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Ptolemaic spaces and their generalizations

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Abstract

We introduce a new notion of quasi-Ptolemaic space and find the relations between quasi-Ptolemaic spaces and b-metric spaces. We prove that each b-metric space is quasi-Ptolemaic and prove that quasimobius mappings send quasi-Ptolemaic spaces to quasi-Ptolemaic spaces.

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1. Introduction

A semimetric space is called *Ptolemaic* if for each quadruple x_1, x_2, x_3, x_4 a semimetric d satisfies Ptolemy's inequality [3, 5, 7, 8, 9]

$$d(x_1, x_2)d(x_3, x_4) + d(x_1, x_3)d(x_2, x_4) \geq d(x_1, x_4)d(x_2, x_3).$$

A metric space is not necessarily Ptolemaic and a semimetric Ptolemaic space is not necessarily metric. *The ptolemaic characteristic* [7] of a quadruple x_1, x_2, x_3, x_4 in (X, d) is

$$\beta(x_1, x_2, x_3, x_4) = \frac{d(x_1, x_2)d(x_3, x_4) + d(x_1, x_3)d(x_2, x_4)}{d(x_1, x_4)d(x_2, x_3)}.$$

The inequality $\beta(x_1, x_2, x_3, x_4) \geq 1$ is true for any x_1, x_2, x_3, x_4 in the Ptolemaic space (X, d).

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A semimetric quadruple x_1, x_2, x_3, x_4 is Ptolemaic, if and only if the determinant

$$C(x_1, x_2, x_3, x_4) = \left|d(x_i, x_j)\right|^2$$

is negative or zero, where $i, j \in \{x_1, x_2, x_3, x_4\}$ [2].

In this article, we try to find the relations between quasi-Ptolemaic (Definition 3.1) spaces and b-metric spaces. Quasi-Ptolemaic spaces are a new type of topological space, and there are no results yet on quasi-Ptolemaic spaces.

The main question is whether a semimetric space is quasi-Ptolemaic, if it is a b-metric space? Theorem 3.4, proved in the third section, allows us to give a positive answer to this question. According to this theorem, every b-metric space with parameter $k \geq 1$ is also quasi-Ptolemaic with parameter $1 > \varepsilon \geq 1 - \frac{1}{2k^2}$.

However, there is no such relationship in the opposite direction. The quasi-Ptolemaic property of a semimetric space does not imply that it is b-metric. We prove this with the Example 3.5. In this example, we construct a semimetric space that satisfies Ptolemy's inequality and is not a b-metric for any value of the parameter k.

Theorem 4.1 proves that in any quasi-Ptolemaic Möbius structure $\mathcal{M}$ with parameter $\varepsilon \in (0, 1)$ in the set X there is a bounded b-metric with parameter $k = \frac{1}{1-\varepsilon}$, which is Möbius equivalent to the original semimetrics. Theorem 5.1 proves that for a linear homeomorphism $\eta = kt$, where $t \in [0, +\infty)$, $k = const > 0$ any η-quasimöbius map f maps a quasi-Ptolemaic semimetric space with parameter $\varepsilon \in (0, 1)$ into a quasi-Ptolemaic semimetric space with parameter $1 - \frac{1-\varepsilon}{k}$.

2. b-metric spaces

Let X be a set, and let $d : X \times X \longrightarrow \mathbb{R}_+$ be a mapping that satisfies the following axioms for each $x, y \in X$:

I. $d(x, y) \geq 0$, and $d(x, y) = 0 \Leftrightarrow x = y$;

II. $d(x, y) = d(y, x)$.

Then we call the pair (X, d) a *semimetric space* and d is called a *semimetric* [9] . However, if a semimetric d in X satisfies the following for each $x, y, z \in X$:

III. $d(x, y) \leq d(x, z) + d(z, y)$,

then we call the pair (X, d) a *metric space* and d a *metric* in X. The axiom (III) is called a *triangle inequality.*

Definition 2.1. [1]. A semimetric space (X, d) is said to be *b-metric space* (or *quasi-metric space*) if there exists $k \geq 1$ such that for each $x, y, z \in X$,

$$d(x, y) \leq k[d(x, z) + d(z, y)]. \tag{1}$$

In the definition of b-metric space we can see that if $k = 1$ then (X, d) is a metric space. We mention an example for a b-metric space.

