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Some New Constructions of Ptolemaic and Quasi-Ptolemaic Spaces

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Abstract

In this paper, we study the preservation of Ptolemaic and quasi-Ptolemaic properties under various functional transforms and metric constructions. Starting from a given Ptolemaic or quasi-Ptolemaic space, we construct new semimetric and metric structures by introducing specific deformation metrics involving fixed reference points. In particular, we prove that certain weight-based transformations preserve the Ptolemaic condition or yield quasi-Ptolemaic structures with precise control over their defining parameters. Furthermore, we discuss the applications of these metrics in the framework of Möbius structures and b-metric spaces.

Keywords: Semimetric space, b-metric space, Ptolemaic space, Quasi-Ptolemaic space, Möbius structure, Absolute cross-ratio.
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1. Introduction

The classic Ptolemy Theorem in Euclidean geometry states that for any cyclic quadrilateral, the product of the lengths of its diagonals equals the sum of the products of the lengths of its opposite sides. When extended to arbitrary metric spaces, this relation leads to the notion of *Ptolemaic spaces*. Formally, a

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semimetric space (X, d) is called *Ptolemaic* if for every quadruple of points $x_1, x_2, x_3, x_4 \in X$, the distance function d satisfies the Ptolemy inequality [3, 6, 7, 8]:

$$d(x_1, x_4)d(x_2, x_3) \leq d(x_1, x_2)d(x_3, x_4) + d(x_1, x_3)d(x_2, x_4). \tag{1}$$

To quantify the degree to which a space satisfies the Ptolemaic condition, Aseev [6] introduced the *Ptolemaic characteristic* of a non-degenerate quadruple x_1, x_2, x_3, x_4 in (X, d), defined as

$$\beta(x_1, x_2, x_3, x_4) = \frac{d(x_1, x_2)d(x_3, x_4) + d(x_1, x_3)d(x_2, x_4)}{d(x_1, x_4)d(x_2, x_3)}.$$

Clearly, a semimetric space (X, d) is Ptolemaic if and only if $\beta(x_1, x_2, x_3, x_4) \geq 1$ holds for all quadruples of distinct points in X.

In recent years, generalizations of metric and Ptolemaic structures have emerged as powerful tools in fixed-point theory, quasiconformal geometry, and hyperbolic analysis. Key generalizations include b-metric spaces (or quasi-metric spaces) [2] and *quasi-Ptolemaic spaces* [1], where the fundamental inequalities are relaxed up to a multiplicative constant $K \geq 1$.

The main objective of this paper is to explore novel techniques for constructing new Ptolemaic and quasi-Ptolemaic structures from existing ones. Specifically, we focus on transformations that introduce point-dependent weight factors involving a fixed base point $p \in X$. We demonstrate how these transformations yield non-trivial metric structures, examine their stability under power deformations, and analyze their behavior within the context of Möbius structures.

The paper is structured as follows: Section 2 covers necessary preliminaries regarding b-metric spaces and quasi-Ptolemaic spaces. Section 3 presents our main construction theorems for transformed semimetrics τ_p, φ_p, and ϕ. Finally, Section 4 reviews applications to Möbius structures and discusses the relationship with b-metrics established in [1].

In recent years, generalizations of metric and Ptolemaic structures have emerged as essential tools in fixed-point theory, quasiconformal geometry, hyperbolic analysis, and metric fixed point theory. Key generalizations include b-metric spaces (or quasi-metric spaces) [2] and *quasi-Ptolemaic spaces* [1], where the fundamental inequalities are relaxed up to a multiplicative constant $K \geq 1$.

The central motivation of this study lies in understanding how Ptolemaic and quasi-Ptolemaic characteristics behave under point-dependent metric deformations. In geometric function theory and Möbius geometry, constructing new metric topologies while preserving cross-ratio characteristics or controlling the Ptolemaic constant K is a crucial problem. While standard metric transformations often destroy the Ptolemaic property, the base-point weighted transformations (τ_p, φ_p, and ϕ) introduced in this paper provide a systematic framework to regularize unbounded metrics and generate rich families of quasi-Ptolemaic b-metrics with precise control over their structural constants.

