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Existence and uniqueness for a fractional differential equation involving Atangana-Baleanu derivative by using a new contraction

Hojjat Afshari[a,*], Vahid Roomi[b], Monireh Nosrati[a]

[a] *Department of Mathematics, Faculty of Science, University of Bonab, Bonab, Iran.*
[b] *Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.*

Abstract

In this study, by using some new contractions, we obtain an existence and uniqueness conclusion for a fractional differential equation with Atangana-Baleanu derivative as follows:

$$\begin{aligned} ({}_0^{ABC}D^{\kappa}\delta)(s) &= h(s,\delta(s)), \qquad s \in J,\ 0 \le \kappa \le 1, \\ &\delta(0) = \delta_0, \end{aligned}$$

where D^{ς} is the Atangana-Baleanu derivative of order ς and f is continuous with $f(0, \hbar(0)) = 0$.

Keywords: Atangana-Baleanu fractional derivative, orbitally complete, fixed point.
2010 MSC: 34K13, 34A34, 34K40

1. Introduction

Fractional calculus is a part of mathematical analysis that studies the performance of derivative and integral operations on non integer orders. In the past years, early works in fractional calculus were limited to mathematics.

*Corresponding author
Email addresses: hojat.afshari@yahoo.com (Hojjat Afshari), roomi@azaruniv.ac.ir (Vahid Roomi), adjoudi@yahoo.com (Monireh Nosrati)

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Recently, this field had found many applications in various directions such as applied mathematics, electrochemistry, tracer in fluid flows, fractional-order multi-poles in electromagnetism, finance, signal processing, bio-engineering, viscoelasticity, fluid mechanics, and fluid dynamics [16].

Definitions of two famous fractional derivatives, namely Riemann-Liouville and Caputo, included a singular kernel. Recently Caputo and Fabrizio provided a definition with a nonsingular kernel which the properties of this new definition can be found in [14].

In 2016, the interesting and new derivatives without singular kernel were introduced by Atangana and Baleanu, which generalized the Caputo-Fabrizio definition [7]. Atangana- Baleanu derivative contains Mittag-Leffler function as a nonlocal and nonsingular kernel. Recently, Atangana et al. provided the numerical approximation to the fractional advection-diffusion equation whose fractional derivatives are Atangana-Baleanu derivative of Riemann-Liouville type [11].

One of the efficient methods in investigating the existence and uniqueness of the solutions of differential equations is using of fixed point theory and for this reason there is a long history of presenting various fixed point theorems (see [1]-[12]).

In [6], Bryant proposed a relaxation of the Banach contractive condition involving the iteration of the function. Recall that, for a positive integer n we denote by h^n the nth iterate of h, so that $u_0 = h^0u$ and $h^{n+1}u = h(h^nu)$ for $u \in X$ and $n \in \mathbb{N}$. Hereupon, the tripled (X, d, h) represent a metric space (X, d) with a self-mapping h on it. We shall use (X^*, d, h) to indicate the corresponding metric space is complete. Shatanawi and Karapınar in [9] introduced F_S-contractions in the sense of Wardowski and Seghal and F_J-contractions in the sense of Wardowski and Jachymski. Then, they ensured some existence and uniqueness fixed point results.

In this work, by using some contractions of mentioned results in [9], we obtain an existence and uniqueness conclusion for a fractional differential equation with Atangana-Baleanu derivative as follows:

$$\begin{aligned} &({}_0^{ABC}D^{\kappa}\delta)(s) = h(s, \delta(s)), && s \in J,\ 0 \le \kappa \le 1, \\ &\delta(0) = \delta_0, \end{aligned} \tag{1.1}$$

where D^{ς} is the Atangana-Baleanu derivative of order ς and f is continuous with $f(0, \hbar(0)) = 0$. Also, Throughout the article $\mathfrak{J}$ denote $[0, 1]$.

Now, here is the following fixed point theorem.

Definition 1.1. [10] Let $\delta \in H^1(a, b),\ a < b$ and $0 \le \kappa \le 1$. The Atangana-Baleanu fractional derivative in Caputo sense of δ of order κ is defined by

$$({}_a^{ABC}D^{\kappa}\delta)(s) = \frac{B(\kappa)}{1-\kappa}\int_a^s \delta'(\nu)E_{\kappa}\left[-\kappa\frac{(s-\nu)^{\kappa}}{1-\kappa}\right]d\nu, \tag{1.2}$$

where E_{κ} is the Mittag-Leffler function defined by $E_{\kappa}(z) = \Sigma_{n=0}^{\infty}\frac{z^n}{\Gamma(n\kappa+1)}$ and $B(\kappa)$ is a normalizing positive function satisfying $B(0) = B(1) = 1$ (see [13]). The associated fractional integral is defined by

$$({}_a{}^{AB}I^{\kappa}\delta)(s) = \frac{1-\kappa}{\kappa}\delta(s) + \frac{\kappa}{B(\kappa)}({}_aI^{\kappa}\delta)(s), \tag{1.3}$$

where ${}_aI^{\kappa}$ is the left Riemann-Liouville fractional integral given as

$$({}_aI^{\kappa}\delta)(s) = \frac{1}{\Gamma(\kappa)}\int_a^s (s-\nu)^{\kappa-1}\delta(\nu)d\nu. \tag{1.4}$$

