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## Some remarks and examples on interpolative metric spaces

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### Abstract

In an article published in 2024, the first named author introduced the concept of an interpolative metric space. These spaces constitute an appealing subclass of the class of  $b$ -metric spaces. Indeed, we recently stated that, in contrast to the  $b$ -metric case, any interpolative metric is a continuous function and "open" balls are open sets in the induced topology. In this note, we also observe that for any interpolative metric space, "closed" balls are closed sets in the induced topology, and show that some essential quasi-normed spaces can be structured as interpolative metric spaces.

*Keywords:* Interpolative metric space;  $b$ -metric space; quasi-normed space.

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### 1. Introduction and preliminaries

Throughout this paper, the sets of real numbers, non-negative real numbers and natural numbers will be denoted by  $\mathbb{R}$ ,  $\mathbb{R}^+$  and  $\mathbb{N}$ , respectively. All topological spaces will assumed to be Hausdorff.

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Following [8, 9, 18], by a quasi-norm on a (real) vector space  $X$  we mean a function  $\|\cdot\| : X \rightarrow \mathbb{R}^+$  that fulfills the following conditions for all  $x, y \in X$  and  $\alpha \in \mathbb{R}$ :

- (qn0)  $\|x\| = 0$  if and only if  $x = \theta$ ;
- (qn1)  $\|\alpha x\| = |\alpha| \|x\|$ ;
- (qn2) there is a constant  $K \geq 1$  such that  $\|x + y\| \leq K[\|x\| + \|y\|]$ .

A quasi-normed space is a pair  $(X, \|\cdot\|)$  such that  $X$  is a (real) vector space and  $\|\cdot\|$  is a quasi-norm on  $X$ . When  $K = 1$  we have the notions of a norm and a normed space, respectively.

Quasi-norms provide a fine structure to characterize those topological vector spaces that are locally bounded. More precisely (see, e.g., [3, 8, 9, 15]), a topological vector space  $(X, \tau)$  is locally bounded if and only if it can be endowed with a quasi-norm whose induced topology agrees with  $\tau$ . Hence, classical spaces as  $l_p(\mathbb{R})$  and  $L_p([0, 1])$ ,  $0 < p < 1$ , yield fundamental examples of non-normable spaces that admit a structure of quasi-normed space (see Section 2, for details).

The expected and reasonable metric generalization of the concepts of quasi-norm and quasi-normed space can be found in [4, 5] under the names of a  $b$ -metric and  $b$ -metric space respectively, and in [1, 7, 17] under the name of a quasi-metric and a quasi-metric space, respectively.

A  $b$ -metric on a set  $X$  is a function  $d : X \times X \rightarrow \mathbb{R}^+$  that fulfills the following conditions for all  $x, y, z \in X$ :

- (bm0)  $d(x, y) = 0$  if and only if  $x = y$ ;
- (bm1)  $d(x, y) = d(y, x)$ ;
- (bm2) there is a constant  $K \geq 1$  such that  $d(x, y) \leq K[d(x, z) + d(z, y)]$ .

A  $b$ -metric space is a pair  $(X, d)$  such that  $X$  is a set and  $d$  is a  $b$ -metric on  $X$ . When  $K = 1$  we have the notions of a metric and a metric space, respectively.

As we might have guessed, a quasi-norm  $\|\cdot\|$ , with constant  $K$ , on a vector space  $X$  induces a  $b$ -metric  $d_{\|\cdot\|}$ , with constant  $K$ , on  $X$ , defined as  $d_{\|\cdot\|}(x, y) = \|y - x\|$ , for all  $x, y \in X$ .

As in the standard metric scenario, each  $b$ -metric  $d$  on a set  $X$  induces a topology  $\tau_d$  given as follows:

$$\tau_d = \{\emptyset\} \cup \{A \subseteq X : \text{for each } x \in A \text{ there is } r > 0 \text{ such that } B_d(x, r) \subseteq A\},$$

where  $B_d(x, r) = \{y \in X : d(x, y) < r\}$  is called the “open” ball of center  $x$  and radius  $r > 0$ .

The “closed” ball  $B_d[x, r]$  of center  $x$  and radius  $r > 0$  is defined by  $B_d[x, r] = \{y \in X : d(x, y) \leq r\}$ , for all  $x \in X$  and  $r > 0$

Although the topology  $\tau_d$  is metrizable ([1, 7, 16, 17]), we emphasize that, contrarily to the case of standard metric spaces, there exist  $b$ -metric spaces  $(X, d)$  with “closed” balls  $B_d[x, r]$  that are not  $\tau_d$ -closed sets ([2]), “open balls”  $B_d(x, r)$  that are not  $\tau_d$ -open sets and, hence,  $d$  is not continuous ([2, 17]). Recall that, in this context,  $d$  is continuous if  $d(x_n, y_n) \rightarrow d(x, y)$  whenever  $d(x, x_n) \rightarrow 0$  and  $d(y, y_n) \rightarrow 0$ .

