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On random fixed point theorems with application to boundary value problem

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Abstract

We establish relation-theoretic random fixed point results for random operators acting on several classes of metric spaces: separable complete metric spaces, Hausdorff L-spaces, and generalized metric spaces with vector-valued distances. The key idea is to impose contraction assumptions on pairs that stand in a given binary relation, and to express measurable conditions explicitly using measurable-space structure and measurable Picard iterates. Our main contributions are threefold. First, we provide a corrected fixed-point argument that avoids circular limit reasoning. Second, we prove uniqueness using an R-directed comparison construction that requires no transitivity assumption on the relation. Third, we formulate a nonnegative measurable matrix-valued contraction principle, grounded in the induced infinity norm. We then apply these abstract results to a coupled random nonlocal boundary value problem. The application is a random integral operator on $C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ and the boundary terms are Riemann—Stieltjes functionals of BV. We include an explicit contraction matrix and a numerical example verifying that the norm condition holds.

Keywords: binary relation, random fixed point, L-space, matrix-valued contraction, random operator, nonlocal boundary value problem

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1. Introduction

The Banach contraction principle is one of the fundamental tools of nonlinear analysis. The Picard iteration is given by a sequence that converges to a unique fixed point of the Picard mapping, which is a strict contraction mapping in a complete metric space. The principle plays an important part in fixed point theory, but is also fundamental in the theory of approximation procedures in numerical analysis, differential equations, functional analysis, and operator equations. It is useful because it provides both a theory of existence and a method for computing it, in the form of an algorithm. It is usefull because it gives not only a theory of existence but also a method of computing, in the form of an algorithm. The Banach contraction principle is essential in both mathematics and applied areas of the world and is used in various ways.

Fixed point theory has subsequently been extended in many directions, including nonlinear contractions, ordered metric spaces, and metric spaces endowed with graphs or arbitrary binary relations. In the relation-theoretic framework, a key innovation is that the contractive condition need only hold for pairs of points that are related by the specified binary relation, rather than for all pairs of points in the space. This more flexible approach was systematically formulated by Alam and Imdad [1], who developed a relation-theoretic contraction principle that generalizes the classical partial-order setting.

Random fixed point theory is needed if the parameters of an operator equation are dependent on an external outcome. Measurable fixed-point approaches to random operators were established by classical works of Hans [6, 7], Hans and Spacek [**?**], Mukherjea [13, 14] and others. More recent works continue to study random approximation, random differential equations, random normed modules, and related probabilistic operator problems [2, 3, 4, 5, 8, 9, 10, 11, 12, 16, 17, 18].

Nieto, Ouahab, and Rodríguez-López [15] established random fixed point results in partially ordered metric spaces and applied them to random differential equations with boundary conditions. The present work replaces the partial-order requirement by a general nonempty binary relation and develops corresponding results in metric, L-space, and vector-valued metric settings.

In this work, we first prove existence of random fixed points for relation-contractive random operators in separable complete metric spaces and use a limit argument that is explicitly compatible with measurability. Second, we give a direct uniqueness proof based on R-directedness and K-closedness; this proof does not require transitivity of the relation. Third, we formulate a matrix-valued random contraction theorem in which componentwise distances are controlled by a nonnegative measurable matrix whose induced infinity norm is strictly below one. This formulation is particularly suitable for coupled systems because each matrix entry describes the influence of one component on another.

Fourth we look at fixed points in Hausdorff L-spaces. An L-space captures the idea of convergence but does not require a metric to generate that convergence. We make clear the compatibility assumption needed for uniqueness and give examples to illustrate the concepts. Finally we use the matrix-valued theorem, on a coupled nonlocal boundary value problem. This application includes a measurability argument, correct Riemann–Stieltjes assumptions, an explicit contraction matrix and a numerical example that checks the norm condition.

2. Preliminaries

Throughout, $(\Gamma, \mathcal{F})$ denotes a measurable space. Whenever A is a metric space, $\mathcal{B}(A)$ denotes its Borel σ-algebra. If a probabilistic formulation is desired, one may equip $(\Gamma, \mathcal{F})$ with a probability measure $\mathbb{P}$; the fixed-point arguments below use only measurability unless an almost-sure formulation is explicitly stated.

Definition 2.1. Let A be a nonempty set and let $K : A \to A$. We write $K^0 = I_A$, where I_A is the identity mapping on A, $K^1 = K$, and

$$K^m = K \circ K^{m-1}, \qquad m \geq 2.$$

A binary relation R on A is a nonempty subset of $A \times A$. We write $\xi R \eta$ when $(\xi, \eta) \in R$ and $\xi \not R \eta$ when $(\xi, \eta) \notin R$. No symmetry, antisymmetry, or transitivity is assumed unless stated explicitly. Two elements $\xi, \eta \in A$ are called R-comparative if $\xi R \eta$ or $\eta R \xi$. If the notation $[\xi, \eta] \in R$ is used, it is only shorthand for this comparability statement. The universal relation $A \times A$ and the equality relation

$$\Delta_A = \{(\xi, \xi) : \xi \in A\}$$

are basic examples. Throughout this paper, R denotes a nonempty binary relation.

