in [a,b].$$

Lemma 2.6. *Let* $\rho \in (0,1)$ *and* $r \in C([a,b])$. *Then* $\mathcal{R} : C([a,b]) \to C([a,b])$ *is bounded and linear. Moreover,*

$$\|\mathcal{R}x - \mathcal{R}y\|_\infty \le \|r\|_\infty \frac{L^\rho}{\Gamma(\rho+1)} \|x-y\|_\infty \tag{8}$$

for all $x, y \in C([a,b])$. *In particular, for all* $x, y \in X$,

$$\|\mathcal{R}x - \mathcal{R}y\|_\infty \le \|r\|_\infty \frac{L^\rho}{\Gamma(\rho+1)} \|x-y\|_X. \tag{9}$$

We also define the nonlocal boundary functional

$$\mathcal{B}x := \int_a^b w(s)x(s)\,ds,$$

where $w \in L^1([a,b])$. Thus the boundary condition can be written as

$$x(b) = \mathcal{B}x.$$

Lemma 2.7. *Let* $w \in L^1([a,b])$. *Then* $\mathcal{B}$ *is a bounded linear functional on* $C([a,b])$, *and*

$$|\mathcal{B}x - \mathcal{B}y| \le \|w\|_{L^1} \|x-y\|_\infty \tag{10}$$

for all $x, y \in C([a,b])$. *In particular, for all* $x, y \in X$,

$$|\mathcal{B}x - \mathcal{B}y| \le \|w\|_{L^1} \|x-y\|_X. \tag{11}$$

For later use, define

$$\psi(t) := \left(\frac{t-a}{L}\right)^2, \qquad t \in [a,b].$$

Lemma 2.8. *Let* $\psi(t) = \dfrac{(t-a)^2}{L^2}$. *Then*

$$\|\psi\|_\infty = 1, \qquad \|\psi'\|_\infty = \frac{2}{L}, \qquad \|\psi''\|_\infty = \frac{2}{L^2}. \tag{12}$$

Consequently,

$$\|\psi(\mathcal{B}x - \mathcal{B}y)\|_X \le \left(1 + \frac{2}{L} + \frac{2}{L^2}\right) \|w\|_{L^1} \|x-y\|_X \tag{13}$$

for all $x, y \in X$.

We recall the fixed point results used later.

Theorem 2.9 (Banach contraction principle)**.** *Let* $(Y, \|\cdot\|_Y)$ *be a Banach space and let* $T : Y \to Y$ *be a contraction. Then* T *has a unique fixed point in* Y.

Theorem 2.10 (Krasnoselskii fixed point theorem)**.** *Let* Y *be a Banach space and let* $\mathcal{M} \subset Y$ *be nonempty, closed, bounded, and convex. Assume that* $A, B : \mathcal{M} \to Y$ *satisfy:*

(i) A *is a contraction on* $\mathcal{M}$;

(ii) B *is continuous and compact on* $\mathcal{M}$;

(iii) $Ax + By \in \mathcal{M}$ *for all* $x, y \in \mathcal{M}$.

Then there exists $x \in \mathcal{M}$ *such that* $x = Ax + Bx$.

Compactness of the integral operators considered below will be verified by the Arzelà–Ascoli theorem.

3. Existence and uniqueness results

In this section, we derive the integral equation associated with (1)–(2) and study the corresponding fixed point problem in the space $X = C^2([a,b])$.

For $x \in X$, define

$$H_x(t) := p(t)x'(t) + q(t)x(t) + \eta(\mathcal{R}x)(t) - g(t) - \lambda f\big(t, x(t)\big), \qquad t \in [a,b], \tag{14}$$

where $(\mathcal{R}x)(t) = r(t)(I_a^\rho x)(t)$.

The next result gives the integral form of the problem.

