



Letters in Nonlinear Analysis and its Applications

Peer Review Scientific Journal

ISSN: 2958-874x

Enriched G-Contractions on Banach Spaces with Directed Graphs

Riyas P^a, Reeja P V^b

^aDepartment of Mathematics, Krishna Menon Govt. Women's College, Pallikunnu, Kannur, India

^bDepartment of Mathematics, Nehru Arts and Science College Kanhangad, Kasaragod, India

Abstract

This paper introduces enriched G-contraction mappings on Banach spaces endowed with a directed graph, generalizing both enriched contractions and G-contractions. We establish the existence and uniqueness of fixed points under suitable conditions by employing an associated averaged operator. It is shown that such mappings are orbitally G-continuous in symmetric graphs, and their iterates yield anchored contractions.

Keywords: Fixed points, Enriched G-contraction, Orbitally G-continuous, Averaged Operator

2010 MSC: 47H10, 47H09, 46B20, 54H25

1. Introduction

The Banach contraction principle [1] is a fundamental result in fixed point theory, ensuring the existence and uniqueness of fixed points together with the convergence of iterative sequences. Over the years, several generalizations of this principle [2, 3] have been developed to widen its scope and applicability. Among these, the class of enriched contractions, introduced by Vasile Berinde and Madalina Pacurar [4], provides a flexible framework that includes classical contractions as well as certain nonexpansive mappings, while still preserving useful approximation properties. In parallel, fixed point theory in a metric space endowed with a directed graph has attracted considerable attention [5, 6, 7]. The introduction of a graph structure

Email addresses: riyasmankadavu@gmail.com (Riyas P), pvreeja@gmail.com (Reeja P V)

allows one to incorporate additional geometric or relational information into the space, thereby extending the applicability of fixed point results to settings where contractive conditions are satisfied only along the edges of the graph. This approach has proven effective in unifying and extending various results in nonlinear analysis. Motivated by these developments, in this paper we introduce and investigate the class of enriched G-contraction mappings on Banach spaces endowed with a directed graph. This class naturally generalizes both classical enriched contractions and G-contractions, and reduces to these well-known notions under appropriate choices of parameters and graph structure.

We first study fundamental properties of enriched G-contractions and establish that, under suitable conditions, iterates of such mappings yield anchored enriched G-contractions at a fixed point. We further prove that enriched G-contractions are orbitally G-continuous when the underlying graph is symmetric. Our main result establishes the existence and uniqueness of fixed points for enriched G-contractions in Banach spaces endowed with directed graphs. The proof is based on the construction of an averaged operator associated with the given mapping, which is shown to be a G-contraction. This enables the use of iterative techniques to obtain convergence to the unique fixed point within the appropriate graph component.

Before presenting the main results, we recall some preliminary definitions and basic concepts related to Banach spaces endowed with directed graphs that will be used throughout the paper.

2. Preliminaries

Definition 2.1. [6] Let X be a Banach space and let $G = (V(G), E(G))$ be a directed graph. We say that X is endowed with the directed graph G if the following conditions hold:

1. $V(G) = X$,
2. $E(G) \subseteq X \times X$,
3. G has no parallel edges, and $(x, x) \in E(G)$ for all $x \in X$.

Thus, a Banach space endowed with a directed graph is represented as a pair (X, G) , where the elements of X serve as vertices, and the graph structure is used to extend notions such as comparison, contraction, and continuity.

Example 2.2. Let $H = \ell^2$, where

$$\ell^2 = \left\{ x = (x_n) : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\},$$

with inner product

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n.$$

Define a directed graph $G = (V(G), E(G))$ as follows:

$$V(G) = H,$$

$$E(G) = \{(x, y) \in H \times H : x_n \leq y_n \text{ for all } n\} \cup \{(x, x) : x \in H\}.$$

Definition 2.3. [6] Let X be a Banach space endowed with a directed graph $G = (X, E(G))$. A mapping $T : X \rightarrow X$ is said to be *edge preserving* if

$$(x, y) \in E(G) \implies (Tx, Ty) \in E(G).$$

Definition 2.4 (Orbitally Continuous Mapping). [6] Let X be a Banach space. A mapping $T : X \rightarrow X$ is said to be *orbitally continuous* if for all $x, y \in X$ and for any sequence $\{k_n\}$ of positive integers,

$$T^{k_n}(x) \rightarrow y \implies T(T^{k_n}(x)) \rightarrow T(y).$$

Definition 2.5. [6] Let X be a Banach space endowed with a directed graph G . A mapping $T : X \rightarrow X$ is said to be G -continuous if for any $x \in X$ and any sequence $(x_n) \subset X$,

$$x_n \rightarrow x \quad \text{and} \quad (x_n, x_{n+1}) \in E(G) \quad \forall n \in \mathbb{N}$$

imply

$$T(x_n) \rightarrow T(x).$$

Definition 2.6. [6] Let X be a Banach space. A mapping $T : X \rightarrow X$ is called *orbitally G -continuous* if for all $x, y \in X$ and for any sequence $(k_n) \subset \mathbb{N}$,

$$T^{k_n}(x) \rightarrow y \quad \text{and} \quad (T^{k_n}(x), T^{k_{n+1}}(x)) \in E(G)$$

imply

$$T(T^{k_n}(x)) \rightarrow T(y).$$

Definition 2.7. [6] Let X be a Banach space endowed with a directed graph $G = (X, E(G))$. A mapping $T : X \rightarrow X$ is called a G -contraction if:

