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Modified double two-step inertial hybrid projection algorithms with two iterative sequences for solving common fixed point problems

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Abstract

This paper is dedicated to the study of common fixed point problem of nonexpansive mappings. Using the double two-step inertial method and hybrid projection algorithm, we design a modified double two-step inertial hybrid projection algorithm with two iterative sequences, and thus, a new variant of the hybrid projection algorithm is obtained. Then the previous two historical information of the two iterative sequences in the algorithm are further utilized simultaneously. Interestingly, a new and extended lemma tool is presented, it generalizes the known tools. This lemma tool plays an important role on the subsequent discussion in this paper. We utiliz this lemma to establish the strong convergence theorem of the proposed algorithm. To further verify the obtained results, two simple numerical experiments are provided. The conclusions obtained in this paper generalize and extend some previous results.

Keywords: Fixed point problem, Double two-step inertial method, Hybrid projection algorithm, Strong convergence, Numerical analysis.
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1. Introduction

In this paper, the goal is dedicated to the study of the common fixed point problem for two nonexpansive mappings. Let H be a real Hilbert space with the inner product $\langle .,. \rangle$ and the norm $\| \cdot \|$ and $C \subset H$ be a nonempty, convex and closed subset. As is well known, the Mann iterative algorithm [1] has weak convergence and a slow convergence speed. Then, modifying the Mann iterative algorithm to make it have strong convergence is one of the research goals of many scholars. The hybrid projection algorithm [2] is a fundamental algorithm that successfully modifies the Mann iterative algorithm to make it strongly convergent and its form is as follows:

$$\begin{cases} \text{Set } x_0, \in C \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n \\ C_n = \{p_* \in C | \|y_n - p_*\| \leq \alpha_n \|x_n - p_*\|\} \\ Q_n = \{p_* \in C | \langle x_n - p_*, x_0 - x_n \rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases}$$

where T is a nonexpansive mapping and $\alpha_n \in (0,1)$. Notice that a mapping T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. Therefore, many scholars have further conducted research on the hybrid projection algorithm. He et al. in [3] studied the problem of numerical experimental implementation of the hybrid projection algorithm. Kim et al. in [4] studied the hybrid projection algorithm for asymptotic non-expanding mappings and established the corresponding strong convergence results. Martinez-Yanes [5] established a Ishikawa-type hybrid projection algorithm for nonexpansive mappings and its form is stated as follows:

$$\begin{cases} \text{Set } x_0, \in C \\ z_n = \alpha_n x_n + (1-\alpha_n)Tx_n \\ y_n = \beta_n x_n + (1-\beta_n)Tz_n \\ C_n = \{p_* \in C | \|y_n - p_*\|^2 \leq \|x_n - p_*\|^2 + (1-\alpha_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, p_* \rangle)\} \\ Q_n = \{p_* \in C | \langle x_n - p_*, x_0 - x_n \rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases}$$

where T is a nonexpansive mapping, $\alpha_n, \beta_n \in (0,1)$ and $\alpha_n \in (0, 1-\delta], \delta \in (0,1]$. Marino et al. [6] established a Ishikawa-type hybrid projection algorithm for nonexpansive mappings and its form is stated as follows:

$$\begin{cases} \text{Set } x_0, \in C \\ z_n = \alpha_n x_n + (1-\alpha_n)Tx_n \\ C_n = \{p_* \in C | \|z_n - p_*\|^2 \leq \|x_n - p_*\|^2 + (1-\alpha_n)(\kappa - \alpha_n)\|x_n - Tx_n\|^2\} \\ Q_n = \{p_* \in C | \langle x_n - p_*, x_0 - x_n \rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases}$$

where T is a κ-strict pseudo-contractions, $\alpha_n \in (0,1)$. Notice that a mapping T is said to be a pseudo-contractions if $\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2$ for all $x, y \in H$; a mapping T is said to be a κ-strict pseudo-contractions if $\|Tx - Ty\|^2 \leq \|x - y\|^2 - (1-\kappa)\|(I-T)x - (I-T)y\|^2$ for all $x, y \in H$. In 2015, Dong et al. [7] introduced a new modified hybrid projection algorithm as follows:

$$\begin{cases} \text{Set } z_0, x_0, \in C \\ z_{n+1} = \alpha_n x_n + (1-\alpha_n)Tz_n \\ C_n = \{p_* \in C | \|z_{n+1} - p_*\|^2 \leq \alpha_n \|x_n - p_*\|^2 + (1-\alpha_n)\|z_n - p_*\|^2\} \\ Q_n = \{p_* \in C | \langle x_n - p_*, x_0 - x_n \rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1}$$

where T is a nonexpansive mapping. In [7], Dong et al. utilized a new lemma tool to prove the strong convergence theorems for their algorithms under some appropriate assumptions $\alpha_n \in [a,b] \subseteq (\frac{1}{2},1)$.

From the heavy ball method of two-order time dynamical system, an inertial extrapolation as an acceleration process was proposed by Polyak [8] firstly to solve convex minimization problem. The inertial algorithm is a two-step iterative method, that is, the next iterate is defined by making use of the previous two iterates. Dong et al. in [9] combined the acceleration technology and hybrid projection algorithm and a highly convergent algorithm was constructed, whose structure is as follows:

$$\begin{cases} \text{Set } x_0, x_1 \in C \\ w_n = x_n + \delta_n(x_n - x_{n-1}) \\ z_n = (1-\alpha_n)w_n + \alpha_n T w_n \\ C_n = \{p_* \in C | \|z_n - p_*\| \le \|w_n - p_*\|\} \\ Q_n = \{p_* \in C | \langle x_n - p_*, x_0 - x_n\rangle \ge 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases}$$

where T is a nonexpansive mapping, $\alpha_n \in [\alpha, 1] \subseteq (0,1]$, $\delta \in [\delta_1, \delta_2]$, $\delta_1 \in (-\infty, 0], \delta_2 \in [0,\infty)$. Tan et al. in [10] combined the acceleration technology and hybrid projection algorithm and a highly convergent algorithm was constructed, whose structure is as follows:

$$\begin{cases} \text{Set } x_0, x_1 \in C \\ w_n = x_n + \delta_n(x_n - x_{n-1}) \\ z_{n+1} = \alpha_n z_n + \alpha_n T w_n \\ C_n = \{p_* \in C | \|z_{n+1} - p_*\|^2 \le \alpha_n\|z_n - p_*\|^2 + (1-\alpha_n)\|w_n - p_*\|^2\} \\ Q_n = \{p_* \in C | \langle x_n - p_*, x_0 - x_n\rangle \ge 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases}$$

where T is a nonexpansive mapping, $\alpha_n \in [\frac{\bar{\alpha}}{1+\bar{\alpha}}, 1], \bar{\alpha} \in (0,1)$, $\delta \in [\delta_1, \delta_2]$, $\delta_1 \in (-\infty, 0], \delta_2 \in [0,\infty)$. In 2025, using the double inertial method, Weng et al. [11] introduced the following proceess:

$$\begin{cases} \text{Set } z_0, x_0, z_1, x_1 \in C \\ w_n = x_n + \delta_n(x_n - x_{n-1}) \\ y_n = z_n + \theta_n(z_n - z_{n-1}) \\ z_{n+1} = \alpha_n w_n + (1-\alpha_n) T y_n \\ C_n = \{p_* \in C | \|z_{n+1} - p_*\|^2 \le \alpha_n\|w_n - p_*\|^2 + (1-\alpha_n)\|y_n - p_*\|^2\} \\ Q_n = \{p_* \in C | \langle x_n - p_*, x_0 - x_n\rangle \ge 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{2}$$

where T is a nonexpansive mapping. In [11], Weng et al. utilized a new improved lemma tool to prove the strong convergence theorems for their algorithms under some appropriate assumptions $\delta_n, \theta_n \in [\delta_1, \delta_2], \delta_1 \in (-\infty, 0], \delta_2 \in [0, +\infty), K = \max\{|\delta_1|, \delta_2\}, \alpha_n \in (\frac{3+12K^2}{4+12K^2}, \gamma), \gamma \in (\frac{3}{4}, 1)$. For more research on the hybrid projection algorithm, please refer to [12, 13]. In 2024, Mewomo et al. [14] studied the two-step inertial Tseng's extragradient method for solving variational inequality problem. In [14], the results of their numerical experiments showed that the two-step inertial method has a positive effect on improving the iterative performance of the proposed algorithm.