Proposition 3.1. *Let $2 < \alpha < 3$. Every classical solution of (1)–(2) satisfies*

$$\begin{aligned} x(t) = \psi(t)\mathcal{B}x &+ \frac{\psi(t)}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} H_x(s)\, ds \\ &- \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} H_x(s)\, ds, \qquad t \in [a,b]. \end{aligned} \tag{15}$$

Conversely, if $x \in C^3([a,b])$ satisfies (15), then x satisfies (1)–(2).

Proof. Assume that x is a classical solution of (1)–(2). Then ${}^C D_a^\alpha x = -H_x$. Applying I_a^α and using Proposition 2.3, we get

$$x(t) = x(a) + x'(a)(t-a) + \frac{x''(a)}{2}(t-a)^2 - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} H_x(s)\, ds.$$

Since $x(a) = x'(a) = 0$, this becomes

$$x(t) = c_2(t-a)^2 - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} H_x(s)\, ds, \qquad c_2 := \frac{x''(a)}{2}.$$

Evaluating this identity at $t = b$ and using $x(b) = \mathcal{B}x$, we obtain

$$c_2 L^2 - \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} H_x(s)\, ds = \mathcal{B}x.$$

Hence

$$c_2 = \frac{1}{L^2} \left(\mathcal{B}x + \frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} H_x(s)\, ds \right).$$

Substituting this expression into the formula for $x(t)$ gives (15).

Conversely, suppose that $x \in C^3([a,b])$ satisfies (15). Evaluating (15) at $t = a$ and differentiating it once at $t = a$, we obtain $x(a) = x'(a) = 0$. Evaluating (15) at $t = b$ gives $x(b) = \mathcal{B}x$. Finally, applying ${}^C D_a^\alpha$ to (15) gives ${}^C D_a^\alpha x = -H_x$, which is exactly (1). □

Definition 3.2. A function $x \in X$ is called an integral solution of (1)–(2) if it satisfies the integral equation (15) for all $t \in [a,b]$.

Motivated by (15), define $\mathcal{J} : C([a,b]) \to X$ by

$$(\mathcal{J}h)(t) := \frac{\psi(t)}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} h(s)\, ds - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)\, ds. \tag{16}$$

Then the fixed point operator $T : X \to X$ can be written as

$$Tx = \psi\, \mathcal{B}x + \mathcal{J} H_x. \tag{17}$$

Thus, fixed points of T are integral solutions of (1)–(2).

For later use, set

$$\mu := \max\{\|p\|_\infty, \|q\|_\infty\}, \qquad \beta := \|g\|_\infty, \qquad C_\rho := \frac{L^\rho}{\Gamma(\rho+1)},$$

and

$$\mu_\rho := \mu + |\eta| \, \|r\|_\infty C_\rho.$$

We also define

$$M_1 := \frac{L^\alpha}{\Gamma(\alpha+1)} + \frac{L^{\alpha-1}}{\Gamma(\alpha)} + \frac{L^{\alpha-2}}{\Gamma(\alpha-1)}, \tag{18}$$

$$M_2 := \frac{L^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{2}{L} + \frac{2}{L^2}\right), \tag{19}$$

and

$$K_w := \left(1 + \frac{2}{L} + \frac{2}{L^2}\right) \|w\|_{L^1}. \tag{20}$$

Remark 3.3. The constant μ_ρ measures the combined size of the lower-order terms and the Riemann–Liouville memory term. The memory contribution enters through

$$|\eta| \, \|r\|_\infty C_\rho.$$

Thus, larger values of $|\eta|$, $\|r\|_\infty$, or L make the contraction conditions more restrictive. This shows explicitly how the memory term affects the solvability estimates.