1. (Edge preserving):

$$(x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G), \quad \forall x, y \in X,$$

2. (Contractive condition): There exists $\alpha \in (0, 1)$ such that

$$(x, y) \in E(G) \Rightarrow \|T(x) - T(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in X.$$

Definition 2.8. Let X be a Banach space endowed with a directed graph $G = (X, E(G))$. We say that G is *convexity compatible* if for all $x, y, u, v \in X$ and $\lambda \in [0, 1]$,

$$(x, u), (y, v) \in E(G) \Rightarrow ((1 - \lambda)x + \lambda y, (1 - \lambda)u + \lambda v) \in E(G).$$

Definition 2.9. [6] A graph G is said to be *connected* if there exists a path between every pair of vertices in G .

A graph G is said to be *weakly connected* if its underlying undirected graph $\tilde{G}$ is connected.

Definition 2.10. [6] Let G be a graph such that $E(G)$ is symmetric, and let $x \in V(G)$. The subgraph G_x consisting of all vertices and edges that lie on some path beginning at x is called the *component of G containing x*

Equivalently, define a relation R on $V(G)$ by

$$yRz \quad \text{if there exists a path in } G \text{ from } y \text{ to } z.$$

Then R is an equivalence relation, and

$$V(G_x) = [x]_G,$$

where $[x]_G$ denotes the equivalence class of x under R .

Remark 2.11. [6] The subgraph G_x is connected.

3. Main Results

Definition 3.1 (Enriched G -Contraction). A mapping $T : X \rightarrow X$ is called an *enriched G -contraction* if there exist constants $b \geq 0$ and $\theta \in [0, b + 1)$ such that:

1. T is edge preserving;
2. for all $(x, y) \in E(G)$,

$$\|b(x - y) + Tx - Ty\| \leq \theta\|x - y\|.$$

We denote such a mapping by saying that T is a G -(b, θ) enriched contraction.

Remark 3.2. 1. If $b = 0$, then T reduces to a classical G -contraction.

2. If $E(G) = X \times X$, then the definition reduces to the classical enriched contraction on Banach spaces.

Definition 3.3. Let X be a Banach space endowed with a directed graph G and let $p \in X$. A mapping $T : X \rightarrow X$ is said to be an enriched G -contraction anchored at p if $Tp = p$ and there exist constants $b \geq 0$ and $\theta \in [0, b + 1)$ such that

$$\|b(x - p) + Tx - p\| \leq \theta\|x - p\|, \quad \forall x \in X \text{ with } (x, p) \in E(G).$$

Theorem 3.4. Let X be a Banach space endowed with a directed graph G and let $T : X \rightarrow X$ be an G -(b, θ) enriched-contraction with $\theta + b < 1$. Assume that T has a fixed point $p \in X$ and $(x, p) \in E(G)$ for all $x \in X$. Then the mapping T^2 is an enriched G -contraction anchored at p .

Proof. Let $x \in X$. Since T is edge-preserving and $(x, p) \in E(G)$, we have $(Tx, p) \in E(G)$. Applying the enriched G -contraction condition to (Tx, p) , we obtain

$$\|b(Tx - p) + T^2x - p\| \leq \theta\|Tx - p\|. \quad (1)$$

Now write

$$Tx - p = (b(x - p) + Tx - p) - b(x - p).$$

Thus

$$\|Tx - p\| \leq \|b(x - p) + Tx - p\| + b\|x - p\|.$$

Applying the enriched G -contraction condition to (x, p) ,

$$\|b(x - p) + Tx - p\| \leq \theta\|x - p\|.$$

Hence, by substituting in the above

$$\|Tx - p\| \leq (\theta + b)\|x - p\|. \quad (2)$$

Substituting this into (1), we get

$$\|b(Tx - p) + T^2x - p\| \leq \theta(\theta + b)\|x - p\|. \quad (3)$$

Now observe that

$$b(Tx - p) = b[(x - p) - (x - Tx)] = b(x - p) - b(x - Tx).$$

Hence

$$b(x - p) = b(Tx - p) + b(x - Tx).$$

Thus,

$$b(x - p) + T^2x - p = [b(Tx - p) + T^2x - p] + b(x - Tx).$$

Taking norms and using triangle inequality,

$$\|b(x - p) + T^2x - p\| \leq \|b(Tx - p) + T^2x - p\| + b\|x - Tx\|.$$

Using (3), we get

$$\|b(x - p) + T^2x - p\| \leq \theta(\theta + b)\|x - p\| + b\|x - Tx\|. \quad (4)$$