Since the two-step inertial method is effective in optimizing and enhancing the iterative performance of algorithm (1), we naturally wonder whether the two-step inertial method is still effective in optimizing and improving the iterative performance of algorithm (1). Compared with the one-step inertial method, the two-step inertia method involves some additional information and calculations. Will this offset the advantages brought by the two-step inertial method?

Inspired by (1) and (2), based on hybrid projection algorithm, this paper uses the double two-step inertial method to explore the common fixed point problem of two nonexpansive mappings. This paper mainly has the following advantages:

(1). Based on this type of algorithm with two iterative sequences, we construct a strongly convergent algorithm that can iteratively approximate the common fixed point of two different nonexpansive mappings.

(2). We adopted a two-step inertial method for both iterative sequences of the algorithm to accelerate them simultaneously. Subsequently, we verify the positive effect of the double two-step inertial method in optimizing the performance of the algorithm by numerical experiments.

(3). In order to successfully establish the convergence theorem of the proposed algorithm, we improved a known inequality tool to enable the theoretical proof to be logically feasible.

The structure of this paper is as follows. Section 2 provides the relevant lemma tools. Section 3 presents the main results of this paper. Section 4 presents two numerical examples to verify the main results of this paper. Section 5 summarizes this paper.

2. Mathematical Preliminaries

In this section, we first define the meanings of some symbols, then state some basic equations or inequalities, and present some known conclusions that are needed as tools for this paper. Finally, we extended some known tool inequalities and presented a new lemma tool.

Let $\mathbb{N}$ represent the set of positive integers. Let $x_n \rightharpoonup x$ denote the weak convergence of a sequence $\{x_n\}$ to a point $x \in H$, $x_n \to x$ denote the strong convergence of a sequence $\{x_n\}$ to a point $x \in H$ and $\omega_w(x_n)$ denote the set of all weak limits of $\{x_n\}$, $Fix(T) = \{x \in C : Tx = x\}$ denote the set of fixed points of nonlinear mapping T. The symbol I denotes the identity mapping. A mapping $T : C \to C$ is called nonexpansive if for all $x, y \in C$ it satisfies the the following inequality $\|Tx - Ty\| \leq \|x - y\|$. For $x, y, z \in H, \lambda, \gamma, \xi \in R$, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$, and

$$\|\lambda x + \gamma y + \xi z\|^2 = \lambda\|x\|^2 + \gamma\|y\|^2 + \xi\|z\|^2 - \lambda\gamma\|x - y\|^2 - \lambda\xi\|x - z\|^2 - \gamma\xi\|y - z\|^2.$$

Suppose that H is a real Hilbert space, the subset $K \subset H$ is convex and closed, P_K is the metric projection from H onto K (that is, for $x \in H$, $P_K(x)$ is the only point in K such that $\|x - P_K(x)\| = \inf\{\|x - z\| : z \in K\}$). Given $x \in H$ and $z \in K$. Then $z = P_K(x)$ if and only if the following relation holds:

$$\langle x - z, y - z\rangle \leq 0 \quad for\ all\ y \in K.$$

Lemma 2.1. *[5] Suppose that H is a real Hilbert space, the subset $C \subset H$ is convex closed and nonempty. Given $x, y, z \in H$ and $a \in R$. Let $D = \{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$, then D is convex and closed.*

Lemma 2.2. *[15] Suppose that H is a real Hilbert space, the subset $C \subset H$ is convex closed and nonempty, the mapping $T : C \to C$ is nonexpansive. Let $\{x_n\}$ be a sequence in C and $x \in H$ such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$ as $n \to +\infty$. Then $x \in Fix(T)$.*

Lemma 2.3. *[5] Suppose that H is a real Hilbert space, the subset $C \subset H$ is convex and closed. Suppose that $\{x_n\} \subset H$ and $u \in H$. Let $q = P_C(u)$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and for $n \in \mathbb{N}$ satisfies the condition*

$$\|x_n - u\| \leq \|u - q\|.$$

Then $x_n \to q$.

Lemma 2.4. *[1] Suppose that* $\{a_n\}$ *and* $\{\delta_n\}$ *are nonnegative real sequences, there exists some constants* $p, s \in [0, +\infty)$ *such that* $p < 1$*, for all* $n \geq 1$*, the following expression holds:*

$$a_{n+1} \leq pa_n + s\delta_n. \tag{3}$$

If $\Sigma_{n=1}^{\infty}\delta_n < +\infty$*, then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. *[7] Suppose that* $\{a_n\}$ *and* $\{\delta_n\}$ *are nonnegative real sequences, there exists some constants* $p, q, s \in [0, +\infty)$ *such that* $p + q < 1$*, for all* $n \geq 1$*, the following expression holds:*

$$a_{n+1} \leq pa_n + qa_{n-1} + s\delta_n. \tag{4}$$

If $\Sigma_{n=1}^{\infty}\delta_n < +\infty$*, then* $\lim_{n\to\infty} a_n = 0$.

In order to be able to successfully complete the theoretical proof of the main result of this paper, we need a new lemma tool, and it is stated as follows.

Lemma 2.6. *Suppose that* $\{a_n\}$ *and* $\{\delta_n\}$ *are nonnegative real sequences, there exists some constants* $p, q, r, s \in [0, +\infty)$ *such that* $p + q + r < 1$*, for all* $n \geq 1$*, the following expression holds:*

$$a_{n+1} \leq pa_n + qa_{n-1} + ra_{n-2} + s\delta_n. \tag{5}$$

If $\Sigma_{n=1}^{\infty}\delta_n < +\infty$*, then* $\lim_{n\to\infty} a_n = 0$.

Proof. Using (5) for all $n = 2, 3, 4, ..., N$, we have

$$\begin{aligned}
&a_3 \leq pa_2 + qa_1 + ra_0 + s\delta_2,\\
&a_4 \leq pa_3 + qa_2 + ra_1 + s\delta_3,\\
&\ldots\ldots\\
&a_{N+1} \leq pa_N + qa_{N-1} + ra_{N-2} + s\delta_N.
\end{aligned} \tag{6}$$

Adding all these inequalities in (6), this implies

$$\Sigma_{n=2}^{N}a_{n+1} \leq p\Sigma_{n=2}^{N}a_n + q\Sigma_{n=2}^{N}a_{n-1} + r\Sigma_{n=2}^{N}a_{n-2} + s\Sigma_{n=2}^{N}\delta_n,$$

which is equivalent to the following expression

$$\begin{aligned}
\Sigma_{n=1}^{N}a_n - a_1 - a_2 + a_{N+1} \leq& p(\Sigma_{n=1}^{N}a_n - a_1) + q[\Sigma_{n=1}^{N}a_n - a_N]\\
&+ r[\Sigma_{n=1}^{N}a_n + a_0 - a_{N-1} - a_N] + s\Sigma_{n=1}^{N}\delta_n,
\end{aligned}$$

that is

$$\begin{aligned}
\Sigma_{n=1}^{N}a_n \leq& p\Sigma_{n=1}^{N}a_n + q\Sigma_{n=1}^{N}a_n + r\Sigma_{n=1}^{N}a_n + s\Sigma_{n=1}^{N}\delta_n\\
&+ a_1 + a_2 - a_{N+1} - pa_1 - qa_N + ra_0 - ra_{N-1} - ra_N.
\end{aligned}$$