Definition 2.2. Let $(A, \mathcal{B}(A))$ be a measurable metric space. An A-valued random variable is an $\mathcal{F}/\mathcal{B}(A)$-measurable mapping $\xi : \Gamma \to A$.

Definition 2.3. A mapping $K : \Gamma \times A \to A$ is a random operator if $\tau \mapsto K(\tau, \xi)$ is measurable for every $\xi \in A$. It is *jointly measurable* if K is measurable from $(\Gamma \times A, \mathcal{F} \otimes \mathcal{B}(A))$ to $(A, \mathcal{B}(A))$. A continuous random operator is a random operator for which $\xi \mapsto K(\tau, \xi)$ is continuous for every $\tau \in \Gamma$.

Definition 2.4. A random fixed point of K is a measurable mapping $\xi : \Gamma \to A$ such that

$$\xi(\tau) = K(\tau, \xi(\tau)), \qquad \tau \in \Gamma.$$

Relation-theoretic terminology. Let R be a binary relation on A and let $K : \Gamma \times A \to A$.

- A sequence $\{\xi_m\}$ in A is R-preserving if $\xi_m R \xi_{m+1}$ for every m.
- The relation R is K-closed if

$$\xi R \eta \quad \Longrightarrow \quad K(\tau, \xi) R K(\tau, \eta), \qquad \tau \in \Gamma.$$

- A subset $E \subseteq A$ is R-directed if for every $\xi, \eta \in E$ there exists $\zeta \in A$ such that $\xi R \zeta$ and $\eta R \zeta$.

For example, if $R = A \times A$, every nonempty subset of A is R-directed. If A is an ordered vector space and $\xi R \eta$ means $\xi \leq \eta$, then any subset in which every pair has a common upper bound is R-directed.

Theorem 2.5. [15] *(Carathéodory joint measurability). Let A be a separable metric space and B a metric space. If $K : \Gamma \times A \to B$ is measurable in τ for each fixed $\xi \in A$ and continuous in ξ for each fixed $\tau \in \Gamma$, then K is jointly measurable with respect to $\mathcal{F} \otimes \mathcal{B}(A)$ and $\mathcal{B}(B)$.*

Lemma 2.6. [15, Lemma 3.1] *(Carathéodory function-space measurability) Let X and Y be locally compact metric spaces and let $f : \Gamma \times X \to Y$. Endow $C(X, Y)$ with the compact-open topology. Then f is a Carathéodory mapping if and only if the induced mapping*

$$\tau \longmapsto f(\tau, \cdot)$$

is measurable from Γ into $C(X, Y)$. This is the precise function-space measurability result used in Section 5 with $X = [0, 1]$ and $Y = \mathbb{R}$.

Definition 2.7. A nonempty subset $M \subseteq A$ is K-invariant if

$$K(\tau, M) \subseteq M, \qquad \tau \in \Gamma.$$

Let $s(A)$ denote the set of all sequences in A.

Definition 2.8. An L-space is a triple $(A, c(A), \lim)$, where $c(A) \subseteq s(A)$ is a family of sequences and $\lim : c(A) \to A$ satisfies:

(i) every constant sequence $\xi_m = \xi$ belongs to $c(A)$ and converges to ξ;
(ii) every subsequence of a sequence converging to ξ also converges to ξ.

We write $\xi_m \to \xi$ when $\{\xi_m\} \in c(A)$ has limit ξ. An L-space is called *Hausdorff* if a convergent sequence has at most one limit.

Definition 2.9. An L-space $(A, \to)$ is equipped compatibly with a relation R if

$$\xi_m \to \xi, \qquad \eta_m \to \eta, \qquad \xi_m R \eta_m \text{ for every } m \quad \Longrightarrow \quad \xi R \eta.$$

Every metric space is a Hausdorff L-space when $\to$ is ordinary metric convergence.

Definition 2.10. Let $M_{n\times 1}(\mathbb{R}_+)$ denote the set of $n \times 1$ column matrices with nonnegative real entries. A vector-valued metric is a mapping

$$\sigma : A \times A \to M_{n\times 1}(\mathbb{R}_+)$$

satisfying, for all $\xi, \eta, \zeta \in A$,

$$\sigma(\xi, \eta) = 0 \iff \xi = \eta, \qquad \sigma(\xi, \eta) = \sigma(\eta, \xi),$$

and

$$\sigma(\xi, \zeta) \le \sigma(\xi, \eta) + \sigma(\eta, \zeta)$$

componentwise. We call (A, σ) complete if every sequence $\{\xi_m\}$ satisfying

$$\|\sigma(\xi_m, \xi_k)\|_\infty \to 0 \quad (m, k \to \infty)$$

converges to some $\xi \in A$ in the sense that

$$\|\sigma(\xi_m, \xi)\|_\infty \to 0.$$

The scalarization

$$d_\sigma(\xi, \eta) = \|\sigma(\xi, \eta)\|_\infty$$

is then a metric.