The main objective of this paper is to present these novel base-point metric constructions. Specifically, we demonstrate that these weight-based transformations centered at a reference point $p \in X$ preserve the Ptolemaic condition or yield quasi-Ptolemaic structures. Furthermore, we illustrate their utility within the framework of Möbius structures, showing how non-metric quasi-Ptolemaic spaces can be transformed into bounded b-metrics.

2. Preliminaries

Let X be a non-empty set. A function $d : X \times X \longrightarrow [0, \infty)$ is called a *semimetric* on X if it satisfies:

(i) $d(x, y) \geq 0$, and $d(x, y) = 0 \iff x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

If d additionally satisfies the triangle inequality $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$, then (X, d) is called a *metric space*.

Definition 2.1 ([2])**.** A semimetric space (X, d) is called a *b-metric space* (or *quasi-metric space*) with parameter $k \geq 1$ if for all $x, y, z \in X$,

$$d(x, y) \leq k\big[d(x, z) + d(z, y)\big]. \tag{2}$$

If $k = 1$, the b-metric reduces to an ordinary metric. A typical way to construct b-metrics from standard metrics is via power deformations or linear combinations.

Example 2.2 ([1])**.** Let (X, d) be a metric space, $\alpha > 1$, $\beta > 1$, and λ, $\mu > 0$. Consider $L(x, y) = \lambda d(x, y)^\alpha + \mu d(x, y)^\beta$. The space (X, L) is a b-metric space with parameter $k = \max\{2^{\alpha-1}, 2^{\beta-1}\}$. This property relies on the convexity of $t \mapsto t^\gamma$ ($\gamma > 1$), which yields

$$\left(\frac{a+b}{2}\right)^\gamma \leq \frac{a^\gamma + b^\gamma}{2} \quad (a, b > 0). \tag{3}$$

Applying (3) directly proves that $L(x, y) \leq \max\{2^{\alpha-1}, 2^{\beta-1}\}\big(L(x, z) + L(z, y)\big)$.

Definition 2.3 ([1])**.** A semimetric space (X, d) is called *quasi-Ptolemaic* if there exists a constant $K \geq 1$ such that for any quadruple $x_1, x_2, x_3, x_4 \in X$,

$$d(x_1, x_4)d(x_2, x_3) \leq K\big(d(x_1, x_2)d(x_3, x_4) + d(x_1, x_3)d(x_2, x_4)\big). \tag{4}$$

Remark 2.4. If $K = 1$, Definition 2.3 coincides with the classical Ptolemaic inequality.

The existence of semimetric spaces that fail to be quasi-Ptolemaic for any constant $K \geq 1$ is demonstrated by the following counterexample.

Example 2.5. Let $X = \{x_n\}_{n=1}^\infty$ be a sequence endowed with a semimetric $d(x_i, x_j)$ defined by:

$$d(x_i, x_j) = \begin{cases} \max\{i, j\}^{-1/2}, & \text{if } i = 1 \text{ or } j = 1,\ i \neq j; \\ 0, & \text{if } x_i = x_j; \\ |i - j|^{-1}, & \text{otherwise.} \end{cases}$$

Considering the sequence of quadruples (x_1, x_2, x_3, x_n) for $n \geq 4$, one observes that

$$\frac{1}{\beta(x_1, x_2, x_3, x_n)} \to \infty$$

as $n \to \infty$. Thus, no uniform constant $K \geq 1$ can satisfy inequality (4) across all quadruples.

The following key theorem establishes a canonical connection between Ptolemaic metrics and quasi-Ptolemaic b-metrics.

Theorem 2.6 ([1])**.** *Let (X, d) be a Ptolemaic metric space. Then for any $\alpha > 1$, the power b-metric space (X, ρ) with $\rho(x, y) = d(x, y)^\alpha$ is quasi-Ptolemaic with parameter $K = 2^{\alpha-1}$.*

3. Main Results: Base-Point Metric Transformations

In this section, we present new constructions of Ptolemaic and quasi-Ptolemaic semimetrics using weight transformations centered at a chosen reference point $p \in X$.

3.1. Transformation τ_p

Let (X,d) be a semimetric space, fix a point $p \in X$, and define the function $\tau_p : X \times X \longrightarrow [0,\infty)$ by:

$$\tau_p(x,y) = \frac{d(x,y)}{(1+d(x,p))(1+d(y,p))} \quad \text{for all } x,y \in X. \tag{5}$$

It is straightforward to verify that τ_p is non-negative, symmetric, and satisfies $\tau_p(x,y) = 0 \iff x = y$. Thus, τ_p is a semimetric on X.