Again, we have

$$\|x - Tx\| = \|(x - p) + (p - Tx)\| \leq \|x - p\| + \|Tx - p\| \leq (1 + \theta + b)\|x - p\|.$$

Substituting in (4)

$$\|b(x - p) + T^2x - p\| \leq \theta'\|x - p\|,$$

where $\theta' = \theta(\theta + b) + b(1 + \theta + b) < \theta + b(1 + 1) = \theta + b + b < 1 + b = b + 1$.

□

Corollary 3.5. *Let X be a Banach space endowed with a directed graph G and let $T : X \rightarrow X$ be an G -(b, θ) enriched-contraction with $\theta + b < 1$. Assume that T has a fixed point $p \in X$ and $(x, p) \in E(G)$ for all $x \in X$. Then the mapping T^n is an enriched G -contraction anchored at p .*

Example 3.6. Let $X = \mathbb{R}$ with the usual norm and define a directed graph G by

$$E(G) = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}.$$

Define $T : X \rightarrow X$ by

$$T(x) = \begin{cases} \frac{1}{2}x, & x \geq 0, \\ 2x, & x < 0. \end{cases}$$

Observe that $p = 0$ is the unique fixed point of T and for $(x, y) \in E(G)$, we have $x, y \geq 0$, and hence

$$Tx - Ty = \frac{1}{2}(x - y).$$

Thus,

$$b(x - y) + Tx - Ty = \left(b + \frac{1}{2}\right)(x - y),$$

and therefore

$$\|b(x - y) + Tx - Ty\| = \left(b + \frac{1}{2}\right)\|x - y\|.$$

Choosing $b = 0.2$, we obtain $\theta = 0.7$, and hence

$$\theta + b = 0.9 < 1.$$

Thus, T is a enriched G -contraction. For $x \geq 0$, we have $T^2(x) = T\left(\frac{1}{2}x\right) = \frac{1}{4}x$. and since $p = 0$, $\|b(x - p) + T^2x - p\| = \left|bx + \frac{1}{4}x\right| = \left(b + \frac{1}{4}\right)|x|$.

Hence,

$$\|b(x - p) + T^2x - p\| = \theta'\|x - p\|,$$

where

$$\theta' = b + \frac{1}{4}.$$

Thus, T^2 is a G -enriched contraction anchored at the fixed point $p = 0$.

Theorem 3.7. *Let $T : X \rightarrow X$ be an enriched G -contraction and let X be a Banach Space endowed with a directed graph G . Suppose that G is symmetric. Then T is orbitally G -continuous.*

Proof. Let $x \in X$ and suppose $(k_n) \subset \mathbb{N}$ is a sequence such that

$$T^{k_n}(x) \rightarrow y \quad \text{as } n \rightarrow \infty,$$

and

$$(T^{k_n}(x), T^{k_{n+1}}(x)) \in E(G).$$

Since G is symmetric, $(u, v) \in E(G)$ implies $(v, u) \in E(G)$. Now, since T is an enriched G -contraction, there exists $b \geq 0$ and $\theta \in [0, b + 1)$ such that: for all $(u, v) \in E(G)$,

$$\|b(u - v) + Tu - Tv\| \leq \theta\|u - v\|.$$

Applying the enriched G -contraction condition to $(T^{k_n}(x), y)$, we obtain

$$\|b(T^{k_n}(x) - y) + T(T^{k_n}(x)) - T(y)\| \leq \theta\|T^{k_n}(x) - y\|.$$

Now,

$$T(T^{k_n}(x)) - T(y) = [b(T^{k_n}(x) - y) + T(T^{k_n}(x)) - T(y)] - b(T^{k_n}(x) - y),$$

and hence,

$$\|T(T^{k_n}(x)) - T(y)\| \leq (\theta + b)\|T^{k_n}(x) - y\|.$$

Since $T^{k_n}(x) \rightarrow y$, it follows that

$$\|T(T^{k_n}(x)) - T(y)\| \rightarrow 0.$$

Therefore,

$$T^{k_{n+1}}(x) = T(T^{k_n}(x)) \rightarrow T(y).$$

Hence, T is orbitally G -continuous □

Theorem 3.8. Let X be a Banach space endowed with a directed graph G and $T : X \rightarrow X$ be a $G - (b, \theta)$ enriched contraction. Assume that:

1. G is symmetric and weakly connected;
2. There exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$;
3. G is convexity compatible.

Then T has a unique fixed point x^* in the G -component of x_0 .