Definition 2.11. A random nonnegative matrix is an entrywise measurable mapping

$$P : \Gamma \to M_{n\times n}(\mathbb{R}_+).$$

Throughout the matrix-valued results we use the induced infinity norm

$$\|P\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |p_{ij}|.$$

Since P has nonnegative entries, multiplication by P preserves $M_{n\times 1}(\mathbb{R}_+)$, and all vector inequalities below are understood componentwise.

3. Random Fixed Point Theorems

Theorem 3.1. *Suppose $(\Gamma, \mathcal{F})$ is a measurable space and (A, σ) is a separable complete metric space endowed with a binary relation R. Suppose $K : \Gamma \times A \to A$ is jointly measurable and satisfies:*

- *(i) $K(\tau, \cdot)$ is continuous for every $\tau \in \Gamma$.*
- *(ii) R is K-closed.*
- *(iii) There is a measurable function $w : \Gamma \to [0, 1)$ such that*

$$\sigma(K(\tau, \xi_1), K(\tau, \xi_2)) \le w(\tau)\sigma(\xi_1, \xi_2)$$

for all $\xi_1, \xi_2 \in A$ with $\xi_1 R \xi_2$.

(iv) There exists a random variable $\xi_0 : \Gamma \to A$ such that

$$\xi_0(\tau) R K(\tau, \xi_0(\tau)), \qquad \tau \in \Gamma.$$

Then K has a random fixed point.

Proof. Define the Picard sequence by

$$\xi_1(\tau) = K(\tau, \xi_0(\tau)), \qquad \xi_{m+1}(\tau) = K(\tau, \xi_m(\tau)), \quad m \geq 1.$$

Because K is jointly measurable and ξ_0 is measurable, every ξ_m is measurable by induction. The initial relation $\xi_0(\tau) R \xi_1(\tau)$ and K-closedness imply inductively that

$$\xi_m(\tau) R \xi_{m+1}(\tau), \qquad m \geq 0.$$

Therefore the contraction is applicable at every consecutive pair. For $m \geq 0$,

$$\begin{aligned} \sigma(\xi_{m+1}(\tau), \xi_{m+2}(\tau)) &= \sigma(K(\tau, \xi_m(\tau)), K(\tau, \xi_{m+1}(\tau))) \\ &\leq w(\tau)\sigma(\xi_m(\tau), \xi_{m+1}(\tau)) \\ &\leq w(\tau)^{m+1}\sigma(\xi_0(\tau), \xi_1(\tau)). \end{aligned}$$

For $p \geq 1$, the triangle inequality gives

$$\begin{aligned} \sigma(\xi_m(\tau), \xi_{m+p}(\tau)) &\leq \sum_{j=m}^{m+p-1} \sigma(\xi_j(\tau), \xi_{j+1}(\tau)) \\ &\leq \sum_{j=m}^{m+p-1} w(\tau)^j \sigma(\xi_0(\tau), \xi_1(\tau)) \\ &\leq \frac{w(\tau)^m}{1 - w(\tau)} \sigma(\xi_0(\tau), \xi_1(\tau)). \end{aligned}$$

For each fixed τ, the right-hand side tends to zero as $m \to \infty$; hence $\{\xi_m(\tau)\}$ is Cauchy. Completeness gives $\xi(\tau) \in A$ such that $\xi_m(\tau) \to \xi(\tau)$. Since A is separable and every ξ_m is measurable, the pointwise limit ξ is measurable.

The shifted sequence $\{\xi_{m+1}(\tau)\}$ also converges to $\xi(\tau)$. On the other hand, continuity of $K(\tau, \cdot)$ gives

$$\xi_{m+1}(\tau) = K(\tau, \xi_m(\tau)) \longrightarrow K(\tau, \xi(\tau)).$$

A convergent sequence in a metric space has a unique limit; therefore

$$K(\tau, \xi(\tau)) = \xi(\tau), \qquad \tau \in \Gamma.$$

Thus ξ is a measurable random fixed point of K. □

Remark 3.2. (Uniqueness) Under the hypotheses of Theorem 3.1, assume in addition that A is R-directed: for every $\xi, \eta \in A$ there exists $\zeta \in A$ with $\xi R \zeta$ and $\eta R \zeta$. Then the random fixed point is unique.