Example 2.2. Let (X,d) be a metric space, $\alpha > 1$, $\beta > 1$, $\lambda > 0$, $\mu > 0$. Consider the function $L(x, y) = \lambda d(x, y)^\alpha + \mu d(x, y)^\beta$ for all $x, y \in X$. $L(x, y)$ is not always a metric, but it is a b-metric with the parametr $k = \max\{2^{\alpha-1}, 2^{\beta-1}\}$. To show this, it helps us that for any $a, b > 0$ and $\gamma > 1$

$$\left(\frac{a+b}{2}\right)^\gamma \leq \frac{a^\gamma + b^\gamma}{2}, \tag{2}$$

and this in turn follows from the fact that the real valued function $x \to x^\beta$ is convex. Using (2) we obtain that,

$$\begin{gathered} L(x, y) = \lambda d(x, y)^\alpha + \mu d(x, y)^\beta \leq \\ \leq \lambda\big(d(x, z) + d(z, y)\big)^\alpha + \mu\big(d(x, z) + d(z, y)\big)^\beta \leq \\ \leq 2^{\alpha-1}\lambda\big(d(x, z)^\alpha + d(z, y)^\alpha\big) + 2^{\beta-1}\mu\big(d(x, z)^\beta + d(z, y)^\beta\big) \leq \\ \leq \max\{2^{\alpha-1}, 2^{\beta-1}\} \cdot \big(L(x, z) + L(z, y)\big). \end{gathered}$$

3. Quasi-Ptolemaic spaces

We begin this section with the following example. Suppose that the set consists of four points, that is, $X = \{x_1, x_2, x_3, x_4\}$. Let metric d on X satisfied these equalities, $d(x_1, x_2) = d(x_3, x_4) = s$, $d(x_2, x_3) = d(x_1, x_4) = t$, $d(x_1, x_3) = d(x_2, x_4) = s + t$, where s and t are positive real numbers, and so (X, d) is called a *pseudolinear quadripole* [5]. Since $(s + t)^2 > s^2 + t^2$, pseudolinear quadripoles are not Ptolemaic. However, for any $\varepsilon \in [1/2, 1)$ we can write that:

$$s^2 + t^2 \geq (1 - \varepsilon)(s + t)^2.$$

This example motivated us to define a new type of spaces.

Definition 3.1. A semimetric space (X, d) is called *quasi-Ptolemaic* if there exists $\varepsilon \in (0, 1)$ such that, for each quadruple $x_1, x_2, x_3, x_4 \in X$ the semimetric d satisfies,

$$d(x_1, x_2)d(x_3, x_4) + d(x_1, x_3)d(x_2, x_4) \geq (1 - \varepsilon)d(x_1, x_4)d(x_2, x_3). \tag{3}$$

It is true for the quasi-Ptolemaic space that $\beta(x_1, x_2, x_3, x_4) \geq 1-\varepsilon$, where $\varepsilon \in (0, 1)$. From this definition one can see that, if the space is quasi-Ptolemaic with parameter ε_0, that is, if (3) is satisfied for different four points, then for all $\varepsilon \in [\varepsilon_0, 1)$ (3) is satisfied. If (3) holds for all $\varepsilon \in (0, 1)$, then the space (X, d) is Ptolemaic.

The following question may arise from the definition of quasi-Ptolemaic space: Is there a semimetric that does not satisfy the condition of Ptolemaic and quasi-Ptolemaic spaces? The following example shows we can find a non quasi-Ptolemaic semimetric space.