Proof. Define a mapping $S : X \rightarrow X$ by

$$Sx = \frac{1}{b+1}(bx + Tx), \quad \forall x \in X.$$

Let $(x, y) \in E(G)$. Since T is edge-preserving, $(Tx, Ty) \in E(G)$. Hence,

$$\|b(x - y) + Tx - Ty\| \leq \theta\|x - y\|.$$

Dividing by $b + 1$, we obtain

$$\|Sx - Sy\| = \frac{1}{b+1}\|b(x - y) + Tx - Ty\| \leq \frac{\theta}{b+1}\|x - y\|.$$

Set $\lambda = \frac{\theta}{b+1}$. Since $\theta < b + 1$, it follows that $\lambda \in [0, 1)$, and therefore S is a G -contraction. Next, since (x_0, x_0) and (x_0, Tx_0) belong to $E(G)$, and the graph structure is convex, it follows that

$$(x_0, Sx_0) \in E(G).$$

Moreover, as T is edge preserving, S is also edge preserving, and hence $(S^n x_0, S^{n+1} x_0)$ is in $E(G)$ for all $n \in \mathbb{N}$. Since S is a G -contraction on the Banach Space X , the sequence $\{S^n x_0\}$ converges to some $x^* \in X$. Furthermore, x^* is the unique fixed point of S as S is an averaged operator.

Thus,

$$x^* = Sx^* = \frac{1}{b+1}(bx^* + Tx^*) \Leftrightarrow Tx^* = x^*.$$

This would imply that x^* is also a unique fixed point of T . Since G is weakly connected and symmetric, the orbit $\{T^n x_0\}$ lies in the same G -component as x_0 , and hence is connected to x^* . $\square$

Example 3.9. Let $X = \mathbb{R}$ with the usual norm and define a directed graph G by $E(G) = X \times X$. Then G is symmetric, weakly connected, and convexity compatible. Define $T : X \rightarrow X$ by $T(x) = -\frac{1}{2}x + \frac{x}{4(1+|x|)}$, $\forall x \in \mathbb{R}$. Then T is a G -(b, θ) enriched contraction for $b = \frac{1}{2}$ and $\theta = \frac{1}{4}$. Let $x, y \in X$. Then

$$\begin{aligned} \|b(x-y) + Tx - Ty\| &= \left\| \frac{1}{2}(x-y) + \left[-\frac{1}{2}(x-y) + \frac{1}{4} \left(\frac{x}{1+|x|} - \frac{y}{1+|y|} \right) \right] \right\| \\ &= \frac{1}{4} \left(\left\| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right\| \right) \\ &\leq \frac{1}{4} \|x-y\| \end{aligned}$$

(Since $\frac{x}{1+|x|}$ is a Lipschitz function with constant 1) Take $x_0 = 0$. Then $(x_0, Tx_0) = (0, 0) \in E(G)$. Solving $x = T(x)$, we get $x = 0$ is the only solution. Hence the unique fixed point is $x^* = 0$. The mapping T is a G -(b, θ) enriched contraction that satisfies all the assumptions of Theorem 3.8 and it admits a unique fixed point.

References

- [1] S. Banach, *Sur Les Operations Dans Les Ensembles Abstraites et Leur Applications aux Equations Integrales*. Fundamenta Mathematicae 3, (1922): 133–181
- [2] R P Aggarwal, Donal O'Regan and Sahu: *Fixed point theory for Lipschitzian-type mappings with applications*, Topological Fixed Point Theory and Its Applications Springer, Vol 6, New York, 2009.
- [3] W A Kirk, *Approximate Fixed Points of nonexpansive maps*, Fixed Point Theory, 10(2)(2009), 275-288 "
- [4] V. Berinde and M. Păcurar, *Approximating fixed points of enriched contractions in Banach spaces*, Journal of Fixed Point Theory and Applications, 22, Article 38, 2020.
- [5] C. Chifu and G. Petrusel, *New results on coupled fixed point theory in metric spaces endowed with a directed graph*, Fixed Point Theory and Applications, 2014 (2014), 151.
- [6] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., 136(4) (2008), 1359–1373.
- [7] M. Younis, D. Singh, M. Asadi and V. Joshi, *Results on contractions of Reich type in graphical b-metric spaces with applications*, Filomat, 33(17) (2019), 5723–5735.
- [8] R P Aggarwal, Maria Meehan and Donal O'Regan : *Fixed point theory And Applications*, Cambridge Tracts in Mathematics, Vol. 141, Cambridge University press 2001
- [9] A Granas and J Dugundji, *Fixed Point Theory*, Springer Monographs in Mathematics, New York 2003
- [10] Riyas P, Harikrishnan P K *Weakly Enriched Contractions and Fixed point Approximations*, Palestine Journal of Mathematics Vol. 00(1)(2026)