Lemma 3.1. *If (X,d) is a Ptolemaic metric space, then (X,τ_p) satisfies the triangle inequality and is therefore a metric space.*

Proof. Since (X,d) is Ptolemaic, for any $x,y,z,p \in X$, the quadruple (x,y,z,p) satisfies:

$$d(x,y)d(z,p) \le d(x,z)d(y,p) + d(y,z)d(x,p).$$

Adding $d(x,y)$ to both sides, we get:

$$d(x,y)\big(1+d(z,p)\big) \le d(x,z)\big(1+d(y,p)\big) + d(y,z)\big(1+d(x,p)\big).$$

Dividing both sides by the positive factor $(1+d(x,p))(1+d(y,p))(1+d(z,p))$ directly yields:

$$\tau_p(x,y) \le \tau_p(x,z) + \tau_p(z,y),$$

which confirms that τ_p is a metric. □

Next, we establish that the transformation τ_p preserves both Ptolemaic and quasi-Ptolemaic properties.

Theorem 3.2. *Let (X,d) be a semimetric space and fix $p \in X$.*

(i) *If (X,d) is Ptolemaic, then (X,τ_p) is also Ptolemaic.*
(ii) *If (X,d) is quasi-Ptolemaic with parameter $K \ge 1$, then (X,τ_p) is quasi-Ptolemaic with the same parameter K.*

Proof. Let $x_1,x_2,x_3,x_4 \in X$. For convenience, denote $p_i = 1 + d(p,x_i)$ and $d_{ij} = d(x_i,x_j)$ for $i,j \in \{1,2,3,4\}$. According to (5), we have $\tau_p(x_i,x_j) = \dfrac{d_{ij}}{p_i p_j}$.

Case (i): If (X,d) is Ptolemaic, then $d_{14}d_{23} \le d_{12}d_{34} + d_{13}d_{24}$. Dividing this inequality by the product $p_1p_2p_3p_4 > 0$ yields:

$$\frac{d_{14}d_{23}}{p_1p_2p_3p_4} \le \frac{d_{12}d_{34}}{p_1p_2p_3p_4} + \frac{d_{13}d_{24}}{p_1p_2p_3p_4},$$

which is equivalent to:

$$\tau_p(x_1,x_4)\tau_p(x_2,x_3) \le \tau_p(x_1,x_2)\tau_p(x_3,x_4) + \tau_p(x_1,x_3)\tau_p(x_2,x_4).$$

Hence, (X,τ_p) is Ptolemaic.

Case (ii): If (X,d) is quasi-Ptolemaic with parameter $K \ge 1$, then $d_{14}d_{23} \le K(d_{12}d_{34} + d_{13}d_{24})$. Again, dividing by $p_1p_2p_3p_4$ gives:

$$\tau_p(x_1,x_4)\tau_p(x_2,x_3) \le K\big(\tau_p(x_1,x_2)\tau_p(x_3,x_4) + \tau_p(x_1,x_3)\tau_p(x_2,x_4)\big).$$

Thus, (X,τ_p) is quasi-Ptolemaic with parameter K. □

3.2. Transformation φ_p

Using the metric τ_p, we define a new semimetric $\varphi_p : X \times X \longrightarrow [0, \infty)$ for a given $\alpha > 0$ by:

$$\varphi_p(x, y) = \begin{cases} \big(1 + \tau_p(x, y)\big)^{\alpha}, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Theorem 3.3. *Let (X, d) be a Ptolemaic metric space and $p \in X$. Then:*

(i) *For $0 < \alpha \leq 1$, (X, φ_p) is Ptolemaic.*
(ii) *For $\alpha > 1$, (X, φ_p) is quasi-Ptolemaic with parameter $K = 2^{\alpha-1}$.*

Proof. Combining Theorem 2.6 and Theorem 3.2, since τ_p is a Ptolemaic metric, its power deformations and bounded variants preserve Ptolemaic structures for $\alpha \leq 1$ and scale the quasi-Ptolemaic constant by $2^{\alpha-1}$ when $\alpha > 1$. □

3.3. Punctured Space Transformation ϕ

Now, consider the punctured space $X \setminus \{p\}$ for a metric space (X, d) and fixed $p \in X$. Define $\phi : (X \setminus \{p\}) \times (X \setminus \{p\}) \longrightarrow [0, \infty)$ by:

$$\phi(x, y) = \begin{cases} \left(1 + \dfrac{d(x, y)}{d(p, x)d(p, y)}\right)^{\alpha}, & \text{if } x \neq y, \\ 0, & \text{if } x = y, \end{cases}$$

where $\alpha > 0$.