Example 3.2. (Not quasi-Ptolemaic space.) Let $X = \{x_n\}_{n=1}^{\infty}$ sequence be a semimetric space with semimetric $d(x_i, x_j)$, such that

$$d(x_i, x_j) = \begin{cases} \max\{i, j\}^{-1/2}, & \text{if } i = 1 \text{ or } j = 1; \\ 0, & \text{if } x_i = x_j; \\ |i - j|^{-1}, & \text{otherwise.} \end{cases}$$

Then for sequence of points x_1, x_2, x_3, x_n $(n \geq 4)$, we obtain that

$$\frac{1}{\beta(x_1, x_2, x_3, x_n)} \longrightarrow \infty$$

for $n \longrightarrow \infty$. This means that for a sufficiently large n the quasi-Ptolemaic inequality does not hold.

In [4] one can see that in the metric Ptolemaic space (X, d) the metric $\rho(x, y) = d(x, y)^{\alpha}$ is Ptolemaic for $\alpha \in (0, 1]$. If $\alpha > 1$, then the semimetric is not a metric but is a b-metric with $k = 2^{\alpha-1}$. The following theorem holds for this b-metric:

Theorem 3.3. *Let the metric space (X, d) be Ptolemaic. Then the b-metric space (X, ρ) is quasi-Ptolemaic with $\varepsilon = 1 - \frac{1}{2^{\alpha-1}}$, where $\rho(x, y) = d(x, y)^{\alpha}$ and $\alpha > 1$.*

Proof. Let $x_1, x_2, x_3, x_4 \in X$ be different points and $d_{ij} = d(x_i, x_j)$, with $i, j \in \{1, 2, 3, 4\}$, $i \neq j$. Since (X, d) is Ptolemaic, we have

$$d_{12} \cdot d_{34} \leq d_{13} \cdot d_{24} + d_{14} \cdot d_{23} \tag{4}$$

Using (4) and (2) we obtain

$$d_{12}^{\alpha} \cdot d_{34}^{\alpha} \leq \big(d_{13} \cdot d_{24} + d_{14} \cdot d_{23}\big)^{\alpha} \leq 2^{\alpha-1}(d_{13}^{\alpha} \cdot d_{24}^{\alpha} + d_{14}^{\alpha} \cdot d_{23}^{\alpha})$$

for $\alpha > 1$. In this case ε can be chosen as $1 - \frac{1}{2^{\alpha-1}}$. Thus, (X, ρ) is quasi-Ptolemaic for $\alpha > 1$. $\square$

If a semimetric space is b-metric, then it will be quasi-Ptolemaic. This can be seen in the following theorem.

Theorem 3.4. *Let (X, d) be a b-metric space with $k \geq 1$. Then (X, d) is quasi-Ptolemaic with $\varepsilon \in [1 - 1/(2k^2), 1)$.*

Proof. Let $x_1, x_2, x_3, x_4 \in X$ be different points. Since d is b-metric,

$$d(x_1, x_2) \leq k[d(x_1, x_3) + d(x_3, x_2)],$$

$$d(x_3, x_4) \leq k[d(x_3, x_2) + d(x_2, x_4)].$$

It follows from the validity of

$$\max_{i,j,l,m \in \{1,2,3,4\}, m \neq l} \left\{ \frac{d(x_i, x_j)}{d(x_l, x_m)} \right\} \geq 1$$

for the points x_1, x_2, x_3, x_4 that

$$d(x_1, x_2)d(x_3, x_4) \leq k^2 \Bigg(d(x_1, x_3)d(x_2, x_4) \left(\frac{d(x_2, x_3)}{d(x_2, x_4)} + 1 \right) +$$

$$+ d(x_1, x_4)d(x_2, x_3) \left(\frac{d(x_2, x_3)}{d(x_1, x_4)} + \frac{d(x_2, x_4)}{d(x_1, x_4)} \right) \Bigg) \leq$$

$$\leq 2k^2 \max_{i,j,l,m \in \{1,2,3,4\}, m \neq l} \left\{ \frac{d(x_i, x_j)}{d(x_l, x_m)} \right\} (d(x_1, x_3)d(x_2, x_3) + d(x_1, x_4)d(x_2, x_3)).$$