Proof. Let ξ^* and $\bar{\xi}^*$ be two random fixed points and fix $\tau \in \Gamma$. Choose $\zeta = \zeta(\tau)$ such that

$$\xi^*(\tau) R \zeta, \qquad \bar{\xi}^*(\tau) R \zeta.$$

Define $\zeta_0 = \zeta$ and $\zeta_{m+1} = K(\tau, \zeta_m)$. Since ξ^* and $\bar{\xi}^*$ are fixed points and R is K-closed,

$$\xi^*(\tau) R \zeta_m, \qquad \bar{\xi}^*(\tau) R \zeta_m$$

for every m. Hence

$$\sigma(\xi^*(\tau), \zeta_m) \leq w(\tau)^m \sigma(\xi^*(\tau), \zeta),$$

and

$$\sigma(\bar{\xi}^*(\tau), \zeta_m) \leq w(\tau)^m \sigma(\bar{\xi}^*(\tau), \zeta).$$

Consequently,

$$\sigma(\xi^*(\tau), \bar{\xi}^*(\tau)) \leq w(\tau)^m \big[\sigma(\xi^*(\tau), \zeta) + \sigma(\bar{\xi}^*(\tau), \zeta)\big] \longrightarrow 0.$$

Thus $\xi^*(\tau) = \bar{\xi}^*(\tau)$ for every τ, and the random fixed point is unique. Notice that transitivity of R is not required. □

Theorem 3.3. *Let (A, σ), R, K, w, and ξ_0 satisfy all assumptions of Theorem 3.1 except continuity of K. Replace continuity by the following R-self-closedness condition: if an R-preserving sequence $\{\xi_m\}$ converges to ξ, then there is a subsequence $\{\xi_{m_j}\}$ with $\xi_{m_j} R \xi$ for every j. Then K has a random fixed point. If A is R-directed, the fixed point is unique.*

Proof. The Picard construction, measurability of the iterates, preservation of the relation, and the Cauchy estimate are identical to those in Theorem 3.1. Thus $\xi_m(\tau) \to \xi(\tau)$. By R-self-closedness, for each fixed τ there is a subsequence $\xi_{m_j}(\tau)$ such that $\xi_{m_j}(\tau) R \xi(\tau)$. Hence

$$\begin{aligned}\sigma(\xi_{m_j+1}(\tau), K(\tau, \xi(\tau))) &= \sigma(K(\tau, \xi_{m_j}(\tau)), K(\tau, \xi(\tau))) \\ &\leq w(\tau)\sigma(\xi_{m_j}(\tau), \xi(\tau)) \longrightarrow 0.\end{aligned}$$

But $\xi_{m_j+1}(\tau)$ is a subsequence of the shifted Picard sequence and therefore also converges to $\xi(\tau)$. Uniqueness of metric limits yields

$$K(\tau, \xi(\tau)) = \xi(\tau).$$

The R-directed comparison argument of Remark 3.2 proves uniqueness. □

Theorem 3.4. *(Matrix-valued random contraction) Suppose $(\Gamma, \mathcal{F})$ is a measurable space and (A, σ) is a separable complete vector-valued metric space with*

$$\sigma : A \times A \to M_{n\times 1}(\mathbb{R}_+),$$

endowed with a binary relation R. Suppose $K : \Gamma \times A \to A$ is jointly measurable and satisfies:

(i) *R is K-closed.*
(ii) *$K(\tau, \cdot)$ is continuous for every $\tau \in \Gamma$.*
(iii) *There exists an entrywise measurable nonnegative matrix $P : \Gamma \to M_{n\times n}(\mathbb{R}_+)$ satisfying, pointwise for every $\tau \in \Gamma$,*

$$\|P(\tau)\|_\infty < 1,$$

and for every $\xi_1, \xi_2 \in A$ with $\xi_1 R \xi_2$,

$$\sigma(K(\tau, \xi_1), K(\tau, \xi_2)) \leq P(\tau)\sigma(\xi_1, \xi_2)$$

componentwise.

(iv) *There exists a random variable $\xi_0 : \Gamma \to A$ such that*

$$\xi_0(\tau) R K(\tau, \xi_0(\tau)), \qquad \tau \in \Gamma.$$

(v) *For all $\xi, \eta \in A$, there exists $\zeta \in A$ such that $\xi R \zeta$ and $\eta R \zeta$.*

Then K has a unique random fixed point.

Proof. Define

$$\xi_{m+1}(\tau) = K(\tau, \xi_m(\tau)).$$

By K-closedness, $\xi_m(\tau) R \xi_{m+1}(\tau)$ for all m. The matrix contraction and induction yield

$$\sigma(\xi_{m+1}(\tau), \xi_{m+2}(\tau)) \leq P(\tau)^{m+1} \sigma(\xi_0(\tau), \xi_1(\tau)).$$

For $p \geq 1$, the componentwise triangle inequality gives

$$\sigma(\xi_m, \xi_{m+p}) \leq \sum_{j=m}^{m+p-1} P(\tau)^j \sigma(\xi_0, \xi_1) \leq P(\tau)^m (I - P(\tau))^{-1} \sigma(\xi_0, \xi_1).$$