Lemma 3.4. *The operator $\mathcal{J} : C([a,b]) \to X$ is linear and satisfies*

$$\|\mathcal{J}h\|_X \le (M_1 + M_2)\|h\|_\infty, \qquad h \in C([a,b]). \tag{21}$$

Proof. Write $\mathcal{J}h = A_h + B_h$, where

$$A_h(t) := \frac{\psi(t)}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} h(s)\, ds, \qquad B_h(t) := -\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)\, ds.$$

For A_h, using Lemma 2.8, we have

$$\|A_h\|_X \le \frac{L^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{2}{L} + \frac{2}{L^2}\right) \|h\|_\infty = M_2 \|h\|_\infty.$$

For B_h, direct differentiation gives

$$B_h'(t) = -\frac{1}{\Gamma(\alpha-1)} \int_a^t (t-s)^{\alpha-2} h(s)\, ds,$$

and

$$B_h''(t) = -\frac{1}{\Gamma(\alpha-2)} \int_a^t (t-s)^{\alpha-3} h(s)\, ds.$$

Hence

$$\|B_h\|_\infty \le \frac{L^\alpha}{\Gamma(\alpha+1)} \|h\|_\infty,$$

$$\|B_h'\|_\infty \le \frac{L^{\alpha-1}}{\Gamma(\alpha)} \|h\|_\infty, \qquad \|B_h''\|_\infty \le \frac{L^{\alpha-2}}{\Gamma(\alpha-1)} \|h\|_\infty.$$

Therefore,

$$\|B_h\|_X \le M_1 \|h\|_\infty.$$

Combining the estimates for A_h and B_h gives (21). □

Lemma 3.5. *The operator $\mathcal{J} : C([a,b]) \to X$ is compact.*

Proof. Let $\mathcal{S}$ be a bounded subset of $C([a,b])$. Then there exists $N > 0$ such that $\|h\|_\infty \le N$ for all $h \in \mathcal{S}$. By Lemma 3.4, $\mathcal{J}(\mathcal{S})$ is bounded in X.

It remains to prove equicontinuity of the families

$$\{\mathcal{J}h : h \in \mathcal{S}\}, \qquad \{(\mathcal{J}h)' : h \in \mathcal{S}\}, \qquad \{(\mathcal{J}h)'' : h \in \mathcal{S}\}.$$

The terms coming from A_h are multiples of ψ, ψ', and ψ'', with uniformly bounded coefficients. Hence they form equicontinuous families. For the terms coming from B_h, we have

$$B_h = -I_a^\alpha h, \qquad B_h' = -I_a^{\alpha-1} h, \qquad B_h'' = -I_a^{\alpha-2} h.$$

Since $\alpha > 2$, the orders α, $\alpha - 1$, and $\alpha - 2$ are all positive. Moreover, for every $\sigma > 0$, the family

$$\{I_a^\sigma h : \ h \in \mathcal{S}\}$$

is uniformly bounded and equicontinuous on $[a,b]$. Indeed, for $a \le t_1 < t_2 \le b$, standard fractional integral estimates give

$$|(I_a^\sigma h)(t_2) - (I_a^\sigma h)(t_1)| \le C_\sigma N |t_2 - t_1|^{\min\{\sigma,1\}},$$

where $C_\sigma > 0$ depends only on σ and L. Thus the three families above are equicontinuous.

By the Arzelà–Ascoli theorem, $\mathcal{J}(\mathcal{S})$ is relatively compact in X. Hence $\mathcal{J}$ is compact. □

Lemma 3.6. *Assume that f is continuous and globally Lipschitz in the second variable, namely*

$$|f(t,u) - f(t,v)| \le L_f |u - v|, \qquad t \in [a,b], \quad u, v \in \mathbb{R}.$$

Then, for all $x, y \in X$,

$$\|H_x - H_y\|_\infty \le \big(\mu_\rho + |\lambda| L_f\big) \|x - y\|_X. \tag{22}$$

