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On some fixed points of integral type contractive mappings in parametric b-metric spaces

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Abstract

In this paper, we establish a new fixed point theorem for nonlinear mappings satisfying an integral-type contractive condition in complete parametric b-metric spaces. The fixed point theorem proved herein extends well-known results of the theory of fixed points. It also generalizes various existing fixed point theorems in metric as well as b-metric spaces. The classical fixed point theorems for Banach, Kannan, and Chatterjea contractions of integral type are derived as corollaries from the main theorem as special cases. To validate these results, several illustrative examples are also provided.

Keywords: Fixed point, parametric b-metric space, integral type contraction, Banach contraction principle, Kannan contraction, and Chatterjea contraction.
2010 MSC: 47H10, 54H25.

1. Introduction

Fixed point theory occupies a very fundamental position in various branches of applied mathematics. One of the most celebrated results of this theory is the Banach contraction principle, established in 1922 [2]. After this, many research papers appeared in the literature of fixed point theory, which proved many

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extensions and generalizations of the Banach contraction principle. Several generalized metric structures have been introduced to extend the Banach contraction principle. One of them is the concept of a b-metric space. Czerwik [8] proposed the concept of a b-metric space. In a b-metric space he replaced the triangle inequality by a relaxed inequality that involves a constant coefficient b. In this way, the introduction of b-metric spaces has opened new avenues for research in fixed point theory.

On the other hand, Branciari [4] introduced integral-type contractive conditions along with a fixed point theorem for mappings satisfying such contractive conditions. This inspired many researchers to investigate the existence of fixed points of such mappings in various generalized metric space structures. In particular, integral analogues of several classical contractions, including those of Kannan and Chatterjea, have been proved in different metric spaces [4, 5, 9, 16].

The theory of parametric b-metric spaces was introduced by Hussain *et al.* [14]. Further, Daheriya *et al.* [10] and Chauhan *et al.* [6] studied and provided another useful framework for nonlinear mappings of different types. To date, many fixed-point results have been established in parametric b-metric spaces. Recently, Kannan *et al.* [17] have established fixed point theorems for integral type contractive mappings in parametric metric spaces.

Inspired by these developments in the literature of fixed point theory, in this paper, we establish a new fixed point theorem for mappings that satisfy a contractive condition of integral-type in complete parametric b-metric spaces. As a result of the main theorem, we obtain Banach, Kannan, and Chatterjea-type fixed point theorems for integral type contractive mappings in a complete parametric b-metric space. These results extend and unify several existing fixed point theorems in the literature.

2. Preliminaries

The notion of parametric b-metric spaces was introduced by Hussain *et al.* [14] as a generalization of b-metric spaces. In this section, we recall several definitions and basic results about parametric b-metric spaces.

Definition 2.1. [14] Let X be a nonempty set and let $d : X \times X \times (0,\infty) \to [0,\infty)$ be a function. Then (X,d) is called a *parametric b-metric space* if there exists a constant $k \geq 1$ such that, for all $x, y, z \in X$ and $t > 0$:

(i) $d(x,y,t) = 0$ if and only if $x = y$.

(ii) $d(x,y,t) = d(y,x,t)$.

(iii) $d(x,z,t) \leq k[d(x,y,t) + d(y,z,t)]$.

The constant k is called the *coefficient* of the parametric b-metric.

Definition 2.2. [14] Let (X,d) be a parametric b-metric space. A sequence $\{x_n\}$ in X is said to *converge* to $x \in X$ if

$$\lim_{n\to\infty} d(x_n, x, t) = 0$$

for every $t > 0$. In this case, we write $x_n \to x$.

Definition 2.3. [14] A sequence $\{x_n\}$ in a parametric b-metric space (X,d) is called a *Cauchy sequence* if

$$\lim_{m,n\to\infty} d(x_m, x_n, t) = 0$$

for every $t > 0$.

Definition 2.4. [14] A parametric b-metric space (X,d) is said to be *complete* if every Cauchy sequence in X converges to a point of X.

Definition 2.5. [14] Let (X,d) be a parametric b-metric space and $T : X \to X$ be a mapping. We say T is *continuous* at x in X if, for any sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$, we have $Tx_n \to Tx$ as $n \to \infty$.

The following Proposition was stated as a Lemma by Daheriya *et al.*[10] without proof. We state it here together with a proof.

Proposition 2.6. *Let (X, d) be a parametric b-metric space with coefficient k. Then the limits of sequences are unique.*

Proof. Suppose $\{x_n\}$ converges to x and y. Then

$$d(x, y, t) \leq k[d(x, x_n, t) + d(x_n, y, t)], \qquad \forall t > 0.$$

Letting $n \to \infty$, we obtain

$$d(x, y, t) = 0, \qquad \forall t > 0.$$

Hence, $x = y$. □

Example 2.7. [14] Let $X = [0, \infty)$ and $d(x, y, t) = t \mid x - y \mid^p$. Then d is a parametric b-metric on X with constant $k = 2^{p-1}$.

