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Relatives of a Theorem of Rus-Hicks-Rhoades

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In memory of Billy Rhoades

Abstract

Let (X, d) be a complete metric space and $f : X \to X$ a map satisfying $d(fx, f^2x) \leq \alpha d(x, fx)$ for every $x \in X$, where $0 < \alpha < 1$. The fixed point theorems due to Rus (1973) and Hicks-Rhoades (1979) on such maps were extended or improved by Park (1980), Harder-Hicks-Saliga (1993), Suzuki (2001), and Jachymski (2003). Moreover, fixed point theorems of Zermelo (1904), Banach (1922), Caristi (1976), extended versions for multimaps due to Nadler (1969) and Covitz-Nadler (1970) are also closely related to the Rus-Hicks-Rhoades theorem. Finally, we unify these theorems based on a particular form of our 2023 Metatheorem.

Keywords: complete metric space, fixed point, poset, fixed point theorem, preorder, stationary point, maximal element.
2010 MSC: 47H10, 54E35, 54H25, 58E30.

1. Introduction

From the late 1970's to the early 1990's we had engaged to study the metric fixed point theory. Since the appearances of the Ekeland variational principle in 1972-79 and the Caristi fixed point theorem in 1976, they are main targets to generalize, modify, imitate, and apply to several directions. Influenced such an

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atmosphere, we obtained a metatheorem, a machinary to find equivalents of the maximal element principle, and applied it until 2000.

During the middle of that period, in 1979-1993, we made eleven joint papers with Billy E. Rhoades. In [1], we collected briefly the contents of all of the joint papers. As Billy once recalled: "Our collaboration has ceased only because our research interests have moved in different directions."

From early 1990's we studied analytical fixed point theory on topological vector spaces including a large number of extensions of the Brouwer fixed point theorem. Later we turned to establish the Knaster-Kuratowski-Mazurkiewicz (KKM) theory or theory of abstract convex spaces with several multimap classes. These have been continued until recently.

Eventually, in Summer of 2022, we came back to our metatheorem and applied it to establish the ordered fixed point theory [2] for ordered sets. While we were doing this we found a theorem of Hicks and Rhoades [3] in the 2003 paper of Jachymski [4]. We noticed that it has an interesting long history. In fact, it was originated from Rus [5] and related to one of our earlier work [6] in 1980. It is natural to think that the theorem of Rus [5] and Hicks-Rhoades [3] is closely related to the Banach contraction principle in 1922, but we found that it is more closer to the multi-valued versions due to Nadler [7] in 1969 and Covitz-Nadler [8] in 1970.

The aim of the present survey article is to trace such history of the Rus-Hicks-Rhoades theorem, and to show its grown-up versions or equivalents or closely related theorems. Such theorems are too many and could be called its relatives.

This article is organized as follows: In Section 2, chronological early history of the theorem in 1973-2001 from its birth to grown-up versions. We introduce theorems due to Rus (1973), Hicks-Rhoades (1979), Park (1980), Park-Rhoades (1980), Harder-Hicks-Saliga (1993), and Suzuki (2001). Section 3 devotes to show its equivalent formulations or relatives given by Jachymski [4] with our comments. In Section 4, we show its equivalent formulations or new relatives based on our 2023 metatheorem. Such equivalent forms include the Nadler type extensions of the Banach contraction principle. Finally, the epilogue is given in Section 5.

2. Early Extensions of the Theorem

In this section, we introduce theorems of Rus and Hicks-Rhoades and their early extensions or relatives.

2.1. Rus [5] in 1973

The origin of our story begins from Rus [5]:

Theorem A. [5] *Let f be a continuous selfmap of a complete metric space X satisfying*

$$d(fx, f^2x) \leq \alpha d(x, fx) \quad \text{for every } x \in X,$$

where $0 < \alpha < 1$. Then f has a fixed point.

Note that f is assumed to be continuous.

2.2. Hicks and Rhoades [3] in 1979

The following is given independently to Rus [5]:

Theorem B. [3] *Let T be a nonexpansive selfmap of a complete metric space (X, d) satisfying*

$$d(Tx, T^2x) \leq \alpha d(x, Tx) \quad \text{for all } x \in X,$$

where $0 < \alpha < 1$. Then T has a fixed point.

Note that every nonexpansive map is continuous. Hence Theorem B follows from Theorem A.