Then we get

$$(1 - p - q - r)\Sigma_{n=1}^{N}a_n \leq s\Sigma_{n=1}^{N}\delta_n + a_1 + a_2 + ra_0.$$

Since $p + q + r < 1$ and $\Sigma_{n=1}^{\infty}\delta_n < +\infty$, these show that

$$\Sigma_{n=1}^{N}a_n \leq \frac{s}{1 - p - q - r}\Sigma_{n=1}^{N}\delta_n + \frac{a_1 + a_2 + ra_0}{1 - p - q - r} < +\infty.$$

Because N is arbitrary, thus the series $\Sigma_{n=1}^{\infty}a_n$ is convergent and $\lim_{n\to\infty} a_n = 0$. □

Remark 2.7. Let $q = 0$ in inequality (4), Lemma 2.5 can be transformed into Lemma 2.4. Let $r = 0$ in inequality (5), Lemma 2.6 can be transformed into Lemma 2.5. Let $q = r = 0$, then Lemma 2.6 can be transformed into Lemma 2.4. In a word, Lemma 2.6 plays an important role on the proof of main results in this paper.

3. Main results

In this section, we elaborate on our ideas, establish the main results of this paper. To begin this part, the following required conditions first are given.

Condition 3.1 Suppose H is a real Hilbert space and the subset $C \subset H$ is nonempty, closed, convex.

Condition 3.2 The mappings T_1 and T_2 are nonexpansive with intersection $Fix(T_1) \cap Fix(T_2) \neq \emptyset$.

Condition 3.3 There exists some real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$.

Condition 3.4 $\delta_i, \theta_i \in [\xi_1, \xi_2], i = 1, 2, \xi_1 \in (-\infty, 0], \xi_2 \in [0, +\infty), \theta_1, \theta_2 \in (-1, 1)$.

Condition 3.5 $\theta_1^2 + \theta_2^2 < 1, \gamma_n \in (0, \dfrac{1 - 24\theta_1^2 - 24\theta_2^2}{4})$.

Next, we state a remark to explain the Condition 3.4 of the above conditions.

Remark 3.1. We require $\delta_i, \theta_i \in [\xi_1, \xi_2], i = 1, 2$, which is a rather broad expression. This indicates that $\delta_1, \theta_1, \delta_2, \theta_2$ will change along with the change of ξ_1, ξ_2. Next, we present the specific example of Conditions 3.4-3.5. Let $\xi_1 = -2,\ \xi_2 = 2$, then $\delta_1, \theta_1, \delta_2, \theta_2 \in [-2, 2]$, then we choose $\theta_1 = -\frac{1}{10}, \theta_2 = \frac{1}{10}$, so

$$\gamma_n \in (0, \frac{1 - 24 \cdot 0.1 - 24 \cdot 0.1}{4}) = (0, 0.13).$$

The above range indicates that γ_n is confined within a relatively large and not very narrow area.

Below, we present the main theorem of this paper.

Theorem 3.2. *Suppose that the Condition 3.1-Condition 3.5 hold. The sequences $\{x_n\}$ and $\{z_n\}$ generated by the following manner:*

$$\begin{cases} Set\ z_0, x_0, z_1, x_1, z_2, x_2 \in C \\ w_n = x_n + \sum_{i=1}^{2} \delta_i (x_{n-i+1} - x_{n-i}) \\ y_n = z_n + \sum_{i=1}^{2} \theta_i (z_{n-i+1} - z_{n-i}) \\ z_{n+1} = \alpha_n x_n + \beta_n T_1 w_n + \gamma_n T_2 y_n \\ C_n = \{p_* \in C | \|z_{n+1} - p_* * \|^2 \leq \alpha_n \|x_n - p_*\|^2 + \beta_n \|w_n - p_*\|^2 + \gamma_n \|y_n - p_*\|^2\} \\ Q_n = \{p_* \in C | \langle x_n - p_*, x_0 - x_n \rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{7}$$

Then the sequences $\{x_n\}$ and $\{z_n\}$ defined by (7) converge strongly to $P_{Fix(T_1) \cap Fix(T_2)}(x_0)$.

Proof. The proof is divided into six steps.

Step 1. We verify that $(Fix(T_1) \cap Fix(T_2)) \subset C_n \cap Q_n$, $n \in \mathbb{N}$.

Case one: We claim that $(Fix(T_1) \cap Fix(T_2)) \subset C_n$, $n \in \mathbb{N}$. Notice that by Lemma 2.1, we can get C_n is convex. Set $p_* \in Fix(T_1) \cap Fix(T_2)$ arbitrarily, based on the definition of z_{n+1} and the nonexpansiveness of T_1 and T_2, this shows

$$\begin{aligned} \|z_{n+1} - p_*\|^2 &= \|\alpha_n (x_n - p_*) + \beta_n (T_1 w_n - p_*) + \gamma_n (T_2 y_n - p_*)\|^2 \\ &= \alpha_n \|x_n - p_*\|^2 + \beta_n \|T_1 w_n - p_*\|^2 + \gamma_n \|T_2 y_n - p_*\|^2 - \alpha_n \beta_n \|T_1 w_n - x_n\|^2 \\ &\quad - \alpha_n \gamma_n \|T_2 y_n - x_n\|^2 - \beta_n \gamma_n \|T_1 w_n - T_2 y_n\|^2 \\ &\leq \alpha_n \|x_n - p_*\|^2 + \beta_n \|w_n - p_*\|^2 + \gamma_n \|y_n - p_*\|^2 - \alpha_n \beta_n \|T_1 w_n - x_n\|^2 \\ &\quad - \alpha_n \gamma_n \|T_2 y_n - x_n\|^2 - \beta_n \gamma_n \|T_1 w_n - T_2 y_n\|^2 \\ &\leq \alpha_n \|x_n - p_*\|^2 + \beta_n \|w_n - p_*\|^2 + \gamma_n \|y_n - p_*\|^2, \end{aligned} \tag{8}$$

it can be derived from (8) that $p_* \in C_n$, then $(Fix(T_1) \cap Fix(T_2)) \subset C_n$ for all $n \geq 0$.

Case two: We claim that $(Fix(T_1) \cap Fix(T_2)) \subset Q_n$, $n \in \mathbb{N}$. Mathematical induction is employed to prove the relation. Set $n = 0$, we obtain $(Fix(T_1) \cap Fix(T_2)) \subset C = Q_0$. Without loss of generality, suppose $(Fix(T_1) \cap Fix(T_2)) \subset Q_n$ holds, then $(Fix(T_1) \cap Fix(T_2)) \subset C_n \cap Q_n$, with the aid of $x_{n+1} = P_{C_n \cap Q_n}(x_0)$ and the properties of projection, one sees

$$\langle p_* - x_{n+1}, x_0 - x_{n+1}\rangle \leq 0, \ \forall p_* \in C_n \cap Q_n,$$

and

$$\langle p_* - x_{n+1}, x_0 - x_{n+1}\rangle \leq 0, \ \forall p_* \in Fix(T_1) \cap Fix(T_2).$$

Then, it can be derived from the definition of Q_{n+1} that $p_* \in Q_{n+1}$ and this implies that $(Fix(T_1) \cap Fix(T_2)) \subset Q_{n+1}$. Hence, so we can get $(Fix(T_1) \cap Fix(T_2)) \subset C_n \cap Q_n$ for all $n \geq 0$.