Example 2.8. Let $X = \mathbb{R}$ and define, for all, $x, y \in X$

$$d(x, y, t) = t|x - y|.$$

Then (X, d) is a parametric b-metric space with coefficient $k = 1$. Clearly,

$$d(x, z, t) = t|x - z| \leq t(|x - y| + |y - z|) = d(x, y, t) + d(y, z, t).$$

Lemma 2.9. *Let $g : [0, \infty) \to (0, \infty)$ be continuous and nonincreasing. Define*

$$G(r) = \int_0^r g(u)\, du, \qquad r \geq 0.$$

Then G is continuous and strictly increasing, and $G(r) = 0$ if and only if $r = 0$. Moreover, G is subadditive and $G(cr) \leq cG(r)$ for all $r \geq 0$ and $c \geq 1$.

3. Main Result

Theorem 3.1. *Let (X, d) be a complete parametric b-metric space with coefficient $k \geq 1$. Suppose that $f : X \to X$ satisfies*

$$\int_0^{d(fx,fy,t)} g(u)\, du \leq \lambda \int_0^{d(x,y,t)} g(u)\, du + \mu \int_0^{d(x,fx,t)} g(u)\, du + \nu \int_0^{d(y,fy,t)} g(u)\, du \tag{1}$$

for all $x, y \in X$ and $t > 0$, where $g : [0, \infty) \to (0, \infty)$ is continuous and nonincreasing. Let $\lambda, \mu, \nu \in [0, 1)$ satisfy $k(\lambda + \nu) + \mu < 1$. Then f has a unique fixed point in X.

Proof. Let $t > 0$ and for any $x_0 \in X$ we define a sequence $\{x_n\}$ as follows:

$$x_{n+1} = fx_n.$$

Putting $x = x_n$ and $y = x_{n-1}$ in (1), we get

$$\int_0^{d(x_{n+1},x_n,t)} g(u)\,du \le \lambda \int_0^{d(x_n,x_{n-1},t)} g(u)\,du + \mu \int_0^{d(x_n,x_{n+1},t)} g(u)\,du + \nu \int_0^{d(x_{n-1},x_n,t)} g(u)\,du.$$

Rewriting this as

$$\int_0^{d(x_{n+1},x_n,t)} g(u)\,du \le \frac{(\lambda+\nu)}{(1-\mu)} \int_0^{d(x_n,x_{n-1},t)} g(u)\,du = \alpha \int_0^{d(x_n,x_{n-1},t)} g(u)\,du,$$

where $\alpha = \dfrac{\lambda+\nu}{1-\mu} < 1$. Continuing in this way, we get

$$\int_0^{d(x_{n+1},x_n,t)} g(u)\,du \le \alpha^{n-1} \int_0^{d(x_1,x_0,t)} g(u)\,du. \tag{2}$$

Hence, by Lemma 2.9, we obtain

$$\lim_{n\to\infty} \int_0^{d(x_n,x_{n-1},t)} g(u)\,du = 0.$$

We claim that $\lim_{n\to\infty} d(x_n, x_{n-1}, t) = 0$. Suppose, on the contrary, that

$$\limsup_{n\to\infty} d(x_n, x_{n-1}, t) = \delta > 0.$$

Then there exists a subsequence $\{d(x_{n_i}, x_{n_i-1}, t)\}$ such that

$$d(x_{n_i}, x_{n_i-1}, t) \ge \frac{\delta}{2} > 0$$

for every i. Hence, by Lemma 2.9, we obtain

$$\int_0^{d(x_{n_i},x_{n_i-1},t)} g(u)\,du \ge \int_0^{\frac{\delta}{2}} g(u)\,du > 0.$$

This contradicts the fact that $\lim_{n\to\infty} \int_0^{d(x_n,x_{n-1},t)} = 0$. Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n-1}, t) = 0, \qquad \forall t > 0. \tag{3}$$

We now prove that $\{x_n\}$ is a Cauchy sequence. Let $m > n$. Repeated application of (iii) of Definition 2.1 gives

$$\begin{aligned} d(x_m, x_n, t) &\le k[d(x_m, x_{n+1}, t) + d(x_{n+1}, x_n, t)] \\ &\le k^2 d(x_m, x_{n+2}, t) + k^2 d(x_{n+2}, x_{n+1}, t) + k d(x_{n+1}, x_n, t) \\ &\le k^{m-n-1} d(x_m, x_{m-1}, t) + k^{m-n-1} d(x_{m-1}, x_{m-2}, t) \\ &\quad + \cdots + k^2 d(x_{n+2}, x_{n+1}, t) + k d(x_{n+1}, x_n, t). \end{aligned}$$

Using Lemma 2.9, it follows that

$$\begin{aligned} \int_0^{d(x_m,x_n,t)} g(u)\,du \le k^{m-n-1} \int_0^{d(x_m,x_{m-1},t)} g(u)\,du + k^{m-n-1} \int_0^{d(x_{m-1},x_{m-2},t)} g(u)\,du + \cdots \\ + k^2 \int_0^{d(x_{n+2},x_{n+1},t)} g(u)\,du + k \int_0^{d(x_{n+1},x_n,t)} g(u)\,du. \end{aligned}$$