2.3. Park [6] in 1980

We gave a Banach type fixed point theorem with respect to contraction pairs of selfmaps on complete metric spaces in [6]:

Theorem C. *Let g and h be selfmaps of a metric space X. If there exists a sequence $\{u_i : i \in \omega\}$ in X such that*

$$u_{2i+1} = gu_{2i}, \quad u_{2i+2} = hu_{2i+1} \quad \textit{for } i \in \omega$$

and $\overline{\{u_i\}}$ is complete, and if there exists a $\lambda \in [0,1)$ such that

$$d(gx, hy) \leq \lambda d(x, y) \tag{1}$$

holds any distinct $x,\ y \in \overline{\{u_i\}}$ satisfying either $x = hy$ or $y = gx$, then either

(i) *g or h has a fixed point in $\{u_i\}$, or*

(ii) *$\{u_i\}$ converges to some $\xi \in X$, and*

$$d(u_i, \xi) \leq \frac{\lambda^i}{1-\lambda} d(u_0, u_i) \quad \textit{for } i > 0.$$

Further, if one of g or h is continuous at ξ and (1) holds for any distinct $x,\ y \in \{\overline{u}_i\}$, then ξ is a common fixed point of g and h.

This seems to be artificial, but this was made to include many existing results of similar nature. In fact, by putting $h = g$ and $y = gx$, (1) becomes $d(gx, g^2x) \leq \lambda d(x, gx)$. Hence Theorem C(i) extends Theorem A, where the continuity is redundant.

2.4. Park and Rhoades [9] in 1980

In this paper we established several fixed point theorems involving hypotheses weak enough to include a number of known theorems as special cases.

Let f be a selfmap of a topological space X. The orbit $O(x)$ of $x \in X$ under f is defined by $O(x) = \{x, f(x), f^2(x), \dots\}$. A function $G : X \to [0, \infty)$ is said to be f-orbitally lower semicontinuous at a point $p \in X$ if, for every $x_0 \in X$, $x_{n_k} \to p$ implies $G(p) \leq \liminf_k G(x_{n_k})$ where $\{x_{n_k}\}_{k=1}^\infty$ is a subsequence of $\{x_n\}_{n=1}^\infty$, which is defined by $x_{n+1} = f(x_n)$, i.e. $\{x_n\}_{n=1}^\infty = O(x_1)$.

Theorem D. *Let d be a nonnegative real valued function defined on $X \times X$ such that $d(x, y) = d(y, x)$ and $d(x, y) = 0$ iff $x = y$. If there exists a point $u \in X$ such that $\lim_n d(f^{n+1}(u), f^n(u)) = 0$, and if $\{f^n(u)\}$ has a convergent subsequence with limit $p \in X$, then p is a fixed point of f iff $G(x) = d(x, f(x))$ is f-orbitally lower semicontinuous at p.*

Theorem E. *Let f be a selfmap of a metric space (X, d) satisfying:*

(1) *$\delta(O(x)) < \infty$ for each $x \in X$, where δ denotes the diameter.*

(2) *There exists a $u \in X$ such that $O(u)$ has a cluster point $p \in X$.*

(3) *There exists a function $\varphi : [0, \infty) \to [0, \infty)$ which is nondecreasing, continuous from the right and satisfies $\phi(t) < t$ for each $t > 0$ and the inequality*

$$d(f(x), f^2(x)) \leq \varphi(\delta(O(x) \cup O(f(y)))) \quad \textit{for each} \quad x, y \in X.$$

Then p is the unique fixed point of f and $\lim_n f^n(u) = p$.

These results extend works of Pal-Maiti, Park, Hegedüs and Daneš. A 2-metric space version of Theorem 2 is added in [9].

Here we add new information related to the above theorems:

Kirk-Saliga [10] in 2001 and Chen-Cho-Yang [11] in 2002 introduced the following concept: We say that $\varphi : M \to \mathbb{R}$ is *lower semicontinuous from above* if given any sequence $\{x_n\}$ in M, the conditions $\lim_n x_n = x$ and $\{\varphi(x_n)\} \downarrow r \Rightarrow \varphi(x) \leq r$.

This concept can be applied to improve Theorem D.

Moreover, Theorem A and B are consequences of Theorem E.

2.5. Harder, Hicks, and Saliga [12] in 1993

Abstract: "Let X be a complete metric space. Fixed point theorems are proved for non-selfmaps. Also several examples are given involving selfmaps and Caristi type conditions."

The following is given as ([12], Corollary 2):

Theorem F. [12] *Suppose C is a closed subset of a complete metric space X and $T : C \to X$ is continuous with $C \subset T(C)$. If there exists $k > 1$ such that for every $x \in T^{-1}(C)$, $d(Tx, T^2x) \geq kd(x, Tx)$, then T has a fixed point in C.*

Set $\alpha = \frac{1}{k}$. Then $d(x, Tx) \leq \alpha d(Tx, T^2x)$. The authors said that their Corollary 2 has many corollaries.

Note that Theorem F is a variant of Theorems A, B and the continuity of T seems to be redundant.