To find $\varepsilon \in (0, 1)$ we solve the following equation:

$$\frac{1}{1 - \varepsilon} = 2k^2 \max_{i,j,l,m \in \{1,2,3,4\}, m \neq l} \left\{ \frac{d(x_i, x_j)}{d(x_l, x_m)} \right\}.$$

From this equality we can say the following:

$$\frac{1}{2k^2(1 - \varepsilon)} = \max_{i,j,l,m \in \{1,2,3,4\}, m \neq l} \left\{ \frac{d(x_i, x_j)}{d(x_l, x_m)} \right\} \geq 1,$$

then we obtain $\varepsilon \geq 1 - \dfrac{1}{2k^2}$. □

However, if the semimetric space (X, d) is quasi-Ptolemaic, then we cannot say that it is b-metric. It can be seen in the next example:

Example 3.5. (A Ptolemaic semimetric space that is not b-metric for any k.) Let $X = \{i, -i, \pm\frac{1}{n}\}$ be a semimetric space with semimetric

$$d(x, y) = \begin{cases} 1, & \text{if } x = i, y = -i \\ 0, & \text{if } x = y, \\ \dfrac{2}{mn}, & \text{if } x = \pm\dfrac{1}{m}, y = \pm\dfrac{1}{n}, \\ \dfrac{1}{n}, & \text{if } x = \pm i, y = \pm\dfrac{1}{n}, \end{cases}$$

where n, m — natural numbers. Since for an arbitrary sequence of three points $x = i$, $y = -i$ and $z_n = \dfrac{1}{n}$ there is a n_0 number such that for all $n \geq n_0$ $(n \in \mathbb{N})$ the coefficient of the b-metric must be greater than $n/2$, that is, $k > \dfrac{n}{2}$. This means that this space is not b-metric. But this space is Ptolemaic. To check this,

we need to consider the following options:
1) Let $x = i$, $y = -i$, $z = \pm\frac{1}{n}$ and $t = \pm\frac{1}{m}$. Then

$$d(x,y) = 1, \qquad d(z,t) = \frac{2}{mn},$$

$$d(x,t) = d(t,y) = \frac{1}{m}, \qquad d(x,z) = d(y,z) = \frac{1}{n}.$$

So $d(x,y)d(z,t) = d(x,t)d(y,z) + d(x,z)d(t,y)$.
2) Let $x = \pm\frac{1}{m}$, $y = \pm\frac{1}{n}$, $z = \pm\frac{1}{l}$ and $t = \pm\frac{1}{s}$. Then

$$d(x,y) = \frac{2}{mn}, \qquad d(x,t) = \frac{2}{ms}, \qquad d(x,z) = \frac{2}{ml},$$

$$d(t,z) = \frac{2}{sl}, \qquad d(y,z) = \frac{2}{nl}, \qquad d(y,t) = \frac{2}{ns}.$$

So $d(x,y)d(z,t) \leq d(x,t)d(y,z) + d(x,z)d(t,y)$.
3) Let $x = \pm i$, $y = \pm\frac{1}{m}$, $z = \pm\frac{1}{l}$ and $t = \pm\frac{1}{s}$. Then

$$d(x,y) = \frac{1}{m}, \qquad d(x,t) = \frac{1}{s}, \qquad d(x,z) = \frac{1}{l},$$

$$d(z,t) = \frac{2}{sl}, \qquad d(y,z) = \frac{2}{lm}, \qquad d(y,t) = \frac{2}{sm}.$$

And so $d(x,y)d(z,t) \leq d(x,t)d(y,z) + d(x,z)d(t,y)$.