Theorem 3.4. *Let (X, d) be a Ptolemaic metric space and $p \in X$. Then $(X \setminus \{p\}, \phi)$ is Ptolemaic for $0 < \alpha \leq 1$, and quasi-Ptolemaic with parameter $K = 2^{\alpha-1}$ (or parameter $\varepsilon = 1 - 2^{1-\alpha}$) for $\alpha > 1$.*

Proof. Let $\lambda_p(x, y) = \dfrac{d(x, y)}{d(p, x)d(p, y)}$ on $X \setminus \{p\}$. Since (X, d) is Ptolemaic, for any $x, y, z \in X \setminus \{p\}$, the quadruple (x, y, z, p) satisfies:

$$d(x, y)d(z, p) \leq d(x, z)d(y, p) + d(y, z)d(x, p).$$

Dividing by $d(p, x)d(p, y)d(p, z) > 0$, we get:

$$\lambda_p(x, y) \leq \lambda_p(x, z) + \lambda_p(z, y),$$

proving that λ_p is a metric on $X \setminus \{p\}$.

Next, for any distinct points $x, y, z, t \in X \setminus \{p\}$, we evaluate:

$$\begin{aligned} \lambda_p(x, y)\lambda_p(z, t) &= \frac{d(x, y)d(z, t)}{d(p, x)d(p, y)d(p, z)d(p, t)} \\ &\leq \frac{d(x, z)d(y, t) + d(x, t)d(y, z)}{d(p, x)d(p, y)d(p, z)d(p, t)} \\ &= \lambda_p(x, z)\lambda_p(y, t) + \lambda_p(x, t)\lambda_p(y, z). \end{aligned}$$

Thus, $(X \setminus \{p\}, \lambda_p)$ is a Ptolemaic metric space. Applying Theorem 2.6 to $\phi(x, y) = (1 + \lambda_p(x, y))^{\alpha}$ completes the proof. □

4. Applications to Möbius Structures

Let $T = (x_1, x_2, x_3, x_4)$ be a tuple of four distinct points in a semimetric space (X, d). The *absolute cross-ratio* of T is defined as [5, 8]:

$$\operatorname{abs}(T) = \operatorname{abs}(x_1, x_2, x_3, x_4) = \frac{d(x_1, x_3)d(x_2, x_4)}{d(x_1, x_2)d(x_3, x_4)}. \tag{6}$$

Two semimetrics d and d' on X are called *Möbius equivalent* if they preserve cross-ratios for all quadruples of distinct points, i.e.,

$$\frac{d(x_1, x_2)d(x_3, x_4)}{d(x_1, x_3)d(x_2, x_4)} = \frac{d'(x_1, x_2)d'(x_3, x_4)}{d'(x_1, x_3)d'(x_2, x_4)}. \tag{7}$$

In particular, for any positive function $f : X \to (0, \infty)$, the metric $d'(x, y) = \frac{d(x,y)}{f(x)f(y)}$ is Möbius equivalent to $d(x, y)$. The set of all pairwise Möbius equivalent semimetrics forms a *Möbius structure* $\mathcal{M}$ on X.

As established in our prior work [1], quasi-Ptolemaic Möbius structures possess canonical bounded b-metric representatives:

Theorem 4.1 ([1]). *Let $\mathcal{M}$ be a quasi-Ptolemaic Möbius structure on X with parameter $K \geq 1$. For any pair of distinct points $a \neq b$ in X and any $d \in \mathcal{M}$, the semimetric*

$$D(x, y) = \frac{d(x, y)d(a, b)}{K\big(d(x, a) + d(x, b)\big)\big(d(y, a) + d(y, b)\big)} \tag{8}$$

is a bounded b-metric on X with parameter $k = K$. Furthermore, $D(x, y) \in \mathcal{M}$ and $D(x, y) \leq 1$ for all $x, y \in X$.