Using (2), we obtain

$$\begin{aligned}\int_0^{d(x_m,x_n,t)} g(u)\,du &\le k^{m-n-1}\alpha^{m-1}\int_0^{d(x_1,x_0,t)} g(u)\,du + k^{m-n-1}\alpha^{m-2}\int_0^{d(x_1,x_0,t)} g(u)\,du + \cdots \\ &\quad + k^2\alpha^{n+1}\int_0^{d(x_1,x_0,t)} g(u)\,du + k\alpha^n\int_0^{d(x_1,x_0,t)} g(u)\,du \\ &\le \alpha^n\left[k + k^2\alpha + \cdots + k^{m-n-1}\alpha^{m-n-2}\right]\int_0^{d(x_1,x_0,t)} g(u)\,du \\ &\le \alpha^n \frac{k}{1-k\alpha}\int_0^{d(x_1,x_0,t)} g(u)\,du \quad (\because k\alpha < 1).\end{aligned}$$

Hence,

$$\lim_{m,n\to\infty}\int_0^{d(x_m,x_n,t)} g(u)\,du = 0. \tag{4}$$

We claim that

$$\lim_{m,n\to\infty} d(x_m,x_n,t) = 0.$$

Suppose, to the contrary, that there exist $\delta > 0$ and subsequences $\{m_i\}$ and $\{n_i\}$ of $\mathbb{N}$ such that

$$d(x_{m_i},x_{n_i},t) \ge \delta$$

for every i. Again, using of Lemma 2.9,

$$\int_0^{d(x_{m_i},x_{n_i},t)} g(u)\,du \ge \int_0^{\delta} g(u)\,du > 0,$$

which contradicts (4). Therefore,

$$\lim_{m,n\to\infty} d(x_m,x_n,t) = 0$$

for every $t > 0$. We conclude that $\{x_n\}$ is a Cauchy sequence. By completeness of (X,d), there exists $x^* \in X$ such that $x_n \to x^*$, that is,

$$\lim_{n\to\infty} d(x_n,x^*,t) = 0, \quad \forall\, t > 0.$$

We now prove that x^* is a fixed point of f. Applying the contractive condition (1) with $x = x_n$, $y = x^*$, and fixed $t > 0$, we obtain

$$\int_0^{d(fx_n,fx^*,t)} g(u)\,du \le \lambda\int_0^{d(x_n,x^*,t)} g(u)\,du + \mu\int_0^{d(x_n,fx_n,t)} g(u)\,du + \nu\int_0^{d(x^*,fx^*,t)} g(u)\,du.$$

It follows that

$$\int_0^{d(x_{n+1},fx^*,t)} g(u)\,du \le \lambda\int_0^{d(x_n,x^*,t)} g(u)\,du + \mu\int_0^{d(x_n,x_{n+1},t)} g(u)\,du + \nu\int_0^{d(x^*,fx^*,t)} g(u)\,du. \tag{5}$$

Using properties of $G(r)$, it follows that

$$\lim_{n\to\infty}\int_0^{d(x_n,x^*,t)} g(u)\,du = 0 \quad \text{and} \quad \lim_{n\to\infty}\int_0^{d(x_n,x_{n+1},t)} g(u)\,du = 0.$$

Now, taking limit as $n \to \infty$ in (5), we obtain

$$\int_0^{d(x^*,fx^*,t)} g(u)\,du \le \nu\int_0^{d(x^*,fx^*,t)} g(u)\,du.$$

This yields

$$(1-\nu)\int_0^{d(x^*,fx^*,t)} g(u)\,du \le 0.$$

As $\nu < 1$, we have $1-\nu > 0$, so

$$\int_0^{d(x^*,fx^*,t)} g(u)\,du = 0.$$

Since $g(u) > 0$ for $u > 0$ and $t > 0$ is arbitrary, we conclude that

$$d(x^*, fx^*, t) = 0, \quad \forall t > 0.$$

Using the Definition 2.1 of parametric b-metric, $fx^* = x^*$. Hence, x^* is a fixed point of f.

To prove uniqueness, suppose that y^* is another fixed point of f. Then

$$fx^* = x^* \quad \text{and} \quad fy^* = y^*.$$

Applying the contractive condition (1) with $x = x^*$ and $y = y^*$, we obtain

$$\int_0^{d(fx^*,fy^*,t)} g(u)\,du \le \lambda \int_0^{d(x^*,y^*,t)} g(u)\,du + \mu \int_0^{d(x^*,fx^*,t)} g(u)\,du + \nu \int_0^{d(y^*,fy^*,t)} g(u)\,du, \quad \forall t > 0.$$

Since $fx^* = x^*$ and $fy^* = y^*$, we have

$$d(x^*, fx^*, t) = 0 \quad \text{and} \quad d(y^*, fy^*, t) = 0, \quad \forall t > 0.$$

Hence, the above inequality reduces to

$$\int_0^{d(x^*,y^*,t)} g(u)\,du \le \lambda \int_0^{d(x^*,y^*,t)} g(u)\,du, \quad \forall t > 0.$$

Rewriting this,

$$(1-\lambda)\int_0^{d(x^*,y^*,t)} g(u)\,du \le 0, \quad \forall t > 0.$$

Since $0 \le \lambda < 1$, we have $1-\lambda > 0$, implying $\displaystyle\int_0^{d(x^*,y^*,t)} g(u)\,du = 0$, $\forall t > 0$.