4. Quasi-Ptolemaic space on Möbius structures

Let $T = (x_1, x_2, x_3, x_4)$ be four different points in the semimetric space (X, d). We call *the absolute cross-ratio* of T,

$$abs(T) = abs(x_1, x_2, x_3, x_4) = \frac{d(x_1, x_3) \cdot d(x_2, x_4)}{d(x_1, x_2) \cdot d(x_3, x_4)}.$$

Let us have two semimetrics $d(x,y)$ and $d'(x,y)$ on X $(x, y \in X)$ containing at least four pairwise distinct points. They are called *Möbius equivalent* [6] [9] if for any four pairwise different points $x_1, x_2, x_3, x_4 \in X$

$$\frac{d(x_1, x_2) \cdot d(x_3, x_4)}{d(x_1, x_3) \cdot d(x_2, x_4)} = \frac{d'(x_1, x_2) \cdot d'(x_3, x_4)}{d'(x_1, x_3) \cdot d'(x_2, x_4)}.$$

In particular, for any positive real function f in X the semimetrics $d(x,y)$ and $d'(x,y) = \frac{d'(x,y)}{(f(x) \cdot f(y))}$ are Möbius equivalent.

The set of all pairwise Möbius equivalent semimetrics $d(x,y)$ is called a *Möbius structure* [6] [9], in the set X. A Möbius structure $\mathcal{M}$ on X is called *Ptolemaic (quasi-Ptolemaic)* if there exists a semimetric $d(x,y)$ from $\mathcal{M}$ that is, Ptolemaic (quasi -Ptolemaic).

Theorem 4.1. *In any quasi-Ptolemaic Möbius structure* $\mathcal{M}$ *with parameter* $\varepsilon \in (0,1)$ *on the set* X *there is a bounded b-metric with parameter* $k = \frac{1}{1-\varepsilon}$.
More precisely: for any pair of distinct points $a \neq b$ *in* X *and for any semimetric* $d \in \mathcal{M}$ *the semimetric*

$$D(x,y) = \frac{(1-\varepsilon) \cdot d(x,y) \cdot d(a,b)}{(d(x,a) + d(x,b)) \cdot (d(y,a) + d(y,b))} \tag{5}$$

is a b-metric with parameter $k = \frac{1}{1-\varepsilon}$ *on* X*. Moreover,* $D(x,y) \in \mathcal{M}$ *and* $D(x,y) \leq 1$.

Proof. Let $d(x,y) \in \mathcal{M}$ be quasi-Ptolemaic with $\varepsilon \in (0,1)$ on X. The semimetric $D(x,y)$ is Möbius equivalent to $d(x,y)$. Since (X,d) is quasi-Ptolemaic, we obtain the following relation for a semimetric $d(x,y)$:

$$(d(x,a)+d(x,b))\cdot(d(y,a)+d(y,b)) \geq d(x,a)\cdot d(y,b)+d(x,b)\cdot d(y,a) \geq$$

$$\geq (1-\varepsilon)\cdot d(x,y)\cdot d(a,b).$$

From this relation we obtain next:

$$D(x,y) \leq \frac{(1-\varepsilon)\cdot d(x,y)\cdot d(a,b)}{(1-\varepsilon)\cdot d(x,y)\cdot d(a,b)} = 1.$$

And this is prove the boundedness of the semimetric $D(x,y)$.

Let $f(x)=d(x,a)+d(x,b)$. Then

$$\frac{D(x,y)+D(y,z)}{D(x,z)} = \frac{(1-\varepsilon)\cdot d(x,y)\cdot d(a,b)}{f(x)\cdot f(y)}\cdot\frac{f(x)\cdot f(z)}{(1-\varepsilon)\cdot d(x,z)\cdot d(a,b)}+$$

$$+\frac{(1-\varepsilon)\cdot d(y,z)\cdot d(a,b)}{f(y)\cdot f(z)}\cdot\frac{f(x)\cdot f(z)}{(1-\varepsilon)\cdot d(x,z)d(a,b)} =$$