Step 2. We state the sequence $\{x_n\}$ is equipped with boundedness. Since $Fix(T_1) \cap Fix(T_2)$ is a nonempty closed convex subset of C, then there exists unique $p_* \in (Fix(T_1) \cap Fix(T_2))$ such that $p_* = P_{Fix(T_1)\cap Fix(T_2)}(x_0)$. Making use of the form of Q_n, this shows $x_n = P_{Q_n}(x_0)$. Since $(Fix(T_1) \cap Fix(T_2)) \subset Q_n$, one sees

$$\|x_n - x_0\| \leq \|q - x_0\|$$

for all $q \in (Fix(T_1) \cap Fix(T_2))$. Since $p_* \in (Fix(T_1) \cap Fix(T_2))$, this shows

$$\|x_n - x_0\| \leq \|p_* - x_0\|, \tag{9}$$

the above (9) shows that the sequence $\{x_n\}$ is a bounded sequence.

Step 3. We mainly prove $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Since $x_{n+1} \in Q_n$, then from the definition of x_{n+1}, this implies that

$$\langle x_n - x_{n+1}, x_0 - x_n\rangle \geq 0. \tag{10}$$

By (10), this shows

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_0\|^2 + \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n\rangle - 2\langle x_n - x_{n+1}, x_0 - x_n\rangle \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_n - x_{n+1}, x_0 - x_n\rangle\end{aligned}$$

and that is

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2. \tag{11}$$

From (9) and (11), by summing for $n = 1, 2, 3, ...$ in (11), we can obtain

$$\Sigma_{n=1}^N \|x_{n+1} - x_n\|^2 \leq \Sigma_{n=1}^N (\|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2) = \|p_* - x_0\|^2 - \|x_1 - x_0\|^2. \tag{12}$$

Since N is arbitrary, using (12), this shows $\Sigma_{n=1}^\infty \|x_{n+1} - x_n\| < +\infty$, in other words, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{13}$$

Step 4. We show $\lim_{n\to\infty} \|z_n - x_n\| = 0$, $\lim_{n\to\infty} \|z_{n+1} - z_n\| = 0$. Since $x_{n+1} \in C_n$, this shows that

$$\|z_{n+1} - x_{n+1}\|^2 \leq \alpha_n \|x_n - x_{n+1}\|^2 + \beta_n \|w_n - x_{n+1}\|^2 + \gamma_n \|y_n - x_{n+1}\|^2. \tag{14}$$

On the one hand, it can be derived from the definition of w_n that

$$\begin{aligned}\|w_n - x_{n+1}\|^2 &= \|x_n + \sum_{i=1}^{2} \delta_i(x_{n-i+1} - x_{n-i}) - x_{n+1}\|^2 \\ &= \|x_n - x_{n+1} + \sum_{i=1}^{2} \delta_i(x_{n-i+1} - x_{n-i})\|^2 \\ &\leq 2\|x_n - x_{n+1}\|^2 + \sum_{i=1}^{2} 4\delta_i^2 \|x_{n-i+1} - x_{n-i}\|^2.\end{aligned} \tag{15}$$

On the other hand, by the definition of y_n, this implies

$$\begin{aligned}\|y_n - x_{n+1}\|^2 &= \|z_n + \sum_{i=1}^{2}\theta_i(z_{n-i+1} - z_{n-i}) - x_{n+1}\|^2 \\ &= \|z_n - x_{n+1} + \sum_{i=1}^{2}\theta_i(z_{n-i+1} - z_{n-i})\|^2 \\ &\le 2\|z_n - x_{n+1}\|^2 + \sum_{i=1}^{2} 4\theta_i^2\|z_{n-i+1} - z_{n-i}\|^2.\end{aligned} \tag{16}$$

Notice that from (14), by means of Cauchy-Schwarz inequality and AM-GM inequality, we have

$$\begin{aligned}\|z_n - z_{n-1}\|^2 &= \|z_n - x_n\|^2 + \|x_n - z_{n-1}\|^2 + 2\langle z_n - x_n, x_n - z_{n-1}\rangle \\ &\le 2(\|z_n - x_n\|^2 + \|x_n - z_{n-1}\|^2) \\ &\le 2\|z_n - x_n\|^2 + 4[\|x_n - x_{n-1}\|^2 + \|x_{n-1} - z_{n-1}\|^2]\end{aligned} \tag{17}$$

and

$$\begin{aligned}\|z_{n-1} - z_{n-2}\|^2 &= \|z_{n-1} - x_{n-1}\|^2 + \|x_{n-1} - z_{n-2}\|^2 + 2\langle z_{n-1} - x_{n-1}, x_{n-1} - z_{n-2}\rangle \\ &\le 2(\|z_{n-1} - x_{n-1}\|^2 + \|x_{n-1} - z_{n-2}\|^2) \\ &\le 2\|z_{n-1} - x_{n-1}\|^2 + 4[\|x_{n-1} - x_{n-2}\|^2 + \|x_{n-2} - z_{n-2}\|^2]\end{aligned} \tag{18}$$

and

$$\begin{aligned}\|x_{n+1} - z_n\|^2 &= \|z_n - x_n + x_n - x_{n+1}\|^2 \\ &\le 2\|z_n - x_n\|^2 + 2\|x_n - x_{n+1}\|^2.\end{aligned} \tag{19}$$

Combining (15), (16), (17), (18) and (19), let $K = \max\{|\delta_1|, \delta_2\}$, we have

$$\begin{aligned}\|z_{n+1} - x_{n+1}\|^2 &\le \alpha_n\|x_n - x_{n+1}\|^2 + 2\beta_n\|x_n - x_{n+1}\|^2 + \beta_n \cdot \sum_{i=1}^{2} 4\delta_i^2\|x_{n-i+1} - x_{n-i}\|^2 \\ &\quad + 2\gamma_n\|z_n - x_{n+1}\|^2 + \gamma_n \cdot \sum_{i=1}^{2} 4\theta_i^2\|z_{n-i+1} - z_{n-i}\|^2 \\ &\le 4\gamma_n\|z_n - x_n\|^2 + 4\gamma_n\|x_n - x_{n+1}\|^2 + \alpha_n\|x_n - x_{n+1}\|^2 \\ &\quad + 2\beta_n\|x_n - x_{n+1}\|^2 + \beta_n \cdot \sum_{i=1}^{2} 4\delta_i^2\|x_{n-i+1} - x_{n-i}\|^2 \\ &\quad + \gamma_n \cdot \sum_{i=1}^{2} 4\theta_i^2\|z_{n-i+1} - z_{n-i}\|^2 \\ &\le 4\gamma_n\|z_n - x_n\|^2 + 4\gamma_n\|x_n - x_{n+1}\|^2 + \alpha_n\|x_n - x_{n+1}\|^2 \\ &\quad + 2\beta_n\|x_n - x_{n+1}\|^2 + \beta_n \cdot \sum_{i=1}^{2} 4\delta_i^2\|x_{n-i+1} - x_{n-i}\|^2 \\ &\quad + 8\theta_1^2\|z_n - x_n\|^2 + 16\theta_1^2[\|x_n - x_{n-1}\|^2 + \|x_{n-1} - z_{n-1}\|^2] \\ &\quad + 8\theta_2^2\|z_{n-1} - x_{n-1}\|^2 + 16\theta_2^2[\|x_{n-1} - x_{n-2}\|^2 + \|x_{n-2} - z_{n-2}\|^2] \\ &\le (4\gamma_n + 8\theta_1^2)\|z_n - x_n\|^2 + (16\theta_1^2 + 8\theta_2^2)\|z_{n-1} - x_{n-1}\|^2 + 16\theta_2^2\|z_{n-2} - x_{n-2}\|^2 \\ &\quad + (4\gamma_n + \alpha_n + 2\beta_n)\|x_{n+1} - x_n\|^2 + (4\beta_n\delta_1^2 + 16\theta_1^2)\|x_n - x_{n-1}\|^2 \\ &\quad + (4\beta_n\delta_2^2 + 16\theta_2^2)\|x_{n-1} - x_{n-2}\|^2,\end{aligned} \tag{20}$$