In [10], it was introduced the notion of interpolative metric space as a generalization of standard metric spaces for getting a “better” results. This structure was further developed in [6, 11, 12, 13].

Let  $X$  be a (non-empty) set and  $\alpha \in (0, 1)$  and  $c \geq 0$  constants. Then, a function  $d : X \times X \rightarrow \mathbb{R}^+$  is called an  $(\alpha, c)$ -interpolative metric on  $X$  if it fulfills the following conditions for all  $x, y, z \in X$  :

- (im0)  $d(x, y) = 0$  if and only if  $x = y$ ;
- (im1)  $d(x, y) = d(y, x)$ ;
- (im2)  $d(x, y) \leq d(x, z) + d(z, y) + c[d(x, z)^\alpha d(z, y)^{1-\alpha}]$ .

An  $(\alpha, c)$ -interpolative metric space is a pair  $(X, d)$  such that  $X$  is a (non-empty) set and  $d$  is an  $(\alpha, c)$ -interpolative metric on  $X$ .

Since an  $(\alpha, c)$ -interpolative metric space  $(X, d)$  is a  $b$ -metric space with constant  $K = 1 + c$  ([13]), we deduce that it can be endowed with the (metrizable) topology  $\tau_d$  induced by its  $b$ -metric structure. Furthermore, the “open” balls are  $\tau_d$ -open sets and  $d$  is continuous ([13]).

The main purpose of this note is to complement [13]. We observe that, for every interpolative metric space, “closed” balls are closed sets in the induced topology, and show that some distinguished quasi-normed spaces, namely  $l_{1/2}(\mathbb{R})$  and  $L_{1/2}[0, 1]$ , can be structured as  $(1/2, 2)$ -interpolative metric spaces.

## 2. Remarks and examples

In [13, Proposition 2.4] we stated that each “open” ball in an  $(\alpha, c)$ -interpolative metric space  $(X, d)$  is  $\tau_d$ -open, and with its help we deduced that  $d$  is continuous ([13, Proposition 2.6]). Although the statement of [13, Proposition 2.4] is correct, there is some annoying typos on line 4 of its proof, so the proof should be as follows:

Let  $x \in X$ ,  $r > 0$  and  $y \in B(x, r)$ . Then  $d(x, y) = s < r$ . Choose  $M > 1$  (sufficiently large) verifying  $cr < (M^{1-\alpha} - 1)(r - s)$ . We show that  $B(y, \delta) \subseteq B(x, r)$ , where  $\delta = \frac{r-s}{M}$ .

Indeed, if  $z \in B(y, \delta)$ , we have

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) + c[d(x, y)^\alpha d(y, z)^{1-\alpha}] \\ &< s + \frac{r-s}{M} + c[r^\alpha \frac{r^{1-\alpha}}{M^{1-\alpha}}] \\ &< s + \frac{r-s+cr}{M^{1-\alpha}} < r. \end{aligned}$$

We also have the following property.

**Proposition 2.1.** *Let  $(X, d)$  be an  $(\alpha, c)$ -interpolative metric space. Then, each “closed” ball is a  $\tau_d$ -closed subset of  $X$ .*

*Proof.* Let  $B_d[x, r]$  be a “closed” ball in  $(X, d)$ , and let  $y \in B_d[x, r]$ . For each  $\varepsilon > 0$  there is  $y_\varepsilon \in B_d(y, \varepsilon) \cap B_d[x, r]$ . Therefore

$$d(x, y) \leq d(x, y_\varepsilon) + d(y_\varepsilon, y) + c[d(x, y_\varepsilon)^\alpha d(y_\varepsilon, y)^{1-\alpha}] < r + \varepsilon + cr^\alpha \varepsilon^{1-\alpha},$$

for all  $\varepsilon > 0$ . Since  $\varepsilon$  is arbitrary we deduce that  $d(x, y) \leq r$ . Hence,  $B_d[x, r]$  is a  $\tau_d$ -closed subset of  $X$ .  $\square$

Fix  $p \in (0, 1)$ . Let

$$l_p(\mathbb{R}) := \{x = \{x_n\}_{n \in \mathbb{N}} : \sum_{n=1}^\infty |x_n|^p < \infty\},$$

with  $x_n \in \mathbb{R}$  for all  $n \in \mathbb{N}$ . It is well known that  $(l_p(\mathbb{R}), \|\cdot\|_{l,p})$  is a quasi-normed space, with constant  $K = 2^{1/p}$ , where

$$\|x\|_{l,p} = \left( \sum_{n=1}^\infty |x_n|^p \right)^{1/p},$$

for all  $x = \{x_n\}_{n \in \mathbb{N}} \in l_p(\mathbb{R})$ . Hence,  $(l_p(\mathbb{R}), d_{\|\cdot\|_{l,p}})$  is a  $b$ -metric space, with constant  $K = 2^{1/p}$  (see, e.g., [14]). For  $p = 1/2$ , we have the following.