Moreover, if $\|x\|_X \le R$, then

$$\|H_x\|_\infty \le \mu_\rho R + \beta + |\lambda| M_f(R), \tag{23}$$

where

$$M_f(R) := \max\{|f(t,u)| : \ t \in [a,b], \ |u| \le R\}.$$

Proof. Using (14), Lemma 2.6, and the Lipschitz condition on f, we obtain

$$\begin{aligned} |H_x(t) - H_y(t)| &\le |p(t)|\,|x'(t) - y'(t)| + |q(t)|\,|x(t) - y(t)| \\ &\quad + |\eta|\,|(\mathcal{R}x)(t) - (\mathcal{R}y)(t)| + |\lambda|\,|f(t,x(t)) - f(t,y(t))| \\ &\le \big(\mu_\rho + |\lambda| L_f\big) \|x - y\|_X. \end{aligned}$$

Taking the supremum over $t \in [a,b]$ gives (22).

If $\|x\|_X \le R$, then $\|x\|_\infty \le R$, and therefore

$$|f(t,x(t))| \le M_f(R).$$

Hence

$$|H_x(t)| \le \mu_\rho R + \beta + |\lambda| M_f(R),$$

which gives (23). □

For $R > 0$, let

$$\mathcal{M}_R := \{x \in X : \ \|x\|_X \le R\}.$$

Define $A, B : \mathcal{M}_R \to X$ by

$$Ax := \psi\,\mathcal{B}x + \mathcal{J}\left(px' + qx + \eta\mathcal{R}x\right), \tag{24}$$

and

$$Bx := \mathcal{J}\left(-g - \lambda f(\cdot, x(\cdot))\right). \tag{25}$$

Then $T = A + B$.

Theorem 3.7 (Existence)**.** *Assume that* $p, q, r, g \in C([a, b])$*,* $w \in L^1([a, b])$*, and* $f \in C([a, b]\times\mathbb{R}, \mathbb{R})$*. Suppose that there exists* $R > 0$ *such that*

$$\Theta_\rho := K_w + \mu_\rho(M_1 + M_2) < 1 \tag{26}$$

and

$$K_w R + (M_1 + M_2)\big(\mu_\rho R + \beta + |\lambda| M_f(R)\big) \le R. \tag{27}$$

Then (1)–(2) *admits at least one integral solution in* $\mathcal{M}_R$*.*

Proof. We verify the assumptions of Theorem 2.10.

Step 1: A is a contraction. Let $x, y \in \mathcal{M}_R$. By Lemmas 2.8, 3.4, and 2.6, we have

$$\|Ax - Ay\|_X \le K_w\|x - y\|_X + \mu_\rho(M_1 + M_2)\|x - y\|_X.$$

Thus

$$\|Ax - Ay\|_X \le \Theta_\rho\|x - y\|_X.$$

Since $\Theta_\rho < 1$, A is a contraction.

Step 2: B is continuous and compact. Continuity follows from the continuity of f and the continuity of $\mathcal{J}$. Let $\{x_n\} \subset \mathcal{M}_R$. Since $|x_n(t)| \le R$, we have

$$|f(t, x_n(t))| \le M_f(R), \qquad t \in [a, b].$$

Therefore,

$$\| - g - \lambda f(\cdot, x_n(\cdot))\|_\infty \le \beta + |\lambda| M_f(R).$$

Thus the inputs of $\mathcal{J}$ form a bounded set in $C([a, b])$. Since $\mathcal{J}$ is compact by Lemma 3.5, $B(\mathcal{M}_R)$ is relatively compact in X. Hence B is compact.

Step 3: invariance of $\mathcal{M}_R$. Let $x, y \in \mathcal{M}_R$. Using (13), Lemma 3.4, and (23), we obtain

$$\begin{aligned}\|Ax + By\|_X &\le K_w R + (M_1 + M_2)\big(\mu_\rho R + \beta + |\lambda| M_f(R)\big)\\ &\le R.\end{aligned}$$

Thus $Ax + By \in \mathcal{M}_R$.