Since $g(u) > 0$ for $u > 0$, it follows that $d(x^*, y^*, t) = 0$ for every $t > 0$. Therefore, $x^* = y^*$, and hence, the fixed point of f is unique. □

Example 3.2. Let $X = [0, 1]$ and define $d(x, y, t) = t|x-y|^2$. Then (X, d) is a complete parametric b-metric space with coefficient $k = 2$. Define

$$fx = \begin{cases} \frac{x}{3}, & 0 \le x < 1, \\ \frac{1}{6}, & x = 1. \end{cases}$$

Let $g(u) = 1$. Then $G(r) = \int_0^r g(u)\,du = r$ is strictly increasing. We want to show that f satisfies (1) with $\lambda = 0, \mu = \frac{1}{4} = \nu$. That is,

$$\int_0^{d(fx,fy,t)} g(u)\,du \le \frac{1}{4}\left\{\int_0^{d(x,fx,t)} g(u)\,du + \int_0^{d(y,fy,t)} g(u)\,du\right\}$$

$$\implies d(fx, fy, t) \le \frac{1}{4}\left(d(x, fx, t) + d(y, fy, t)\right). \tag{6}$$

Case 1: If $x = 1$ and $y = 1$, then $fx = \frac{1}{6}$, $fy = \frac{1}{6}$.

$$d(fx, fy, t) = t\left|\frac{1}{6} - \frac{1}{6}\right|^2 = 0.$$

Therefore, the contractive condition holds trivially.

Case 2: If $x = 1$ and $y \in [0, 1)$, then $fx = \frac{1}{6}, fy = \frac{y}{3}$.

$$d(fx, fy, t) = t\left|\frac{1}{6} - \frac{y}{3}\right|^2,$$

$$d(x, fx, t) = t\left|1 - \frac{1}{6}\right|^2,$$

$$d(y, fy, t) = t\left|y - \frac{y}{3}\right|^2$$

$$d(x, y, t) = t\left|1 - y\right|^2.$$

Consider

$$\begin{aligned} d(fx, fy, t) = t\left|\frac{1}{6} - \frac{y}{3}\right|^2 &\leq t\left(\frac{1}{6}\right)^2 \\ &\leq \frac{1}{4}t\left(\frac{5}{6}\right)^2 \\ &\leq \frac{1}{4}\left(d(x, fx, t)\right) + \left(d(y, fy, t)\right). \end{aligned}$$

Case 3: If $x, y \in [0, 1)$, then $fx = \frac{x}{3}$, $fy = \frac{y}{3}$.

$$d(fx, fy, t) = t\left|\frac{x}{3} - \frac{y}{3}\right|^2,$$

$$d(x, fx, t) = t\left|x - \frac{x}{3}\right|^2 = t\left(\frac{2x}{3}\right)^2,$$

$$d(y, fy, t) = t\left|y - \frac{y}{3}\right|^2 = t\left(\frac{2y}{3}\right)^2$$

$$d(x, y, t) = t\left|x - y\right|^2.$$

Consider

$$\begin{aligned} d(fx, fy, t) = t\left(\frac{(x-y)^2}{9}\right) &\leq t\left(\frac{x^2 + y^2}{9}\right) \\ &= \frac{1}{4}t\left(\left(\frac{2x}{3}\right)^2 + \left(\frac{2y}{3}\right)^2\right) \\ &= \frac{1}{4}t\left(\left(d(x, fx, t)\right) + \left(d(y, fy, t)\right)\right). \end{aligned}$$

is a valid contraction with

$$\lambda = 0, \quad \mu = \nu = \frac{1}{4}, \quad 2(\lambda + \nu) + \mu = 2\left(\frac{1}{4}\right) + \frac{1}{4} = \frac{3}{4} < 1.$$

Hence, f has a unique fixed point $x = 0$.

Now we show that several well-known contractions follow as special cases of Theorem 3.6.

Corollary 3.3 (Banach Type Contraction). *[4] Let (X, d) be a complete parametric b-metric space and let $f : X \to X$ satisfy*

$$\int_0^{d(fx,fy,t)} g(u)\,du \le \lambda \int_0^{d(x,y,t)} g(u)\,du$$

for all $x, y \in X$ and $t > 0$, where $0 \le \lambda < 1$. Then f has a unique fixed point in X.

Proof. Taking $\mu = \nu = 0$ in Theorem 3.6, inequality (1) reduces to

$$\int_0^{d(fx,fy,t)} g(u)\,du \le \lambda \int_0^{d(x,y,t)} g(u)\,du.$$

Since $\lambda < 1$, all conditions of Theorem 3.6 are satisfied. Hence, f has a unique fixed point.