Consider $d : M \times M \to [0, \infty)$ given by

$$d(\delta, \sigma) = \| (\delta - \sigma)^2 \|_{\infty} = \sup_{s\in J}(\delta(s) - \sigma(s))^2,$$

where $M = C(J, \mathbb{R})$ denote the set of continuous functions, (M, d) is a complete b-metric space with $s_1 = 2$. We discuss about problem

$$\begin{aligned} &({}_0^{ABC}D^\kappa \delta)(s) = h(s, \delta(s)), && s \in J,\ 0 \le \kappa \le 1, \\ &\delta(0) = \delta_0, \end{aligned} \tag{1.5}$$

where D^κ is the Atangana-Baleanu derivative in Caputo sense of order κ and
$h : J \times M \to M$ is continuous with $h(0, \delta(0)) = 0$.

Proposition 1.2. *[5] For $0 < \kappa < 1$, we have*

$$({}^{AB}I_b^{\kappa\,ABC}D^\kappa \delta)(s) = \delta(s) - \delta(b). \tag{1.6}$$

2. Main results

For a positive integer n we denote by h^n the nth iterate of h, so that $y = h^0 y$ and $h^{n+1}y = h(h^n y)$ for $y \in \mathrm{X}$ and $n \in \mathrm{N}$. The tripled (X, d, h) represent a metric space (X, d) with a self-mapping h on it. We shall use $(\mathrm{X}*, d, h)$ to indicate the corresponding metric space is complete. Also on (X, d, h) an orbit of $y_0 \in \mathrm{X}$ is the set

$$O(y_0) = \{h^n y_0 : n = 0, 1, 2, ...\},$$

and $\rho(y_0)$ denote to the diameter of the set $O(y_0)$. Note that for any subset B of X, $\rho(B) = \sup\{d(u, y) : u, y \in B\}$ is the diameter of B. We shall use the tripled (X_{0*}, d, h) if for some $y \in \mathrm{X}$, every Cauchy sequence from $O(y)$ converges in X. In this case, the corresponding space is called orbitally complete.

Corollary 2.1. *[9] For $(\mathrm{X}, d*, h)$ with $p : \mathrm{X} \to \mathrm{N}$, we suppose there exists $\tau > 0$ such that for $v, w \in \mathrm{X}$*

$$d(h^{p(y)}y, h^{p(y)}w \le e^{-\tau} d(y, w). \tag{2.1}$$

Assume there exists $y_0 \in \mathrm{X}$ such that $0 < \rho < \infty$. Moreover, (X, d) is $h-$orbitally complete.Then h has a unique fixed point.

Theorem 2.2. *Suppose for the problem given (1), we have the following:*

$$|h(p, \delta(p)) - h(p, \sigma(p))| \le \frac{B(\xi)\Gamma(\xi)}{(1-\xi)\Gamma(\xi) + 1} e^{-\tau} \left| \sqrt{|\delta(p)|} - \sqrt{|\sigma(p)|} \right|, \quad p \in J,$$

and

$$|h(p, \delta(p))| + |h(p, \sigma(p))| \le \frac{B(\xi)\Gamma(\xi)}{(1-\xi)\Gamma(\xi) + 1} e^{-\tau} \left| \sqrt{|\delta(p)|} - \sqrt{|\sigma(p)|} \right|, \quad p \in J, \tau > 0.$$

Then the problem (1) has a unique solution.