The inverse exists through the Neumann series because $\|P(\tau)\|_\infty < 1$. Taking infinity norms,

$$\|\sigma(\xi_m, \xi_{m+p})\|_\infty \leq \frac{\|P(\tau)\|_\infty^m}{1 - \|P(\tau)\|_\infty} \|\sigma(\xi_0, \xi_1)\|_\infty \longrightarrow 0.$$

Thus the Picard sequence is Cauchy and converges pointwise to a map ξ. Joint measurability of K implies inductively that every Picard iterate is measurable; since A is separable with respect to the scalar metric d_σ, the pointwise limit ξ is measurable. Continuity gives

$$K(\tau, \xi_m(\tau)) \longrightarrow K(\tau, \xi(\tau)),$$

while $K(\tau, \xi_m(\tau)) = \xi_{m+1}(\tau) \to \xi(\tau)$. Hence

$$K(\tau, \xi(\tau)) = \xi(\tau).$$

For uniqueness, let ξ^* and $\bar{\xi}^*$ be fixed points. Choose ζ with $\xi^* R \zeta$ and $\bar{\xi}^* R \zeta$, and set $\zeta_m = K^m(\tau, \zeta)$. Then

$$\sigma(\xi^*, \zeta_m) \leq P(\tau)^m \sigma(\xi^*, \zeta), \qquad \sigma(\bar{\xi}^*, \zeta_m) \leq P(\tau)^m \sigma(\bar{\xi}^*, \zeta).$$

Using $\|P(\tau)^m\|_\infty \leq \|P(\tau)\|_\infty^m$ and the triangle inequality gives

$$\left\|\sigma(\xi^*, \bar{\xi}^*)\right\|_\infty = 0.$$

Hence $\xi^* = \bar{\xi}^*$. □

Remark 3.5. The matrix formulation is not merely a scalar contraction written componentwise. It permits different contraction strengths and cross-component couplings. In a system with n components, the entry $p_{ij}(\tau)$ measures how the jth component of an input difference contributes to the ith component of the output difference. This interpretation is used in the coupled system in Section 5.

4. Random Fixed Point Theorems in L-Spaces

Metric convergence is only one way to encode an iterative limiting process. L-spaces retain the two sequential features used by Picard methods—convergence of constant sequences and inheritance of the limit by subsequences—while allowing the ambient convergence to be specified abstractly. Because the fixed-point identification compares two limits of the same orbit, we work with Hausdorff L-spaces so that limits are unique.

Theorem 4.1. *Let $(A, \to)$ be a Hausdorff L-space equipped compatibly with a binary relation R, and let $K : \Gamma \times A \to A$ be jointly measurable. Assume:*

(*i*) *For every $\xi_1, \xi_2 \in A$ such that $\xi_1 R \xi_2$ does not hold, there exists $\xi_3 \in A$ such that*

$$\xi_1 R \xi_3, \qquad \xi_3 R \xi_2.$$

(*ii*) *There exist random variables* $\xi_0, \xi^* : \Gamma \to A$ *such that the Picard orbit*

$$\xi_{m+1}(\tau) = K(\tau, \xi_m(\tau))$$

converges to $\xi^*(\tau)$*, and for some* $p \in \mathbb{N}$ *one has*

$$\xi_m(\tau) R \xi^*(\tau), \qquad m \geq p, \quad \tau \in \Gamma.$$

(*iii*) K *is orbitally* R*-continuous: whenever* $\eta_m(\tau) = K(\tau, \eta_{m-1}(\tau)) \to \eta^*(\tau)$ *and* $\eta_m(\tau) R \eta^*(\tau)$ *eventually, one has*

$$K(\tau, \eta_m(\tau)) \to K(\tau, \eta^*(\tau)).$$

(*iv*) *Asymptotic compatibility holds: if* $\xi R \eta$ *and the orbit* $K^m(\tau, \xi)$ *converges to* $\xi^*(\tau)$*, then the orbit* $K^m(\tau, \eta)$ *also converges to* $\xi^*(\tau)$*.*

Then ξ^* *is the unique random fixed point of* K*.*

Proof. By assumption, $\xi_m \to \xi^*$. The shifted sequence ξ_{m+1} has the same limit ξ^*. Orbital R-continuity gives

$$K(\tau, \xi_m(\tau)) \to K(\tau, \xi^*(\tau)).$$

Since $K(\tau, \xi_m(\tau)) = \xi_{m+1}(\tau)$, the Hausdorff property yields

$$K(\tau, \xi^*(\tau)) = \xi^*(\tau).$$

The map ξ^* is measurable by hypothesis, so it is a random fixed point.