□

The following corollary may be viewed as an integral-type Kannan contraction in complete parametric b-metric spaces. It extends the classical Kannan contraction [16] to complete parametric b-metric spaces.

Corollary 3.4 (Kannan Type Contraction). *Let (X, d) be a complete parametric b-metric space and suppose $f : X \to X$ satisfies*

$$\int_0^{d(fx,fy,t)} g(u)\,du \le \alpha \left[\int_0^{d(x,fx,t)} g(u)\,du + \int_0^{d(y,fy,t)} g(u)\,du \right]$$

for all $x, y \in X$, where $0 \le \alpha < \frac{1}{k+1}$. Then f has a unique fixed point in X.

Proof. Choosing $\lambda = 0, \quad \mu = \nu = \alpha$ in Theorem 3.6, inequality (1) reduces to

$$\int_0^{d(fx,fy,t)} g(u)\,du \le \alpha \left[\int_0^{d(x,fx,t)} g(u)\,du + \int_0^{d(y,fy,t)} g(u)\,du \right].$$

Note here that $k(\lambda + \nu) + \mu = k\alpha + \alpha = \alpha(k + 1) < 1$.

Thus the conditions of Theorem 3.6 are satisfied and the mapping f has a unique fixed point.

□

The following corollary may be called an integral-type Chatterjea contraction in complete parametric b-metric spaces. It follows from Theorem 3.6 and extends the classical Chatterjea [5] contraction to complete parametric b-metric spaces.

Corollary 3.5 (Chatterjea Type Contraction). *Let (X, d) be a complete parametric b-metric space with constant $k \ge 1$ and let $f : X \to X$ satisfy*

$$\int_0^{d(fx,fy,t)} g(u)\,du \le \beta \left[\int_0^{d(x,fy,t)} g(u)\,du + \int_0^{d(y,fx,t)} g(u)\,du \right]$$

for all $x, y \in X$, where $0 \le \beta < \frac{1}{3k^2+k}$. Then f has a unique fixed point in X.

Proof. Since (X, d) is a parametric b-metric space with constant $k \ge 1$, the condition (iii) of Definition 2.1 gives

$$d(x, fy, t) \le k\,[d(x, y, t) + d(y, fy, t)]$$

and

$$d(y, fx, t) \le k\,[d(x, y, t) + d(x, fx, t)] \qquad t > 0.$$

Substituting these relations into the given contractive condition, we obtain

$$\begin{aligned}\int_0^{d(fx,fy,t)} g(u)\,du &\le \beta\left[\int_0^{k[d(x,y,t)+d(y,fy,t)]} g(u)\,du + \int_0^{k[d(x,y,t)+d(x,fx,t)]} g(u)\,du\right]\\ &\le \beta\int_0^{kd(x,y,t)} g(u)\,du + \beta\int_0^{kd(y,fy,t)} g(u)\,du + \beta\int_0^{kd(x,y,t)} g(u)\,du\\ &\quad + \beta\int_0^{kd(x,fx,t)} g(u)\,du\\ &\le 2k\beta\int_0^{d(x,y,t)} g(u)\,du + k\beta\int_0^{d(x,fx,t)} g(u)\,du + k\beta\int_0^{d(y,fy,t)} g(u)\,du\end{aligned}$$

$$\int_0^{d(fx,fy,t)} g(u)\,du \le \lambda\int_0^{d(x,y,t)} g(u)\,du + \mu\int_0^{d(x,fx,t)} g(u)\,du + \nu\int_0^{d(y,fy,t)} g(u)\,du,$$

where $\lambda = 2k\beta,\ \mu = k\beta,\ \nu = k\beta \ge 0$. Moreover, since $0 \le \beta < \dfrac{1}{3k^2+k}$, these constants satisfy $k(\lambda+\nu)+\mu < 1$.

Hence, all the conditions of Theorem 3.6 are satisfied. Therefore, by Theorem 3.6, the mapping f has a unique fixed point in X. □

In 2025, Kannan *et al.* [17] presented results on fixed points of integral-type contractive mappings in complete parametric metric spaces. One result, which is close to one of our theorems, is given below.

Theorem 3.6. *[17] Let (X,d) be a complete parametric metric space and let $f : X \to X$ be a continuous self-mapping. Suppose there exists a constant $\beta \in [0,1)$ and a nonnegative Lebesgue-integrable function $\psi : [0,\infty) \to [0,\infty)$ satisfying*

$$\int_0^{\infty} \psi(r)\,dr < \infty,$$

such that, for all $x, y \in X$ with $x \ne y$ and for every $t > 0$, we have

$$\int_0^{d(fx,fy,t)} \psi(u)\,du \ \le\ \beta\max\left\{\int_0^{d(x,y,t)} \psi(u)\,du,\ \int_0^{d(x,fx,t)} \psi(u)\,du,\ \int_0^{d(y,fy,t)} \psi(u)\,du\right\}.$$

Then f has a unique fixed point in X.