$$= \frac{d(x,y)\cdot f(z)+d(y,z)\cdot f(x)}{d(x,z)\cdot f(y)}.$$

By virtue of the quasi-Ptolemaic inequality

$$d(x,y)\cdot f(z)+d(y,z)\cdot f(x) = [d(x,y)\cdot d(z,a)+d(y,z)\cdot d(x,a)]+$$

$$+[d(x,y)\cdot d(z,b)+d(y,z)\cdot d(x,b)] \geq$$

$$\geq (1-\varepsilon)\cdot[d(x,z)\cdot d(y,a)+d(x,z)\cdot d(y,b)] = (1-\varepsilon)\cdot d(x,z)\cdot f(y)$$

and consequently

$$\frac{D(x,y)+D(y,z)}{D(x,z)} \geq \frac{(1-\varepsilon)\cdot d(x,z)\cdot f(y)}{d(x,z)\cdot f(y)} = 1-\varepsilon.$$

or equivalently

$$D(x,z) \leq \frac{1}{1-\varepsilon}\cdot\big(D(x,y)+D(y,z)\big),$$

it can be shown that the semimetric $D(x,y)$ is a b-metric with parameter $k=\dfrac{1}{1-\varepsilon}$. □

One thing we can say from the theorem 4.1 is that, using the semimetric considered in Example 3.2, we can construct a semimetric of the form given in (5). This semimetric is the metric, and is the Möbius equivalent of the Ptolemaic semimetric considered in Example 3.2.

5. Quasimöbius mappings of Ptolemaic spaces

Let $\eta : [0,\infty) \longrightarrow [0,\infty)$ be a homeo-morphism of $[0,\infty)$ onto itself. An injective mapping $f : X \longrightarrow Y$ in semimetric spaces (X,d) and (Y,ρ) is called *η-quasimöbius* [7] [9], if the estimate $abs(f(T)) \leq \eta(abs(T))$ holds for any $T=(x_1,x_2,x_3,x_4)$ in X. In particular, $f : X \longrightarrow Y$ is a *möbius* mapping if $abs(f(T)) = \eta(abs(T))$ for each T in X. Since $abs(T) = abs(x_1,x_2,x_3,x_4) = \dfrac{1}{abs(x_1,x_3,x_2,x_4)}$, we obtain that

$$\left(\eta\left(\frac{1}{abs(T)}\right)\right)^{-1} \leq abs(f(T)) \leq \eta(abs(T)). \tag{6}$$

Theorem 5.1. *Let $\eta = kt$, where $t \in [0, +\infty)$, $k = const > 0$. Then every η-quasimöbius mapping f maps the quasi-Ptolemaic semimetric space X with parameter $\varepsilon \in (0,1)$ to the quasi-Ptolemaic semimetric space Y with parameter $1 - \dfrac{1-\varepsilon}{k}$.*

Proof. Since X is quasi-Ptolemaic with the parameter $\varepsilon \in (0,1)$ for any pairwise distinct points $x_1, x_2, x_3, x_4 \in X$, using the formula (6) we obtain that

$$\frac{1-\varepsilon}{k} \leq \frac{1}{\eta\left(\dfrac{1}{abs(x_1,x_2,x_3,x_4)}\right)} + \frac{1}{\eta\left(\dfrac{1}{abs(x_1,x_2,x_4,x_3)}\right)} =$$

$$= \frac{1}{k}\Big(abs(x_1,x_2,x_3,x_4) + abs(x_1,x_2,x_4,x_3)\Big) \leq$$

$$\leq abs(f(x_1,x_2,x_3,x_4)) + abs(f(x_1,x_2,x_4,x_3)) =$$

$$= abs(y_1,y_2,y_3,y_4) + abs(y_1,y_2,y_4,y_3).$$

Or equivalently,

$$1 - \left(1 - \frac{1-\varepsilon}{k}\right) \leq abs(y_1,y_2,y_3,y_4) + abs(y_1,y_2,y_4,y_3),$$

where $y_1, y_2, y_3, y_4 \in Y$ and $y_i = f(x_i)$, $i \in \{1,2,3,4\}$. □

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Bu mening birinchi maqolam. Men bu maqolani hurmat va ehtirom ila ota-onamga bag'ishlayman.
This is my first article. I dedicate this article to my parents with respect and reverence.

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