For uniqueness, let ξ be another random fixed point. If $\xi_0(\tau) R \xi(\tau)$ holds, asymptotic compatibility implies that the orbit starting from ξ converges to ξ^*. But this orbit is the constant sequence ξ, which converges to ξ; Hausdorffness gives $\xi = \xi^*$. If $\xi_0(\tau) R \xi(\tau)$ does not hold, choose ζ as in assumption (*i*). The relation $\xi_0 R \zeta$ transfers the limit ξ^* to the orbit of ζ, and $\zeta R \xi$ then transfers the same limit to the constant orbit of ξ. Again $\xi = \xi^*$. □

Remark 4.2. Assumption (*i*) may be replaced by the stronger bridge condition that for every $\xi_1, \xi_2 \in A$ there exists $\xi_3 \in A$ satisfying

$$\xi_1 R \xi_3, \qquad \xi_3 R \xi_2.$$

Example 4.3. (Metric L-space) Let (A, d) be any metric space and define $\xi_m \to \xi$ by $d(\xi_m, \xi) \to 0$. Then $(A, \to)$ is a Hausdorff L-space. If R has a sequentially closed graph, the compatibility condition in Definition 2.9 holds. Every continuous K is orbitally R-continuous.

Example 4.4. (Ordered Banach space) Let E be a Banach space and $C \subset E$ a closed cone. Define $\xi R \eta$ when $\eta - \xi \in C$, and use norm convergence. Since C is closed, $\xi_m R \eta_m$, $\xi_m \to \xi$, and $\eta_m \to \eta$ imply $\eta - \xi \in C$; hence R is compatible with convergence.

Remark 4.5. Assumption (*iv*) is deliberately strong: it requires R-related initial data to lie in the same asymptotic basin of the Picard iteration. In metric examples it is verifiable, for instance, when a relation-restricted contraction gives $\sigma(K^m \xi, K^m \eta) \to 0$ and one of the two orbits is known to converge. Stating this role explicitly separates the abstract L-space hypothesis from the metric contraction arguments of Section 3.

5. Application to a Boundary Value Problem

We apply the vector-valued fixed-point theorem to a coupled random nonlocal boundary value problem. Let $\alpha, \beta : [0,1] \to \mathbb{R}$ be functions of bounded variation, with total variations V_α and V_β. Thus, for every $h \in C([0,1], \mathbb{R})$, the Riemann–Stieltjes integrals exist and satisfy

$$\left| \int_0^1 h(s)\, d\alpha(s) \right| \le V_\alpha \left\| h \right\|_\infty, \qquad \left| \int_0^1 h(s)\, d\beta(s) \right| \le V_\beta \left\| h \right\|_\infty .$$

For $\tau \in \Gamma$, consider the coupled system

$$\begin{cases} \xi''(\theta,\tau) = k_1(\theta, \xi(\theta,\tau), \eta(\theta,\tau), \tau), \\ \eta''(\theta,\tau) = k_2(\theta, \xi(\theta,\tau), \eta(\theta,\tau), \tau), \qquad 0 < \theta < 1, \\ \xi(0,\tau) = 0, \qquad \xi(1,\tau) = T_1\left(\displaystyle\int_0^1 \xi(s,\tau)\, d\alpha(s) \right), \\ \eta(0,\tau) = 0, \qquad \eta(1,\tau) = T_2\left(\displaystyle\int_0^1 \eta(s,\tau)\, d\beta(s) \right). \end{cases} \tag{1}$$

Here $T_1, T_2 : \mathbb{R} \to \mathbb{R}$ are continuous. A random solution is a measurable mapping

$$\tau \longmapsto (\xi(\cdot,\tau), \eta(\cdot,\tau))$$

from Γ into $C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R})$ satisfying (1) for every τ.

Lemma 5.1. *Let $k_1, k_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous. A pair $(\xi, \eta) \in C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R})$ solves the deterministic version of (1) if and only if*

$$\begin{aligned} \xi(\theta) &= \int_0^1 Q(\theta,s) k_1(s, \xi(s), \eta(s))\, ds + \theta T_1\left(\int_0^1 \xi(s)\, d\alpha(s) \right), \\ \eta(\theta) &= \int_0^1 Q(\theta,s) k_2(s, \xi(s), \eta(s))\, ds + \theta T_2\left(\int_0^1 \eta(s)\, d\beta(s) \right), \end{aligned} \tag{2}$$

where

$$Q(\theta,s) = \begin{cases} -\theta(1-s), & 0 \le \theta \le s \le 1, \\ -s(1-\theta), & 0 \le s \le \theta \le 1. \end{cases}$$

Proof. Integrating $u'' = h$ twice and imposing $u(0) = 0$ and $u(1) = b$ gives

$$u(\theta) = \int_0^1 Q(\theta,s) h(s)\, ds + \theta b.$$

Applying this formula to the two equations with

$$b = T_1\left(\int_0^1 \xi\, d\alpha \right) \quad \text{and} \quad b = T_2\left(\int_0^1 \eta\, d\beta \right)$$

yields (2). Conversely, differentiating twice and evaluating at $\theta = 0, 1$ recovers the differential equations and boundary conditions. □