We have extended this result to the setting of parametric b-metric spaces as follows:

Theorem 3.7. *Let (X,d) be a complete parametric b-metric space and let $g : [0,\infty) \to (0,\infty)$ be continuous and nonincreasing. Let $f : X \to X$ be a mapping satisfying*

$$\int_0^{d(fx,fy,t)} g(u)\,du \le \phi\left(\int_0^{d(x,y,t)} g(u)\,du, \int_0^{d(x,fx,t)} g(u)\,du, \int_0^{d(y,fy,t)} g(u)\,du\right), \tag{7}$$

for all $x, y \in X$, where $\phi : [0,\infty)^3 \to [0,\infty)$ is a continuous function satisfying $\phi(a,b,c) \le \lambda a + \mu b + \nu c$ with $0 \le \lambda, \mu, \nu < 1$ and $k(\lambda+\nu)+\mu < 1$. Then f has a unique fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary and construct a sequence $\{x_n\}$ in X by $x_{n+1} = fx_n, \quad n = 0, 1, 2, \ldots$. Using the contractive condition with $x = x_n$ and $y = x_{n-1}$, we obtain

$$\int_0^{d(fx_n, fx_{n-1}, t)} g(u)\,du \;\leq\; \phi\left(\begin{array}{l} \displaystyle\int_0^{d(x_n, x_{n-1}, t)} g(u)\,du, \\ \displaystyle\int_0^{d(x_n, fx_n, t)} g(u)\,du, \\ \displaystyle\int_0^{d(x_{n-1}, fx_{n-1}, t)} g(u)\,du \end{array}\right).$$

Since $fx_n = x_{n+1}$ and $fx_{n-1} = x_n$, the above inequality becomes

$$\int_0^{d(x_{n+1},x_n,t)} g(u)\,du \leq \phi\Bigg(\int_0^{d(x_n,x_{n-1},t)} g(u)\,du, \int_0^{d(x_n,x_{n+1},t)} g(u)\,du, \int_0^{d(x_{n-1},x_n,t)} g(u)\,du\Bigg).$$

Using the property of ϕ, we obtain

$$\begin{aligned}\int_0^{d(x_{n+1},x_n,t)} g(u)\,du \leq \lambda &\int_0^{d(x_n,x_{n-1},t)} g(u)\,du \\ &+ \mu \int_0^{d(x_n,x_{n+1},t)} g(u)\,du \\ &+ \nu \int_0^{d(x_{n-1},x_n,t)} g(u)\,du. \end{aligned} \tag{8}$$

Rewriting,

$$(1-\mu)\int_0^{d(x_{n+1},x_n,t)} g(u)\,du \leq (\lambda+\nu)\int_0^{d(x_n,x_{n-1},t)} g(u)\,du = \alpha \int_0^{d(x_n,x_{n-1},t)} g(u)\,du,$$

where $\alpha = \frac{\lambda+\nu}{1-\mu}$. Since $k(\lambda+\nu)+\mu < 1$, we have $\alpha < 1$. Thus

$$\int_0^{d(x_{n+1},x_n,t)} g(u)\,du \leq \alpha^{n-1} \int_0^{d(x_1,x_0,t)} g(u)\,du. \tag{9}$$

Hence, by Lemma 2.9,

$$\lim_{n\to\infty} \int_0^{d(x_n,x_{n-1},t)} g(u)\,du = 0.$$

We claim that

$$\lim_{n\to\infty} d(x_n, x_{n-1}, t) = 0.$$

Suppose, on the contrary, that

$$\limsup_{n\to\infty} d(x_n, x_{n-1}, t) = \delta > 0.$$

Then there exists a subsequence $\{d(x_{n_i}, x_{n_i-1}, t)\}$ such that

$$d(x_{n_i}, x_{n_i-1}, t) \geq \frac{\delta}{2} > 0$$

for every i. Using Lemma 2.9,

$$\int_0^{d(x_{n_i},x_{n_i-1},t)} \geq \int_0^{\frac{\delta}{2}} > 0.$$

This contradicts the fact that

$$\lim_{n\to\infty}\int_0^{d(x_n,x_{n-1},t)} = 0.$$

Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n-1}, t) = 0. \tag{10}$$

We now prove that $\{x_n\}$ is a Cauchy sequence. Let $m > n$. Repeated application of (iii) of Definition 2.1 gives

$$\begin{aligned} d(x_m, x_n, t) &\le k[d(x_m, x_{n+1}, t) + d(x_{n+1}, x_n, t)] \\ &\le k^2 d(x_m, x_{n+2}, t) + k^2 d(x_{n+2}, x_{n+1}, t) + kd(x_{n+1}, x_n, t) \\ &\le k^{m-n-1} d(x_m, x_{m-1}, t) + k^{m-n-1} d(x_{m-1}, x_{m-2}, t) \\ &\quad + \cdots + k^2 d(x_{n+2}, x_{n+1}, t) + kd(x_{n+1}, x_n, t). \end{aligned}$$