which is equivalent to

$$\|z_{n+1}-x_{n+1}\|^2 \le p\|z_n-x_n\|^2+q\|z_{n-1}-x_{n-1}\|^2+r\|z_{n-2}-x_{n-2}\|^2+s\zeta_n, \tag{21}$$

where $p=\sup_{n\ge0}\{4\gamma_n+8\theta_1^2\}$, $q=16\theta_1^2+8\theta_2^2$, $r=16\theta_2^2$, $s=\max_{n\ge0}\{4\gamma_n+\alpha_n+2\beta_n, 4\beta_n\delta_1^2+16\theta_1^2, 4\beta_n\delta_2^2+16\theta_2^2\}$, $\zeta_n=\|x_{n+1}-x_n\|^2+\|x_n-x_{n-1}\|^2+\|x_{n-1}-x_{n-2}\|^2$. It is noted from *Condition 3.4* that

$$p+q+r=\sup_{n\ge0}\{4\gamma_n+8\theta_1^2\}2+16\theta_1^2+8\theta_2^2+16\theta_2^2=\sup_{n\ge0}\{4\gamma_n+24\theta_1^2+24\theta_2^2\}<1,$$

then $p+q+r<1$. By (10), it is easy to know that $\Sigma_{n=1}^{\infty}\zeta_n<+\infty$. Thus, by Lemma 2.6 and (16), for $n\to\infty$ we can obtain

$$\|z_n-x_n\|\to 0. \tag{22}$$

Step 5. We show $\lim_{n\to\infty}\|Tx_n-x_n\|=0$. By (13) and (15), we have

$$\lim_{n\to\infty}\|w_n-x_{n+1}\|=0. \tag{23}$$

By (13) and (22), we have

$$\|z_{n+1}-z_n\|\le\|z_{n+1}-x_{n+1}\|+\|x_{n+1}-x_n\|+\|x_n-z_n\|,$$

that is

$$\lim_{n\to\infty}\|z_{n+1}-z_n\|=0. \tag{24}$$

By (13) and (22), we have

$$\lim_{n\to\infty}\|x_{n+1}-z_n\|=0. \tag{25}$$

From (24) and (25), by (16), it gets

$$\lim_{n\to\infty}\|y_n-x_{n+1}\|=0. \tag{26}$$

Using the fact that $x_{n+1}\in C_n$, we can extract the inequality from (8) as follows

$$\begin{aligned}&\alpha_n\beta_n\|T_1w_n-x_n\|^2+\alpha_n\gamma_n\|T_2y_n-x_n\|^2+\beta_n\gamma_n\|T_1w_n-T_2y_n\|^2\\ &\le\alpha_n\|x_n-x_{n+1}\|^2+\beta_n\|w_n-x_{n+1}\|^2+\gamma_n\|y_n-x_{n+1}\|^2-\|z_{n+1}-x_{n+1}\|^2.\end{aligned} \tag{27}$$

Make use of (13), (22), (23) and (26), by (27) one sees

$$\lim_{n\to\infty}\|T_1w_n-x_n\|=\lim_{n\to\infty}\|T_2y_n-x_n\|=\lim_{n\to\infty}\|T_1w_n-T_2y_n\|=0. \tag{28}$$

Since $\|x_n-w_n\|\le\|x_n-x_{n+1}\|+\|x_{n+1}-w_n\|$ and $\|x_n-y_n\|\le\|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|$, then we can derive from (13), (23) and (26) that

$$\lim_{n\to\infty}\|x_n-w_n\|=0,\ \lim_{n\to\infty}\|x_n-y_n\|=0. \tag{29}$$

With the help of (28) and (29), by the nonexpansiveness of T_1, then for $n\to\infty$ we can get

$$\begin{aligned}\|T_1x_n-x_n\|&\le\|T_1x_n-T_1w_n\|+\|T_1w_n-x_n\|\\ &\le\|x_n-w_n\|+\|T_1w_n-x_n\|\to 0,\end{aligned}$$

this implies that

$$\lim_{n\to\infty}\|T_1x_n-x_n\|=0. \tag{30}$$

With the help of (28) and (29), by the nonexpansiveness of T_2, then for $n\to\infty$ we can get

$$\begin{aligned}\|T_2x_n-x_n\|&\le\|T_2x_n-T_2y_n\|+\|T_2y_n-x_n\|\\ &\le\|x_n-y_n\|+\|T_2y_n-x_n\|\to 0,\end{aligned}$$

this implies that

$$\lim_{n\to\infty}\|T_2x_n - x_n\| = 0. \tag{31}$$

Step 6. We show $\{x_n\}$ strongly converges to $P_{Fix(T_1)\cap Fix(T_2)}(x_0)$. Since $\{x_n\}$ is a bounded sequence, by Lemma 2.2 and (30), (31), this shows that

$$\omega_w(x_n) \subset (Fix(T_1)\cap Fix(T_2)).$$

This together with (9) by Lemma 2.3 guarantees the strong convergence of $\{x_n\}$ to $P_{Fix(T_1)\cap Fix(T_2)}(x_0)$. Since the following inequality by (22) holds

$$\|z_n - P_{Fix(T_1)\cap Fix(T_2)}(x_0)\| \le \|z_n - x_n\| + \|x_n - P_{Fix(T_1)\cap Fix(T_2)}(x_0)\| \to 0,$$

then this implies that the sequence $\{z_n\}$ strongly converges to $P_{Fix(T_1)\cap Fix(T_2)}(x_0)$.

This completes the proof. □

Remark 3.3. Set $\delta_i = 0$, $\theta_i = 0$, $i = 1, 2$, $\alpha_n = 0$, $T_1 = I$ in (7), then the algorithm (7) can degrade into algorithm (1). Set $\delta_2 = 0$, $\theta_2 = 0$, $\alpha_n = 0$, $T_1 = I$ in (7), then the algorithm (7) can degrade into algorithm (2). Furthermore, Lemma 2.6 is applied in the inequality (20) of step 4 on this theorem.

Next, we considered some special cases and obtained some results from the combined use of the one-step inertial method and the two-step inertial method.

Condition 3.6 $\delta_1, \theta_1, \theta_2 \in [\xi_1, \xi_2], \xi_1 \in (-\infty, 0], \xi_2 \in [0, +\infty), \theta_1, \theta_2 \in (-1, 1), \theta_1^2 + \theta_2^2 < 1,$

$$\gamma_n \in \left(0, \frac{1 - 24\theta_1^2 - 24\theta_2^2}{4}\right).$$

If set $\delta_2 = 0$, then we can have the following Corollary 3.4.