A direct calculation gives

$$C_Q := \sup_{0 \le \theta \le 1} \int_0^1 |Q(\theta,s)|\, ds = \sup_{0 \le \theta \le 1} \frac{\theta(1-\theta)}{2} = \frac{1}{8}. \tag{3}$$

Theorem 5.2. *Let $k_1, k_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \times \Gamma \to \mathbb{R}$ satisfy the Carathéodory conditions: for each (θ, ξ, η), the map $\tau \mapsto k_i(\theta, \xi, \eta, \tau)$ is measurable, and for each τ, the map $(\theta, \xi, \eta) \mapsto k_i(\theta, \xi, \eta, \tau)$ is continuous. Suppose:*

(*i*) *There exist measurable nonnegative functions $q_1, q_2, q_3, q_4 : \Gamma \to [0, \infty)$ such that*

$$|k_1(\theta, \xi_1, \eta_1, \tau) - k_1(\theta, \xi_2, \eta_2, \tau)| \leq q_1(\tau)|\xi_1 - \xi_2| + q_2(\tau)|\eta_1 - \eta_2|,$$
$$|k_2(\theta, \xi_1, \eta_1, \tau) - k_2(\theta, \xi_2, \eta_2, \tau)| \leq q_3(\tau)|\xi_1 - \xi_2| + q_4(\tau)|\eta_1 - \eta_2|.$$

(*ii*) *There exist constants $D_1, D_2 \geq 0$ such that*

$$|T_1(a) - T_1(b)| \leq D_1|a - b|, \qquad |T_2(a) - T_2(b)| \leq D_2|a - b|$$

for all $a, b \in \mathbb{R}$.

(*iii*) *α and β are functions of bounded variation with total variations V_α and V_β.*

(*iv*) *For every $\tau \in \Gamma$, the nonnegative matrix*

$$P(\tau) = \begin{pmatrix} C_Q q_1(\tau) + D_1 V_\alpha & C_Q q_2(\tau) \\ C_Q q_3(\tau) & C_Q q_4(\tau) + D_2 V_\beta \end{pmatrix}$$

satisfies the pointwise condition

$$\|P(\tau)\|_\infty < 1.$$

Then the problem (1) has a unique random solution.

Proof. Set

$$A = C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R})$$

and define the vector-valued metric

$$\sigma((\xi_1, \eta_1), (\xi_2, \eta_2)) = \begin{pmatrix} \|\xi_1 - \xi_2\|_\infty \\ \|\eta_1 - \eta_2\|_\infty \end{pmatrix}.$$

Then A is a separable complete vector-valued metric space. For this application define the binary relation R on A to be the universal relation

$$R = A \times A.$$

Hence every pair is R-related, R is automatically M-closed for any self-mapping M of A, and A is R-directed. The constant zero pair $(0,0)$ is a measurable A-valued random variable and, because R is universal, it automatically satisfies the initial relation required in Theorem 3.4.

Define $M : A \times \Gamma \to A$ by

$$M(\xi, \eta, \tau) = \big(M_1(\cdot, \xi, \eta, \tau), M_2(\cdot, \xi, \eta, \tau)\big),$$

where

$$M_1(\theta, \xi, \eta, \tau) = \int_0^1 Q(\theta, s) k_1(s, \xi(s), \eta(s), \tau)\, ds + \theta T_1\left(\int_0^1 \xi(s)\, d\alpha(s)\right),$$
$$M_2(\theta, \xi, \eta, \tau) = \int_0^1 Q(\theta, s) k_2(s, \xi(s), \eta(s), \tau)\, ds + \theta T_2\left(\int_0^1 \eta(s)\, d\beta(s)\right).$$

Measurability. Fix $(\xi, \eta) \in A$ and $\theta \in [0,1]$. By the Carathéodory hypotheses, the map

$$(s, \tau) \longmapsto Q(\theta, s) k_i(s, \xi(s), \eta(s), \tau)$$

is measurable, and for each fixed τ it is continuous in s. Therefore, the standard parameter-integral measurability theorem implies that

$$\tau \longmapsto \int_0^1 Q(\theta, s) k_i(s, \xi(s), \eta(s), \tau)\, ds$$

is measurable. The Riemann-Stieltjes boundary term is deterministic once (ξ, η) is fixed. Hence $\tau \mapsto M_i(\theta, \xi, \eta, \tau)$ is measurable for every θ. For fixed τ, the map $\theta \mapsto M_i(\theta, \xi, \eta, \tau)$ is continuous. Lemma 2.6 therefore yields measurability of

$$\tau \longmapsto M_i(\cdot, \xi, \eta, \tau)$$

as a $C([0,1], \mathbb{R})$-valued mapping. Thus M is a random operator.