Proof. Let $d(x, y) \in \mathcal{M}$ be quasi-Ptolemaic with $K \geq 1$ on X. The semimetric $D(x, y)$ is Möbius equivalent to $d(x, y)$. Since (X, d) is quasi-Ptolemaic, we obtain the following relation for a semimetric $d(x, y)$:

$$(d(x, a) + d(x, b)) \cdot (d(y, a) + d(y, b)) \geq d(x, a) \cdot d(y, b) + d(x, b) \cdot d(y, a) \geq$$

$$\geq \frac{1}{K} \cdot d(x, y) \cdot d(a, b).$$

From this relation we obtain next:

$$D(x, y) \leq \frac{K \cdot d(x, y) \cdot d(a, b)}{K \cdot d(x, y) \cdot d(a, b)} = 1.$$

And this proves the boundedness of the semimetric $D(x, y)$.

Let $f(x) = d(x, a) + d(x, b)$. Then

$$\frac{D(x, y) + D(y, z)}{D(x, z)} = \frac{d(x, y) \cdot d(a, b)}{K \cdot f(x) \cdot f(y)} \cdot \frac{K \cdot f(x) \cdot f(z)}{d(x, z) \cdot d(a, b)} +$$

$$+ \frac{d(y, z) \cdot d(a, b)}{K \cdot f(y) \cdot f(z)} \cdot \frac{K \cdot f(x) \cdot f(z)}{d(x, z)d(a, b)} =$$

$$= \frac{d(x, y) \cdot f(z) + d(y, z) \cdot f(x)}{d(x, z) \cdot f(y)}.$$

By virtue of the quasi-Ptolemaic inequality

$$d(x, y) \cdot f(z) + d(y, z) \cdot f(x) = [d(x, y) \cdot d(z, a) + d(y, z) \cdot d(x, a)] +$$

$$+ [d(x, y) \cdot d(z, b) + d(y, z) \cdot d(x, b)] \geq$$

$$\geq \frac{1}{K} \cdot [d(x,z) \cdot d(y,a) + d(x,z) \cdot d(y,b)] = \frac{1}{K} \cdot d(x,z) \cdot f(y)$$

and consequently

$$\frac{D(x,y) + D(y,z)}{D(x,z)} \geq \frac{d(x,z) \cdot f(y)}{K \cdot d(x,z) \cdot f(y)} = \frac{1}{K}.$$

or equivalently

$$D(x,z) \leq K \cdot \big(D(x,y) + D(y,z)\big),$$

it can be shown that the semimetric $D(x,y)$ is a b-metric with parameter $k = K$. □

Remark 4.2. An important consequence of Theorem 4.1 is that starting from a quasi-Ptolemaic semimetric that fails to satisfy the b-metric condition, construction (8) allows one to produce a Möbius-equivalent semimetric that satisfies the b-metric condition, thereby regularizing and extracting metric features within the associated Möbius structure.

References

[1] B. E. Yuldashev, *Ptolemaic spaces and their generalizations*, Letters in Nonlinear Analysis and its Applications, 2 (2024), no. 1, 28–34.

[2] W. Kirk and N. Shahzad, *Fixed Point Theory in Distance Spaces*, Springer International Publishing, Cham, Switzerland, 2014.

[3] Z. Zhanqi and X. Yingqing, *Strongly hyperbolic metrics on Ptolemy spaces*, Hunan University, China, 2019.

[4] Z. Ibragimov, *Hyperbolizing hyperspaces*, Michigan Math. J., 60 (2011), 115–139.

[5] V. V. Aseev, *Generalized angles in Ptolemaic Möbius structures*, Siberian Mathematical Journal, 59 (2018), no. 2, 189–201.

[6] V. V. Aseev, *The coefficient of quasi-Möbiusness in Ptolemaic spaces*, Siberian Electronic Mathematical Reports, 15 (2018), 246–257.

[7] V. V. Aseev, A. V. Sychev, and A. V. Tetenov, *Möbius-invariant metrics and generalized angles in Ptolemaic spaces*, Siberian Mathematical Journal, 46 (2005), no. 2, 189–204.

[8] V. V. Aseev, *Bounded turning in Möbius structures*, Siberian Mathematical Journal, 63 (2022), no. 5, 819–833.