Corollary 3.4. *Suppose that the Condition 3.1-Condition 3.3 and Conditin 3.6 hold. The sequences $\{x_n\}$ and $\{z_n\}$ generated by the following manner:*

$$\begin{cases} Set\ z_0, x_0, z_1, x_1, z_2 \in C \\ w_n = x_n + \delta_1(x_n - x_{n-1}) \\ y_n = z_n + \sum_{i=1}^{2}\theta_i(z_{n-i+1} - z_{n-i}) \\ z_{n+1} = \alpha_n x_n + \beta_n T_1 w_n + \gamma_n T_2 y_n \\ C_n = \{p_* \in C \,|\, \|z_{n+1} - p_*\|^2 \le \alpha_n\|x_n - p_*\|^2 + \beta_n\|w_n - p_*\|^2 + \gamma_n\|y_n - p_*\|^2\} \\ Q_n = \{p_* \in C \,|\, \langle x_n - p_*, x_0 - x_n\rangle \ge 0\} \\ x_{n+1} = P_{C_n\cap Q_n}(x_0). \end{cases} \tag{32}$$

Then the sequences $\{x_n\}$ and $\{z_n\}$ defined by (32) converge strongly to $P_{Fix(T_1)\cap Fix(T_2)}(x_0)$.

Next, we are going to state a condition that will be required subsequently.

Condition 3.7 The parameters $\delta_i, \theta_1 \in [\xi_1, \xi_2], i = 1, 2, \xi_1 \in (-\infty, 0], \xi_2 \in [0, +\infty),$

$$\theta_1 \in \left(-\sqrt{\frac{1}{24}}, \sqrt{\frac{1}{24}}\right), \gamma_n \in \left(0, \frac{1 - 24\theta_1^2}{4}\right).$$

If set $\theta_2 = 0$, then we can have the following Corollary 3.5.

Corollary 3.5. *Suppose that the Condition 3.1-Condition 3.3 and Condition 3.7 hold. The sequences $\{x_n\}$ and $\{z_n\}$ generated by the following manner:*

$$\begin{cases} Set\ z_0, x_0, z_1, x_1, x_2 \in C \\ w_n = x_n + \sum_{i=1}^{2} \delta_i (x_{n-i+1} - x_{n-i}) \\ y_n = z_n + \theta_1 (z_n - z_{n-1}) \\ z_{n+1} = \alpha_n x_n + \beta_n T_1 w_n + \gamma_n T_2 y_n \\ C_n = \{p_* \in C | \|z_{n+1} - p_*\|^2 \leq \alpha_n \|x_n - p_*\|^2 + \beta_n \|w_n - p_*\|^2 + \gamma_n \|y_n - p_*\|^2\} \\ Q_n = \{p_* \in C | \langle x_n - p_*, x_0 - x_n \rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{33}$$

Then the sequences $\{x_n\}$ and $\{z_n\}$ defined by (33) converge strongly to $P_{Fix(T_1)\cap Fix(T_2)}(x_0)$.

If set $T_1 = T_2 = T$, then we can have the following Corollary 3.6.

Corollary 3.6. *Suppose that the Condition 3.1-Condition 3.4 and Condition 3.5 hold. The sequences $\{x_n\}$ and $\{z_n\}$ generated by the following manner:*

$$\begin{cases} Set\ z_0, x_0, z_1, x_1, z_2, x_2 \in C \\ w_n = x_n + \sum_{i=1}^{2} \delta_i (x_{n-i+1} - x_{n-i}) \\ y_n = z_n + \sum_{i=1}^{2} \theta_i (z_{n-i+1} - z_{n-i}) \\ z_{n+1} = \alpha_n x_n + \beta_n T w_n + \gamma_n T y_n \\ C_n = \{p_* \in C | \|z_{n+1} - p_*\|^2 \leq \alpha_n \|x_n - p_*\|^2 + \beta_n \|w_n - p_*\|^2 + \gamma_n \|y_n - p_*\|^2\} \\ Q_n = \{p_* \in C | \langle x_n - p_*, x_0 - x_n \rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{34}$$

Then the sequences $\{x_n\}$ and $\{z_n\}$ defined by (34) converge strongly to $P_{Fix(T)}(x_0)$.

Remark 3.7. It is worth noting that the algorithm given in Corollaries 3.4 and 3.5, in its structure, employs a combination of the Polyak inertial method and the two-step inertial method. Such a situation has not been presented in many studies. The construction of the algorithm in Corollary 3.6 is also uncommon, and many researchers have not considered such an approach.

4. Application to Common Split Feasible Problem

The split feasible problem is a classic type of optimization problem, with the core being to find points that satisfy a specific containment relationship in two different spaces. The split feasibility problem has important applications in signal processing, machine learning, medical imaging and other fields. Censor and Elfving [16] introduced split feasible problem (SFP) as follows:

$$find\ x^* \in C\ such\ that\ Ax^* \in Q, \tag{35}$$

where $A : H_1 \to H_2$ is a bounded linear operator, C, Q stand for the nonempty closed convex subset of H_1, H_2 respectively. Throughout the paper, we use Γ to denote the solution set of SFP, $\Gamma := \{x^* \in C | Ax^* \in Q\}$. Note that A^* represents the adjoint of A.

In [17], based on split feasible problem (35), commen split feasible problem (CSFP) was introduced as follows:

$$find\ x^* \in C\ such\ that\ A_1 x^* \in Q\ and\ A_2 x^* \in K, \tag{36}$$

where $A_i : H_1 \to H_2, i = 1, 2$ are a bounded linear operators, C and Q, K stand for the nonempty closed convex subset of H_1, H_2 respectively. Throughout the paper, we use $\bar{\Gamma}$ to denote the solution set of CSFP, $\bar{\Gamma} := \{x^* \in C | A_1 x^* \in Q\ and\ A_2 x^* \in K\}$.

Lemma 4.1. *[18] The u be a solution of SFP (35) (that is $u \in \Gamma$) if and only if u is the fixed point of the operator $P_C(I-\gamma A^*(I-P_Q)A)$, where $\gamma > 0$.*

Lemma 4.2. *[18] For every $\gamma \in R$, the operator $P_C(I-\gamma A^*(I-P_Q)A)$ is nonexpansive, where $0 < \gamma < \frac{2}{\|A\|^2}$.*

It can be obtained from the above lemma that the operator $P_C(I-\gamma A^*(I-P_K)A)$ is nonexpansive, where $0 < \gamma < \frac{2}{\|A\|^2}$. Then we can get the following theorem.

Theorem 4.3. *Let H_1, H_2 be two real Hilbert spaces, the subset $C \subset H_1, Q \subset H_2$ be nonempty closed and convex. The linear operators $A_i : H_1 \to H_2,\ i = 1, 2$ are bounded. P_C is the projection of H_1 into C, P_Q is the projection of H_2 into Q. $\bar{\Gamma} := \{x^* \in C | A_1x^* \in Q \text{ and } A_2x^* \in Q\} \neq \emptyset$, $0 < \gamma < \frac{2}{\|A_i\|^2}, i = 1, 2$. Suppose that the Condition 3.3-Condition 3.5 hold. The sequences $\{x_n\}$ and $\{z_n\}$ generated by the following manner:*

$$\begin{cases} \text{Set } z_0, x_0, z_1, x_1, z_2, x_2 \in C \\ w_n = x_n + \sum_{i=1}^{2} \delta_i(x_{n-i+1} - x_{n-i}) \\ y_n = z_n + \sum_{i=1}^{2} \theta_i(z_{n-i+1} - z_{n-i}) \\ z_{n+1} = \alpha_n x_n + \beta_n P_C(I-\gamma A_1^*(I-P_Q)A_1)w_n + \gamma_n P_C(I-\gamma A_2^*(I-P_K)A_2)y_n \\ C_n = \{p_* \in C | \|z_{n+1} - p_*\|^2 \leq \alpha_n\|x_n - p_*\|^2 + \beta_n\|w_n - p_*\|^2 + \gamma_n\|y_n - p_*\|^2\} \\ Q_n = \{p_* \in C | \langle x_n - p_*, x_0 - x_n\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{37}$$

Then the sequences $\{x_n\}$ and $\{z_n\}$ defined by (37) converge strongly to $P_{\bar{\Gamma}}(x_0)$.