**Example 2.2.** Consider the quasi-normed space  $(l_{1/2}(\mathbb{R}), \|\cdot\|_{l,1/2})$ . Next, we show that the  $b$ -metric space  $(l_{1/2}(\mathbb{R}), d_{\|\cdot\|_{l,1/2}})$  is an  $(1/2, 2)$ -interpolative metric space. Indeed, it suffices to show that for each  $x, y, z \in l_{1/2}(\mathbb{R})$  condition (im2) is satisfied. We have

$$|x_n - y_n|^{1/2} \leq |x_n - z_n|^{1/2} + |z_n - y_n|^{1/2},$$

for all  $n \in \mathbb{N}$ . Hence

$$\sum_{n=1}^\infty |x_n - y_n|^{1/2} \leq \sum_{n=1}^\infty |x_n - z_n|^{1/2} + \sum_{n=1}^\infty |z_n - y_n|^{1/2},$$

and consequently

$$\begin{aligned} d_{\|\cdot\|_{l,1/2}}(x, y) &= \|x - y\|_{l,1/2} = \left( \sum_{n=1}^{\infty} |x_n - y_n|^{1/2} \right)^2 \\ &\leq \left( \sum_{n=1}^{\infty} |x_n - z_n|^{1/2} + \sum_{n=1}^{\infty} |z_n - y_n|^{1/2} \right)^2 \\ &= \|x - z\|_{l,1/2} + \|z - y\|_{l,1/2} + 2 \left[ (\|x - z\|_{l,1/2})^{1/2} (\|z - y\|_{l,1/2})^{1/2} \right] \\ &= d_{\|\cdot\|_{l,1/2}}(x, z) + d_{\|\cdot\|_{l,1/2}}(z, y) + 2 \left[ d_{\|\cdot\|_{l,1/2}}(x, z)^{1/2} d_{\|\cdot\|_{l,1/2}}(z, y)^{1/2} \right]. \end{aligned}$$

Once again, fix  $p \in (0, 1)$ . Let  $L_p[0, 1]$  be the real vector space of all functions  $f : [0, 1] \rightarrow \mathbb{R}$  that are measurable and verify  $\int_0^1 |f(x)|^p dx < \infty$ , with functions equal almost everywhere identified. It is well known that  $(L_p[0, 1], \|\cdot\|_{L,p})$  is a quasi-normed space, with constant  $K = 2^{1/p}$ , where

$$\|f\|_{L,p} = \left( \int_0^1 |f(x)|^p dx \right)^{1/p},$$

for all  $f \in L_p[0, 1]$ . Hence,  $(L_p[0, 1], d_{\|\cdot\|_{L,p}})$  is a  $b$ -metric space, with constant  $K = 2^{1/p}$ . For  $p = 1/2$ , we have the following.

**Example 2.3.** Consider the quasi-normed space  $(L_{1/2}[0, 1], \|\cdot\|_{L,1/2})$ . Next, we show that the  $b$ -metric space  $(L_{1/2}[0, 1], d_{\|\cdot\|_{L,1/2}})$  is an  $(1/2, 2)$ -interpolative metric space. Indeed, as in Example 2.2 above, it suffices to show that for each  $f, g, h \in L_{1/2}[0, 1]$  condition (im2) is satisfied. We have

$$|f(x) - g(x)|^{1/2} \leq |f(x) - h(x)|^{1/2} + |h(x) - g(x)|^{1/2},$$

for all  $x \in [0, 1]$ . Hence

$$\int_0^1 |f(x) - g(x)|^{1/2} dx \leq \int_0^1 |f(x) - h(x)|^{1/2} dx + \int_0^1 |h(x) - g(x)|^{1/2} dx,$$

and consequently

$$\begin{aligned} d_{\|\cdot\|_{L,1/2}}(f, g) &= \|f - g\|_{L,1/2} = \left( \int_0^1 |f(x) - g(x)|^{1/2} dx \right)^2 \\ &\leq \left( \int_0^1 |f(x) - h(x)|^{1/2} dx + \int_0^1 |h(x) - g(x)|^{1/2} dx \right)^2 \\ &= \|f - h\|_{L,1/2} + \|h - g\|_{L,1/2} + 2 \left[ (\|f - h\|_{L,1/2})^{1/2} (\|h - g\|_{L,1/2})^{1/2} \right] \\ &= d_{\|\cdot\|_{L,1/2}}(f, h) + d_{\|\cdot\|_{L,1/2}}(h, g) + 2 \left[ d_{\|\cdot\|_{L,1/2}}(f, h)^{1/2} d_{\|\cdot\|_{L,1/2}}(h, g)^{1/2} \right]. \end{aligned}$$

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