Again, using Lemma 2.9,

$$\begin{aligned} \int_0^{d(x_m,x_n,t)} g(u)\,du &\le k^{m-n-1}\int_0^{d(x_m,x_{m-1},t)} g(u)\,du + k^{m-n-1}\int_0^{d(x_{m-1},x_{m-2},t)} g(u)\,du + \cdots \\ &\quad + k^2\int_0^{d(x_{n+2},x_{n+1},t)} g(u)\,du + k\int_0^{d(x_{n+1},x_n,t)} g(u)\,du. \end{aligned}$$

Using (9), we obtain

$$\begin{aligned} \int_0^{d(x_m,x_n,t)} g(u)\,du &\le k^{m-n-1}\alpha^{m-1}\int_0^{d(x_1,x_0,t)} g(u)\,du + k^{m-n-1}\alpha^{m-2}\int_0^{d(x_1,x_0,t)} g(u)\,du + \cdots \\ &\quad + k^2\alpha^{n+1}\int_0^{d(x_1,x_0,t)} g(u)\,du + k\alpha^n\int_0^{d(x_1,x_0,t)} g(u)\,du \\ &\le \alpha^n\left[k + k^2\alpha + \cdots + k^{m-n-1}\alpha^{m-n-2}\right]\int_0^{d(x_1,x_0,t)} g(u)\,du \\ &\le \alpha^n \frac{k}{1-k\alpha}\int_0^{d(x_1,x_0,t)} g(u)\,du \quad (\because k\alpha < 1). \end{aligned}$$

Hence,

$$\lim_{m,n\to\infty}\int_0^{d(x_m,x_n,t)} g(u)\,du = 0. \tag{11}$$

We claim that

$$\lim_{m,n\to\infty} d(x_m, x_n, t) = 0.$$

Suppose, to the contrary, that there exist $\delta > 0$ and subsequences $\{m_i\}$ and $\{n_i\}$ of $\mathbb{N}$ such that

$$d(x_{m_i}, x_{n_i}, t) \ge \delta$$

for every i. Using Lemma 2.9,

$$\int_0^{d(x_{m_i},x_{n_i},t)} g(u)\,du \ge \int_0^{\delta} g(u)\,du > 0,$$

which contradicts (11). Therefore,

$$\lim_{m,n\to\infty} d(x_m, x_n, t) = 0$$

for every $t > 0$. We conclude that $\{x_n\}$ is a Cauchy sequence in X and since (X, d) is a complete parametric b-metric space, there exists $x^* \in X$ such that $x_n \to x^*$, that is,

$$\lim_{n\to\infty} d(x_n, x^*, t) = 0 \quad \text{for every } t > 0.$$

Now we show that x^* is a fixed point of f. Using the contractive condition with $x = x_n$ and $y = x^*$, we obtain

$$\begin{aligned}\int_0^{d(fx_n,fx^*,t)} g(u)\,du &\le \phi\left(\int_0^{d(x_n,x^*,t)} g(u)\,du, \int_0^{d(x_n,fx_n,t)} g(u)\,du, \int_0^{d(x^*,fx^*,t)} g(u)\,du\right)\\ &\le \lambda\int_0^{d(x_n,x^*,t)} g(u)\,du + \mu\int_0^{d(x_n,fx_n,t)} g(u)\,du + \nu\int_0^{d(x^*,fx^*,t)} g(u)\,du.\end{aligned}$$

Thus

$$\int_0^{d(x_{n+1},fx^*,t)} g(u)\,du \le \lambda\int_0^{d(x_n,x^*,t)} g(u)\,du + \mu\int_0^{d(x_n,x_{n+1},t)} g(u)\,du + \nu\int_0^{d(x^*,fx^*,t)} g(u)\,du. \tag{12}$$

We know that

$$\lim_{n\to\infty} d(x_n, x^*, t) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(x_n, x_{n+1}, t) = 0, \quad \forall t > 0.$$

From Lemma 2.9, it follows that

$$\lim_{n\to\infty}\int_0^{d(x_n,x^*,t)} g(u)\,du = 0 \quad \text{and} \quad \lim_{n\to\infty}\int_0^{d(x_n,x_{n+1},t)} g(u)\,du = 0.$$

Now taking limit as $n \to \infty$ in the above inequality (12), we obtain

$$\int_0^{d(x^*,fx^*,t)} g(u)\,du \le \nu\int_0^{d(x^*,fx^*,t)} g(u)\,du.$$

This yields

$$(1-\nu)\int_0^{d(x^*,fx^*,t)} g(u)\,du \le 0.$$

Since $\nu < 1$, we have $1 - \nu > 0$, so

$$\int_0^{d(x^*,fx^*,t)} g(u)\,du = 0.$$

Since $g(u) > 0$ for $u > 0$ and $t > 0$ is arbitrary, we conclude that

$$d(x^*, fx^*, t) = 0, \quad \forall t > 0.$$

Using (i) of Definition 2.1 of the parametric b-metric, $fx^* = x^*$. Hence, x^* is a fixed point of f.