Next, while a special case $Q = K \subseteq H_2$, we also can get the following conclusion.

Theorem 4.4. *Let H_1, H_2 be two real Hilbert spaces, the subset $C \subset H_1, Q \subset H_2$ be nonempty closed and convex. The linear operators $A_i : H_1 \to H_2,\ i = 1, 2$ are bounded. P_C is the projection of H_1 into C, P_Q is the projection of H_2 into Q. $\bar{\Gamma} := \{x^* \in C | A_1x^* \in Q \text{ and } A_2x^* \in Q\} \neq \emptyset$, $0 < \gamma < \frac{2}{\|A_i\|^2}, i = 1, 2$. Suppose that the Condition 3.3-Condition 3.5 hold. The sequences $\{x_n\}$ and $\{z_n\}$ generated by the following manner:*

$$\begin{cases} \text{Set } z_0, x_0, z_1, x_1, z_2, x_2 \in C \\ w_n = x_n + \sum_{i=1}^{2} \delta_i(x_{n-i+1} - x_{n-i}) \\ y_n = z_n + \sum_{i=1}^{2} \theta_i(z_{n-i+1} - z_{n-i}) \\ z_{n+1} = \alpha_n x_n + \beta_n P_C(I-\gamma A_1^*(I-P_Q)A_1)w_n + \gamma_n P_C(I-\gamma A_2^*(I-P_Q)A_2)y_n \\ C_n = \{p_* \in C | \|z_{n+1} - p_*\|^2 \leq \alpha_n\|x_n - p_*\|^2 + \beta_n\|w_n - p_*\|^2 + \gamma_n\|y_n - p_*\|^2\} \\ Q_n = \{p_* \in C | \langle x_n - p_*, x_0 - x_n\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{38}$$

Then the sequences $\{x_n\}$ and $\{z_n\}$ defined by (38) converge strongly to $P_{\bar{\Gamma}}(x_0)$.

5. Numerical experiment

In this section, some numerical tests are used to demonstrate the performance of the proposed algorithms. All our numercial experiments are implemented in MATLAB R2018b on a 11th Gen Intel(R) Core(TM) i5-1155G7 @ 2.50GHz computer with RAM 16.0GB (64-bit operating system, x64-based processor).

In 2010, He et al. [3] studied the numerical implementation problem of the hybrid projection algorithm. Their research results are highly beneficial for scholars to explore the numerical performance of different hybrid projection algorithms. The Euclidean projection of x_0 onto a halfspace $H_{a,b}^- = \{x : \langle a, x\rangle \leq b\}$ is given by $P_{H_{a,b}^-}x = x - \frac{[\langle a,x\rangle - b]_+}{\|a\|^2}a$. Based on the research results in [3], we adopted the comparison method to study and compare the numerical performance of algorithms (1), (2) and (7). We provide two numerical examples to support the main results of this paper. Notice that *DongLu2015* denotes the algorithm (1), *WengYu2025* denotes the algorithm (2), *WengSQ2025* denotes the algorithm (7) in this part. The settings of some parameters are as follows:

- Let $x_0 = k*rand(2,1)$, $z_0 = k*rand(2,1)$, $x_1 = \alpha_n x_0 + \beta_n T_1 x_0 + \gamma_n T_2 x_0$, $z_1 = \alpha_n z_0 + \beta_n T_1 z_0 + \gamma_n T_2 z_0$, $x_2 = \alpha_n x_1 + \beta_n T_1 x_1 + \gamma_n T_2 x_1$, $z_2 = \alpha_n z_1 + \beta_n T_1 z_1 + \gamma_n T_2 z_1$, $\delta_1 = \theta_1 = 0.06$, $\delta_2 = \theta_2 = 0.08$, $\gamma_n = 0.1 \in (0, 0.19)$, $\alpha_n = 0.1$, $\beta_n = 0.8$ in algorithm (7) (WengSQ2025);
- Let $x_0 = k * rand(2,1)$, $z_0 = k * rand(2,1)$, $\alpha_n = 0.8$ in algorithm (1) (DongLu2015);
- Let $x_0 = k * rand(2,1)$, $z_0 = k * rand(2,1)$, $x_1 = \alpha_n x_0 + (1-\alpha_n)Tx_0$, $z_1 = \alpha_n z_0 + (1-\alpha_n)Tz_0$, $\delta_1 = \theta_1 = 0.06$, $\alpha_n = 0.8$ in algorithm (2) (WengYu2025);
- Let error termination criterion as $E_n = \|x_n - x_{n-1}\| < Error$, $Error = 1e-6$, maximum number of iterations as 1000.

Example 5.1. We respectively present two nonexpansive mappings in R^2, which have a common fixed point.

Part 1-1: Define $T_1 : R^2 \to R^2$ by

$$T_1(z) = (\frac{1}{2}z_1, \frac{1}{2}z_2);$$

Part 1-2: Define $T_2 : R^2 \to R^2$ by

$$T_2(z) = (\frac{z_1 + z_2}{2}, \frac{z_1 - z_2}{2}), z = (z_1, z_2) \in R^2.$$

For this example, it is easy to verify that $Fix(T_1) \cap Fix(T_2) \neq \emptyset$, that is $x_* = (0,0) \in Fix(T_1) \cap Fix(T_2)$. The numerical result of this example is reported in Fig.1.