Continuity and matrix contraction. Let $(\xi_1, \eta_1), (\xi_2, \eta_2) \in A$. From the Lipschitz assumptions, the Green-kernel bound (3), and the bounded-variation estimates,

$$\begin{aligned} \|M_1(\xi_1, \eta_1, \tau) - M_1(\xi_2, \eta_2, \tau)\|_\infty &\le [C_Q q_1(\tau) + D_1 V_\alpha] \|\xi_1 - \xi_2\|_\infty + C_Q q_2(\tau) \|\eta_1 - \eta_2\|_\infty, \\ \|M_2(\xi_1, \eta_1, \tau) - M_2(\xi_2, \eta_2, \tau)\|_\infty &\le C_Q q_3(\tau) \|\xi_1 - \xi_2\|_\infty + [C_Q q_4(\tau) + D_2 V_\beta] \|\eta_1 - \eta_2\|_\infty. \end{aligned}$$

Equivalently,

$$\sigma(M(\xi_1, \eta_1, \tau), M(\xi_2, \eta_2, \tau)) \le P(\tau)\sigma((\xi_1, \eta_1), (\xi_2, \eta_2))$$

componentwise. The same estimates show continuity of $M(\cdot, \cdot, \tau)$. Since the q_i are measurable, P is an entrywise measurable nonnegative matrix. The hypothesis $\|P(\tau)\|_\infty < 1$ is imposed pointwise for every τ. Therefore all assumptions of Theorem 3.4 are satisfied, and M has a unique random fixed point. By Lemma 5.1, the random fixed points of M are precisely the random solutions of (1). □

Example 5.3. (Numerical verification of the norm condition) Take $\alpha(\theta) = \beta(\theta) = \theta$, so $V_\alpha = V_\beta = 1$, and choose

$$D_1 = 0.20, \qquad D_2 = 0.25,$$

with

$$q_1 = 0.80, \qquad q_2 = 0.40, \qquad q_3 = 0.32, \qquad q_4 = 0.64.$$

These functions are constant in τ and therefore measurable. Since $C_Q = 1/8$,

$$P = \begin{pmatrix} 0.80/8 + 0.20 & 0.40/8 \\ 0.32/8 & 0.64/8 + 0.25 \end{pmatrix} = \begin{pmatrix} 0.30 & 0.05 \\ 0.04 & 0.33 \end{pmatrix}.$$

Therefore,

$$\|P\|_\infty = \max\{0.30 + 0.05,\ 0.04 + 0.33\} = 0.37 < 1.$$

For example, these bounds are realized by

$$k_1(\theta, \xi, \eta, \tau) = f_1(\theta, \tau) + 0.80 \tanh(\xi) + 0.40 \tanh(\eta),$$

$$k_2(\theta, \xi, \eta, \tau) = f_2(\theta, \tau) + 0.32 \tanh(\xi) + 0.64 \tanh(\eta),$$

together with $T_1(s) = 0.20s$ and $T_2(s) = 0.25s$, because tanh is 1-Lipschitz. If f_1 and f_2 satisfy the stated Carathéodory conditions, Theorem 5.2 guarantees a unique random solution.

Remark 5.4. The argument extends directly to an m-component system. Replace σ by the m-vector of componentwise sup-norm distances and $P(\tau)$ by an $m \times m$ nonnegative measurable matrix collecting the componentwise Lipschitz constants. If $\|P(\tau)\|_\infty < 1$ for every τ, the proof of Theorem 3.4 applies without further structural change.

6. Conclusion

We have developed random fixed point results in spaces endowed with a general binary relation and have made the measurability assumptions explicit. In the complete metric-space result, K-closedness is preserved along the Picard orbit, the scalar random contraction yields a geometric Cauchy estimate, and continuity allows to recognize the shifted Picard limit with the image of the limiting point. This is a direct; non-circular proof of the fixed-point identity.

The uniqueness is achieved by an R-directed comparison construction. There is no transitivity assumption that is used in this argument and the relation-restricted contraction is only used for pairs in which the relation has been established. The second theorem gives a relationship-theoretic subsequence condition instead of the continuity condition. We also form a vector-valued object controlled by a nonnegative measurable contraction matrix, and we define the induced infinity norm and the cone operations and the pointwise norm requirement.

The L-space section shows how to set up the fixed-point mechanism, in a sequential setting. We made the Hausdorff assumption clear because the uniqueness of limits is required when the shifted orbit is compared to the continuous image. Metric. Ordered Banach spaces give concrete examples while the asymptotic compatibility condition records the stronger assumption needed for the uniqueness of limits when the metric contraction estimate is missing.

Finally the matrix-valued theorem is used on a coupled nonlocal boundary value problem. The corrected integral formulation includes the second nonlinearity and the boundary operator. It also uses functions α and β that're of bounded variation along, with the Green kernel that comes from the second-order problem. A function-valued measurability argument is. An explicit contraction matrix is provided. The numerical example shows that the norm condition is not empty and the method works naturally for finite systems. Future work could look at L-space compatibility assumptions almost-sure matrix contraction conditions and applications to higher-dimensional or fractional random boundary value systems.

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