All assumptions of Theorem 2.10 are satisfied. Hence there exists $x \in \mathcal{M}_R$ such that $x = Ax + Bx = Tx$. By Definition 3.2, x is an integral solution of (1)–(2). □

Theorem 3.8 (Uniqueness)**.** *Assume that* $p, q, r, g \in C([a, b])$*,* $w \in L^1([a, b])$*, and* $f \in C([a, b] \times \mathbb{R}, \mathbb{R})$ *is globally Lipschitz in the second variable with Lipschitz constant* L_f*. If*

$$\Lambda_\rho := K_w + \big(\mu_\rho + |\lambda| L_f\big)(M_1 + M_2) < 1, \tag{28}$$

then (1)–(2) *admits a unique integral solution in* X*.*

Proof. Let $x, y \in X$. By (13), Lemma 3.4, and Lemma 3.6, we obtain

$$\begin{aligned}\|Tx - Ty\|_X &\le \|\psi(\mathcal{B}x - \mathcal{B}y)\|_X + \|\mathcal{J}(H_x - H_y)\|_X \\ &\le K_w\|x - y\|_X + (M_1 + M_2)\|H_x - H_y\|_\infty \\ &\le \Lambda_\rho\|x - y\|_X.\end{aligned}$$

Since $\Lambda_\rho < 1$, T is a contraction on X. The conclusion follows from the Banach contraction principle and Definition 3.2. ☐

Corollary 3.9 (Problem without memory term)**.** *Assume that* $\eta = 0$*. Then* $\mu_\rho = \mu$*. In this case, the existence condition becomes*

$$K_w + \mu(M_1 + M_2) < 1,$$

while the uniqueness condition becomes

$$K_w + \big(\mu + |\lambda|L_f\big)(M_1 + M_2) < 1.$$

Thus the above results reduce to the corresponding criteria for Caputo fractional boundary value problems without the Riemann–Liouville memory term.

Theorem 3.10 (Continuous dependence on the source term)**.** *Assume that the hypotheses of Theorem 3.8 hold. Let* x_1 *and* x_2 *be the unique integral solutions corresponding to* $g_1, g_2 \in C([a, b])$*, respectively, while all other data are fixed. Then*

$$\|x_1 - x_2\|_X \le \frac{M_1 + M_2}{1 - \Lambda_\rho}\|g_1 - g_2\|_\infty.$$

Proof. Let T_1 and T_2 be the fixed point operators corresponding to g_1 and g_2, respectively. Since $x_i = T_i x_i$, $i = 1, 2$, we have

$$\|x_1 - x_2\|_X \le \|T_1x_1 - T_1x_2\|_X + \|T_1x_2 - T_2x_2\|_X.$$

The first term is bounded by $\Lambda_\rho\|x_1 - x_2\|_X$. For the second term, Lemma 3.4 gives

$$\|T_1x_2 - T_2x_2\|_X \le (M_1 + M_2)\|g_1 - g_2\|_\infty.$$

Therefore,

$$(1 - \Lambda_\rho)\|x_1 - x_2\|_X \le (M_1 + M_2)\|g_1 - g_2\|_\infty,$$

which proves the estimate. ☐

4. Ulam–Hyers and Ulam–Hyers–Rassias stability

In this section, we study Ulam–Hyers and Ulam–Hyers–Rassias type stability for the integral problem associated with (1)–(2). Throughout this section, $T : X \to X$ denotes the fixed point operator defined in (17).