Proof. Considering $d : M \times M \to \mathbb{R}^+$, define a metric d as follows: $d(\delta, \sigma) = \sup_{p \in J} |\delta(p) - \sigma(p)|$. Applying the Atangana-Baleanu integral and using Proposition 1.2, we have

$$(T\delta)(p) = \delta_0 +_0^{AB} I^\xi h(p, \delta(p)).$$

We show that the equation (1) has a unique solution.

$$\begin{aligned}
|T\delta(p) - T\sigma(p)| &= \left|{}^{AB}I^{\xi}\left[h(p,\delta(p)) - h(p,\sigma(p))\right]\right| \\
&\leq \frac{1-\xi}{B(\xi)}|h(p,\delta(p)) - h(p,\sigma(p))| \\
&+ \frac{\xi}{B(\xi)}\, {}_0I^{\xi}|h(p,\delta(p)) - h(p,\sigma(p))| \\
&\leq \frac{1-\xi}{B(\xi)}.\frac{B(\xi).\Gamma(\xi)}{(1-\xi)\Gamma(\xi)+1}e^{-\tau}\left|\sqrt{|\delta(p)|} - \sqrt{|\sigma(p)|}\right| \\
&+ \frac{\xi}{B(\xi)}.\frac{B(\xi).\Gamma(\xi)}{(1-\xi)\Gamma(\xi)+1}\, {}_0I^{\xi}(1)e^{-\tau}\left|\sqrt{|\delta(p)|} - \sqrt{|\sigma(p)|}\right| \\
&= \left(\frac{B(\xi).\Gamma(\xi)}{(1-\xi)\Gamma(\xi)+1}e^{-\tau}\left|\sqrt{|\delta(p)|} - \sqrt{|\sigma(p)|}\right|\right) \times \left(\frac{1-\xi}{B(\xi)} + \frac{\xi}{B(\xi)}.\frac{1}{\xi.\Gamma(\xi)}\right) \\
&\leq e^{-\tau}\sup\left|\sqrt{|\delta(p)|} - \sqrt{|\sigma(p)|}\right|.
\end{aligned}$$

Also we have:

$$\begin{aligned}
|T\delta(p)| + |T\sigma(p)| &= \left|{}^{AB}I^{\xi}\left[h(p,\delta(p))\right]\right| + \left|{}^{AB}I^{\xi}\left[h(p,\sigma(p))\right]\right| \\
&\leq^{AB} I^{\xi}\left[|h(p,\delta(p))| + |h(p,\sigma(p))|\right] \\
&\leq \frac{1-\xi}{B(\xi)}.\frac{B(\xi).\Gamma(\xi)}{(1-\xi)\Gamma(\xi)+1}e^{-\tau}\left|\sqrt{|\delta(p)|} + \sqrt{|\sigma(p)|}\right| \\
&+ \frac{\xi}{B(\xi)}.\frac{B(\xi).\Gamma(\xi)}{(1-\xi)\Gamma(\xi)+1}\, {}_0I^{\xi}(1)e^{-\tau}\left|\sqrt{|\delta(p)|} + \sqrt{|\sigma(p)|}\right| \\
&= \left(\frac{B(\xi).\Gamma(\xi)}{(1-\xi)\Gamma(\xi)+1}e^{-\tau}\left|\sqrt{|\delta(p)|} + \sqrt{|\sigma(p)|}\right|\right) \times \left(\frac{1-\xi}{B(\xi)} + \frac{\xi}{B(\xi)}.\frac{1}{\xi.\Gamma(\xi)}\right) \\
&\leq e^{-\tau}\sup\left|\sqrt{|\delta(p)|} + \sqrt{|\sigma(p)|}\right| \leq \sup\left|\sqrt{|\delta(p)|} + \sqrt{|\sigma(p)|}\right|
\end{aligned}$$

On the other hand, we have:

$$[\sup\left(|T\delta(p)| + |T\sigma(p)|\right) \leq \sup\left|\sqrt{|\delta(p)|} + \sqrt{|\sigma(p)|}\right|]$$

Also we have the following relation:

$$\begin{aligned}
d(T^2\delta, T^2\sigma) &= \sup(|T^2\delta(p) - T^2\sigma(p)|) \\
&= \sup(|T\delta(p) - T\sigma(p)|) \times \sup(|T\delta(p) + T\sigma(p)|) \\
&\leq \sup(|T\delta(p) - T\sigma(p)|) \times \sup(|T\delta(p)| + |T\sigma(p)|) \\
&\leq e^{-\tau}(\sup\sqrt{|\delta(p)|} - \sqrt{|\sigma(p)|}) \times (\sup\sqrt{|\delta(p)|} + \sqrt{|\sigma(p)|}) \\
&= e^{-\tau}\sup||\delta(p)| - |\sigma(p)|| \\
&\leq e^{-\tau}\sup|\delta(p) - \sigma(p)| \\
&= e^{-\tau}d(\delta,\sigma).
\end{aligned}$$

So, the condition (2.1) for $p : \mathrm{X} \to \mathbb{N}$, $p(\delta) = 2$ and $\delta \in M$ is true. Hence by Corollary 2, the problem (1), has a unique solution. □

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