To prove uniqueness, suppose that y^* is another fixed point of f. Then $fx^* = x^*, \quad fy^* = y^*$. Applying the contractive condition with $x = x^*$ and $y = y^*$, we obtain

$$\int_0^{d(fx^*,fy^*,t)} g(u)\,du \le \phi\left(\int_0^{d(x^*,y^*,t)} g(u)\,du, \int_0^{d(x^*,fx^*,t)} g(u)\,du, \int_0^{d(y^*,fy^*,t)} g(u)\,du\right).$$

Since $fx^* = x^*$ and $fy^* = y^*$, we have

$$d(x^*, fx^*, t) = 0, \qquad d(y^*, fy^*, t) = 0.$$

Thus the above inequality reduces to

$$\int_0^{d(x^*,y^*,t)} g(u)\,du \le \phi\left(\int_0^{d(x^*,y^*,t)} g(u)\,du, 0, 0\right).$$

Using the condition

$$\phi(a,b,c) \le \lambda a + \mu b + \nu c,$$

we obtain

$$\int_0^{d(x^*,y^*,t)} g(u)\,du \le \lambda \int_0^{d(x^*,y^*,t)} g(u)\,du.$$

Hence,

$$(1-\lambda)\int_0^{d(x^*,y^*,t)} g(u)\,du \le 0.$$

Since $0 \le \lambda < 1$, it follows that

$$\int_0^{d(x^*,y^*,t)} g(u)\,du = 0.$$

Because $g(u) > 0$ for $u > 0$, we conclude that

$$d(x^*, y^*, t) = 0.$$

Therefore, $x^* = y^*$, and the fixed point of f is unique. □

Example 3.8. Let $X = [0,1]$ and define, for $x, y \in X$ and $t > 0$,

$$d(x,y,t) = \frac{|x-y|^2}{t}.$$

Now $|x-z|^2 \le 2\big(|x-y|^2 + |y-z|^2\big)$ gives the triangle inequality with $k = 2$. Since $[0,1] \subset \mathbb{R}$ is closed and bounded, (X,d) is a complete parametric b-metric space. Take

$$g(u) = 1, \qquad u \ge 0, \qquad \text{so that} \qquad \int_0^r g(u)\,du = r.$$

Define $f : X \to X$ by

$$fx = \frac{x}{2}.$$

Let

$$\phi(a,b,c) = \frac{a}{4} + \frac{\sqrt{bc}}{4} \qquad (a,b,c \ge 0),$$

$$\phi(a,b,c) = \frac{a}{4} + \frac{\sqrt{bc}}{4} \le \frac{a}{4} + \frac{b+c}{8} = \lambda a + \mu b + \nu c, \qquad \lambda = \frac{1}{4}, \mu = \nu = \frac{1}{8},$$

and

$$2(\lambda+\nu) + \mu = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} < 1.$$

So ϕ satisfies the hypothesis $\phi(a,b,c) \le \lambda a + \mu b + \nu c$ with $k(\lambda+\nu) + \mu < 1$. Since $\int_0^r g(u)\,du = r$, let

$$a = d(x,y,t) = \frac{|x-y|^2}{t}, \qquad b = d(x,fx,t), \qquad c = d(y,fy,t).$$

With minor calculations, we get

$$d(fx,fy,t) = \frac{|x-y|^2}{4t} = \frac{d(x,y,t)}{4}.$$

Hence

$$\int_0^{d(fx,fy,t)} g(u)\,du = d(fx,fy,t) = \frac{d(x,y,t)}{4} = \frac{a}{4}.$$

Now

$$\int_0^{d(fx,fy,t)} g(u)\,du = \frac{a}{4} \;\le\; \frac{a}{4} + \frac{\sqrt{bc}}{4} = \phi(a,b,c),$$

which holds for all $x, y \in X$ and $t > 0$, since $\sqrt{bc} \ge 0$. If $x = fx$ or $y = fy$, then either $b = d(x, fx, t) = 0$ or $c = d(y, fy, t) = 0$ that is only at the fixed point. Thus

$$\int_0^{d(fx,fy,t)} g(u)\,du \le \phi\left(\int_0^{d(x,y,t)} g(u)\,du,\ \int_0^{d(x,fx,t)} g(u)\,du,\ \int_0^{d(y,fy,t)} g(u)\,du\right)$$

holds for all $x, y \in X$. All hypotheses of the theorem are satisfied. Therefore, f has a unique fixed point in X.

Simple computation confirms that

$$x^* = 0$$

is the unique fixed point of $f(x) = \dfrac{x}{2}$ on $X = [0, 1]$.

Remark 3.9. Choosing $k = 1$, $\psi \equiv g$ integrable, and

$$\phi(a,b,c) = \beta \max\{a,b,c\} \le \beta(a+b+c), \quad \lambda = \mu = \nu = \beta$$

in Theorem 3.7, with $\beta \in [0, 1/3)$, recovers a fixed point theorem of Kannan [17] (refer Theorem 3.6) on parametric metric spaces, showing that Theorem 3.7 is a genuine generalization of Theorem 3.6 to the wider class of parametric b-metric spaces.

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