For an admissible function $y \in C^3([a, b])$, define the residual

$$\mathcal{E}_y(t) := {}^C D_a^\alpha y(t) + p(t)y'(t) + q(t)y(t) + \eta r(t)(I_a^\rho y)(t) - g(t) - \lambda f(t, y(t)), \qquad t \in [a, b]. \tag{29}$$

Definition 4.1 (Ulam–Hyers stability)**.** The problem (1)–(2) is said to be *Ulam–Hyers stable* if there exists a constant $k > 0$ such that, for every $\varepsilon > 0$ and every $y \in C^3([a, b])$ satisfying

$$|\mathcal{E}_y(t)| \le \varepsilon, \qquad t \in [a, b], \tag{30}$$

together with

$$y(a) = y'(a) = 0, \qquad y(b) = \mathcal{B}y, \tag{31}$$

there exists an integral solution $x \in X$ of (1)–(2) such that

$$\|y - x\|_X \le k\varepsilon.$$

Definition 4.2 (Ulam–Hyers–Rassias type stability)**.** Let $\varphi \in C([a,b],\mathbb{R}_+)$ with $\varphi(t) > 0$ on $[a,b]$. The problem (1)–(2) is said to be *Ulam–Hyers–Rassias type stable with respect to* φ if there exists a constant $k_\varphi > 0$ such that, for every $\varepsilon > 0$ and every $y \in C^3([a,b])$ satisfying

$$|\mathcal{E}_y(t)| \le \varepsilon\varphi(t), \qquad t \in [a,b], \tag{32}$$

together with (31), there exists an integral solution $x \in X$ of (1)–(2) such that

$$\|y - x\|_X \le k_\varphi \varepsilon.$$

Lemma 4.3. *Let* $y \in C^3([a,b])$ *satisfy* (30) *and* (31)*. Then*

$$(Ty)(t) - y(t) = \frac{\psi(t)}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1}\mathcal{E}_y(s)\,ds - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\mathcal{E}_y(s)\,ds, \tag{33}$$

and

$$\|Ty - y\|_X \le (M_1 + M_2)\varepsilon. \tag{34}$$

Proof. Set $h(t) := \mathcal{E}_y(t)$. Then $h \in C([a,b])$ and $|h(t)| \le \varepsilon$ on $[a,b]$. By the definition of H_y, we have

$${}^C D_a^\alpha y(t) = -H_y(t) + h(t).$$

Applying I_a^α and using Proposition 2.3, we obtain

$$y(t) = c_2(t-a)^2 - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} H_y(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)\,ds,$$

where $c_2 = y''(a)/2$. Using the boundary condition $y(b) = \mathcal{B}y$, we determine c_2. Comparing the resulting expression with the definition of Ty, we get

$$(Ty)(t) - y(t) = \frac{\psi(t)}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} h(s)\,ds - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)\,ds.$$

This gives (33).

Since $|h(t)| \le \varepsilon$, Lemma 3.4 gives

$$\|Ty - y\|_X = \|\mathcal{J}h\|_X \le (M_1 + M_2)\varepsilon.$$

This proves (34). □

Theorem 4.4 (Ulam–Hyers stability)**.** *Assume that the hypotheses of Theorem 3.8 hold and that* $\Lambda_\rho < 1$*. Then* (1)*–*(2) *is Ulam–Hyers stable. More precisely, if* $y \in C^3([a,b])$ *satisfies* (30) *and* (31)*, and if* $x \in X$ *is the unique integral solution of* (1)*–*(2)*, then*

$$\|y - x\|_X \le \frac{M_1 + M_2}{1 - \Lambda_\rho}\,\varepsilon. \tag{35}$$

Proof. Since x is the unique fixed point of T, we have $x = Tx$. Hence

$$\|x - y\|_X = \|Tx - y\|_X \le \|Tx - Ty\|_X + \|Ty - y\|_X.$$

By Theorem 3.8, T is a contraction with constant Λ_ρ. Therefore,

$$\|x - y\|_X \le \Lambda_\rho \|x - y\|_X + \|Ty - y\|_X.$$

It follows that

$$\|x - y\|_X \le \frac{1}{1 - \Lambda_\rho} \|Ty - y\|_X.$$

Using Lemma 4.3, we obtain (35). □

For the weighted perturbation, we use the following assumption.