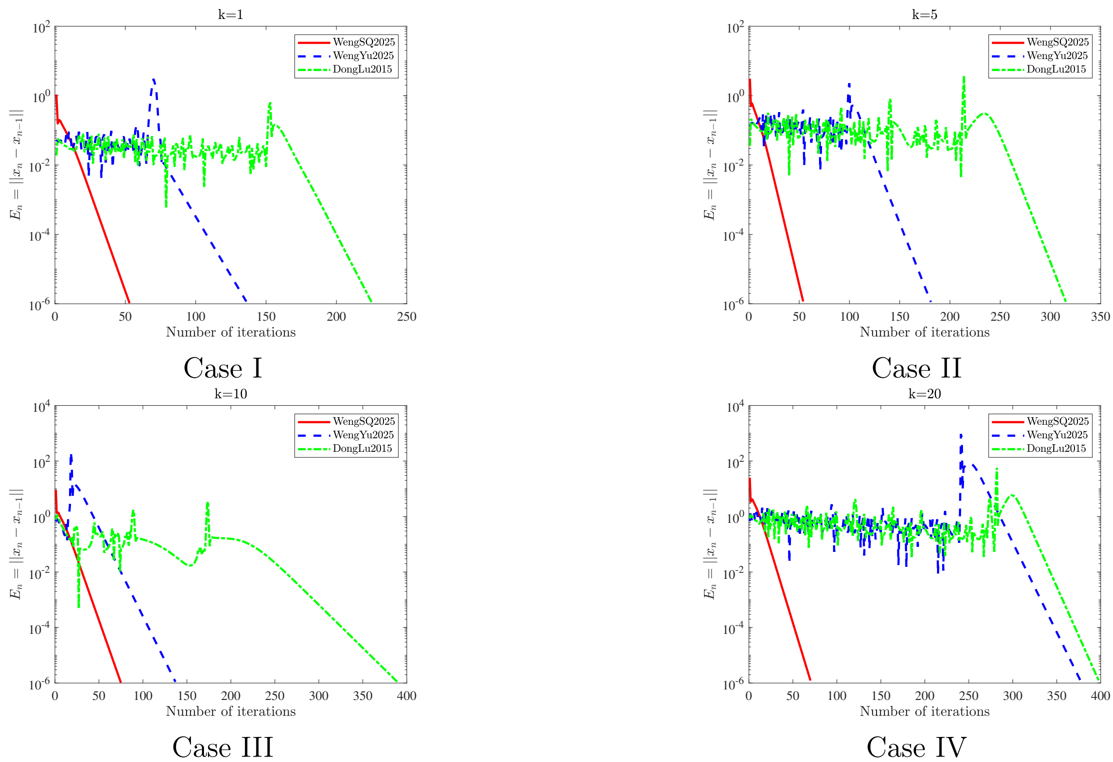

Figure 1: The iterative processes of the three algorithms on Example 5.1

Example 5.2. We respectively present two nonexpansive mappings in R^3, which have a common fixed point.

Part 1-1: Define $T_1 : R^3 \to R^3$ by

$$T_1(z) = (\frac{1}{3}z_1, \frac{1}{3}z_2, \frac{1}{3}z_3);$$

Part 1-2: Define $T_2 : R^3 \to R^3$ by

$$T_2(z) = (\frac{sin(z_1)}{2}, \frac{z_2}{3} + \frac{z_1 - z_3}{5}, \frac{sin(z_3)}{2}), z = (z_1, z_2, z_3) \in R^3.$$

For this example, it is easy to verify that $Fix(T_1) \cap Fix(T_2) \neq \emptyset$, that is $x_* = (0, 0, 0) \in Fix(T_1) \cap Fix(T_2)$. The numerical result of this example is reported in Fig.2.

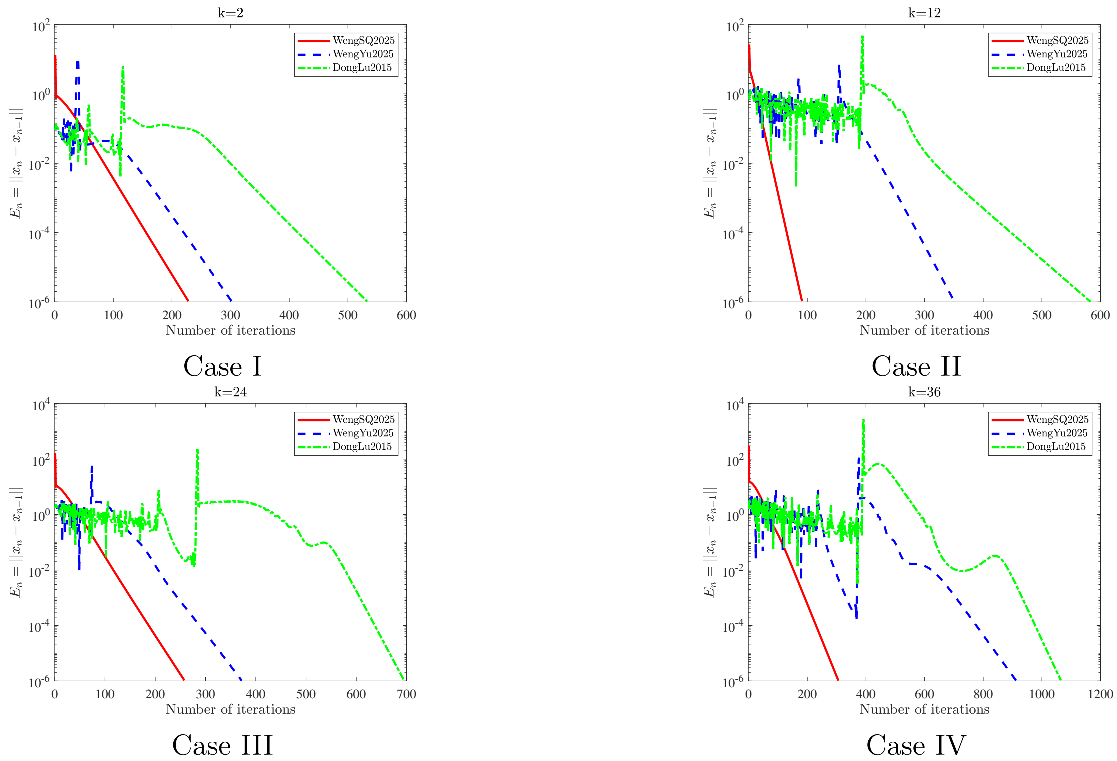

Figure 2: The iterative processes of the three algorithms on Example 5.2

Remark 5.3.

- The selection of parameters in the above-mentioned numerical experiments was made by us based on the parameter conditions. The selection of these parameters is all in line with the underlying assumptions.
- In order to present the fact that both different mappings have the same fixed point, that is, the common fixed point, in a convenient and intuitive way, we have made the definitions of the operators relatively simple, but still effective.
- From the obtained numerical experimental results in Figs. 1-2, it can be seen that the numerical performance of the hybrid projection algorithm with the double two-step inertial method proposed by us is relatively more outstanding.
- In the two numerical examples mentioned above, the initial values of the algorithm we provided were different. However, the selection of the initial values can also have an impact on the iterative efficiency of the algorithm. What we are presenting here is just a part of the demonstration showing the competitive advantage of our algorithm.

- According to the results of our numerical experiments, the two-step inertial method [algorithm (7)] performs relatively better than the single-step inertial method [algorithm (2)].

6. Conclusions

In this paper, we study the iterative approximation of the solutions to the common fixed point problem of two nonexpansive mappings. Based on the classical hybrid projection algorithm and the two-step inertial method, we have constructed a double two-step inertial hybrid projection algorithm. The main research objective of this paper is to explore the feasibility of iterative approximation of the solution to the common fixed point problem of two nonexpansive mappings based on a hybrid projection algorithm with two iterative sequences. The second is to explore the improvement effect of the numerical performance of the algorithm itself after simultaneously applying the two-step inertial method to two iterative sequences. This paper theoretically and successfully establishes the strong convergence theorems of the proposed algorithms, these mean that the iterative approximation of the solution to the common fixed point problem of two nonexpansive mappings based on the hybrid projection algorithm with two iterative sequences is feasible. Furthermore, the numerical experiment results presented in this paper show that by simultaneously accelerating the two iterative sequences using the double two-step inertial method, it plays a positive role in optimizing and improving the iterative performance of the algorithm. The strong convergence algorithms of the two non-expanding operators studied in this paper have more extensive applicable scenarios in theory.

7. Future Work

Regarding this research, based on aspects such as space, operators, and algorithm structure, it can be considered to expand in the following directions.

- First, from the perspective of acceleration methods, we can consider extending the double two-step inertial method to the double multi-step inertial method. The strong convergence of this method has been theoretically proven, and then numerical experiments will be conducted to study the optimization effect of the double multi-step inertial method on the algorithm.
- Secondly, it is worth considering attempting to extend the research results of this paper to uniformly convex Banach spaces, uniformly smooth Banach spaces, and CAT(0) spaces, etc.
- Thirdly, one can further delve into more extensive public fixed-point problems. For instance, the algorithm proposed in this paper can be appropriately modified and restructured to solve problems involving two non-expansive mappings and monotone inclusion, variational inequality problems, etc., thereby further extending the results presented in this paper.

Data vailability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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