Assumption 4.5. Let $\varphi \in C([a,b],\mathbb{R}_+)$ with $\varphi(t) > 0$ on $[a,b]$. Assume that there exists a constant $C_\varphi > 0$ such that

$$(I_a^\tau \varphi)(b) \le C_\varphi, \qquad \tau \in \{\alpha, \alpha-1, \alpha-2\}. \tag{36}$$

Lemma 4.6. *Let $y \in C^3([a,b])$ satisfy (32) and (31). Suppose that Assumption 4.5 holds. Then*

$$\|Ty - y\|_X \le \widehat{C}_\varphi \varepsilon, \tag{37}$$

where

$$\widehat{C}_\varphi := C_\varphi \left(4 + \frac{2}{L} + \frac{2}{L^2}\right). \tag{38}$$

Proof. Set $h(t) := \mathcal{E}_y(t)$. Then

$$|h(t)| \le \varepsilon\varphi(t), \qquad t \in [a,b].$$

As in the proof of Lemma 4.3, we have

$$(Ty)(t) - y(t) = \frac{\psi(t)}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} h(s)\, ds - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)\, ds.$$

Write $Ty - y = A + B$, where

$$A(t) := \frac{\psi(t)}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} h(s)\, ds,$$

and

$$B(t) := -\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)\, ds.$$

For the first term, Assumption 4.5 gives

$$\frac{1}{\Gamma(\alpha)} \left| \int_a^b (b-s)^{\alpha-1} h(s)\, ds \right| \le \varepsilon (I_a^\alpha \varphi)(b) \le \varepsilon C_\varphi.$$

Therefore, using Lemma 2.8,

$$\|A\|_X \le \left(1 + \frac{2}{L} + \frac{2}{L^2}\right) C_\varphi \varepsilon.$$

For the second term, we estimate B, B', and B''. We have

$$\|B\|_\infty \le \varepsilon (I_a^\alpha \varphi)(b) \le \varepsilon C_\varphi,$$

$$\|B'\|_\infty \le \varepsilon (I_a^{\alpha-1} \varphi)(b) \le \varepsilon C_\varphi,$$

and

$$\|B''\|_\infty \le \varepsilon (I_a^{\alpha-2} \varphi)(b) \le \varepsilon C_\varphi.$$

Thus

$$\|B\|_X \le 3 C_\varphi \varepsilon.$$

Combining the estimates for A and B, we get

$$\|Ty - y\|_X \le C_\varphi \left(4 + \frac{2}{L} + \frac{2}{L^2}\right) \varepsilon = \widehat{C}_\varphi \varepsilon.$$

This proves (37). ☐

Theorem 4.7 (Ulam–Hyers–Rassias type stability)**.** *Assume that the hypotheses of Theorem 3.8 hold and that* $\Lambda_\rho < 1$*. Let* φ *satisfy Assumption 4.5. Then* (1)*–*(2) *is Ulam–Hyers–Rassias type stable with respect to* φ*. More precisely, if* $y \in C^3([a, b])$ *satisfies* (32) *and* (31)*, and if* $x \in X$ *is the unique integral solution of* (1)*–*(2)*, then*

$$\|y - x\|_X \leq \frac{\widehat{C}_\varphi}{1 - \Lambda_\rho}\,\varepsilon. \tag{39}$$

Proof. Since $x = Tx$, we have

$$\|x - y\|_X \leq \|Tx - Ty\|_X + \|Ty - y\|_X.$$

Using the contraction property of T, we get

$$\|x - y\|_X \leq \Lambda_\rho \|x - y\|_X + \|Ty - y\|_X.$$

Therefore,

$$\|x - y\|_X \leq \frac{1}{1 - \Lambda_\rho}\|Ty - y\|_X.$$

Applying Lemma 4.6, we obtain

$$\|x - y\|_X \leq \frac{\widehat{C}_\varphi}{1 - \Lambda_\rho}\varepsilon.$$

This proves (39). □

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