2.6. Suzuki [13] in 2001

For metric spaces, there have appeared thousands of generalizations, modifications, imitations, extensions, and their applications. One of them for our present work is the following:

Theorem G. [13] *Let X be a complete metric space and let T be a mapping from X into itself. Suppose that there exist a τ-distance p on X and $r \in [0, 1)$ such that $p(Tx, T^2x) \leq r \cdot p(x, Tx)$ for all $x \in X$. Assume that either of the following holds:*

(i) *If $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, and $\lim_n p(x_n, Tx_n) = 0$, and $\lim_n p(x_n, y) = 0$, then $Ty = y$;*

(ii) *If $\{x_n\}$ and $\{Tx_n\}$ converge to y, then $Ty = y$;*

(iii) *T is continuous.*

Then there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $p(x_0, x_0) = x_0$.

Theorems A and B are particular case of (iii) and we showed several times that the continuity is redundant. It is open that whether this also holds for Theorem G.

There may be similar theorems for thousands of artificial metric type spaces.

3. Jachymski's 2003 Theorem [4]

From now on Max($\preceq$) (resp. Min($\preceq$)) denotes the set of maximal (resp. minimal) elements of an ordered set $(X, \preceq)$, and Fix(f) (resp. Per(f)) denotes the set of all fixed (resp. periodic) points of a map $f : X \to X$ for a nonempty set X.

In our previous articles, we introduced many examples of maps $f : X \to X$ satisfying $\mathrm{Per}(f) = \mathrm{Fix}(f) \neq \emptyset$. Such sets X can have more rich properties by applying the following main theorem of Jachymski ([4], Theorem 2):

Jachymski's 2003 Theorem. *Let X be a nonempty abstract set and $T : X \to X$. The following statements are equivalent:*

(a) $\mathrm{Per}(T) = \mathrm{Fix}(T) \neq \emptyset$.

(b) (Zermelo) *There exists a partial ordering $\preccurlyeq$ such that every chain in $(X, \preccurlyeq)$ has a supremum and T is progressive with respect to $\preccurlyeq$ (that is, $x \preccurlyeq Tx$ for all $x \in X$).*

(c) (Caristi) *There exists a complete metric d and a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$ such that T satisfies the Caristi condition $d(x, Tx) \leq \varphi(x) - \varphi(Tx)$ for all $x \in X$.*

(d) *There exists a complete metric d and a d-Lipschitzian function $\varphi : X \to \mathbb{R}^+$ such that T satisfies the Caristi condition and T is nonexpansive with respect to d; i.e.*

$$d(Tx, Ty) \le d(x, y) \quad \textit{for all} \quad x, y \in X.$$

(e) (Hicks-Rhoades) *For each $\alpha \in (0, 1)$, there exists a complete metric d such that T is nonexpansive with respect to d and*

$$d(Tx, T^2x) \le \alpha d(x, Tx) \quad \textit{for all} \quad x \in X.$$

(f) *There exists a complete metric d such that T is continuous with respect to d and for each $x \in X$, the sequence $(T^n x)_{n=1}^{\infty}$ is convergent (the limit may depend on x).*

(g) *There exists a partition of X, $X = \bigcup_{\gamma \in \Gamma} X_\gamma$, such that all the sets X_γ are nonempty, T-invariant and pairwise disjoint, and for all $\gamma \in \Gamma$, $T|_{X_\gamma}$ has a unique periodic point.*

(h) *For each $\alpha \in (0, 1)$, there exists a partition of X, $X = \bigcup_{\gamma \in \Gamma} X_\gamma$, and complete metrics d_γ on X_γ such that all the sets X_γ are nonempty; T-invariant and pairwise disjoint; and*

$$d_\gamma(Tx, Ty) \le \alpha d_\gamma(x, y) \quad \textit{for all } x, y \in X.$$

Remark 3.1. [4] Implication (a) $\Longrightarrow$ (b) is a converse to Zermelo's theorem [14]. Implication (a) $\Longrightarrow$ (c) is a reciprocal to Caristi's theorem [15]; in fact, a stronger result, (a) $\Longrightarrow$ (d) can be obtained here. Implication (a) $\Longrightarrow$ (e) is a converse to a fixed point theorem of Hicks-Rhoades. Finally (a) $\Longrightarrow$ (f) answers a question posed by Matkowski.

Remark 3.2. Jachymski's 2003 Theorem is a rich source of relatives of the Rus-Hicks-Rhoades theorem, and each of (b)–(h) seems to be order theoretic fixed point theorems. For them, we state our own comments in [17] as follows:

(a) This could be $\mathrm{Max}(\preccurlyeq) \subset \mathrm{Fix}(T) = \mathrm{Per}(T) \neq \emptyset$ by defining $\preccurlyeq$ on X.

(b) Zermelo's theorem is improved in [17] and has many equivalents there. Note that its conclusion should be as above in (a).

(c) Caristi's theorem is improved in [17] and its conclusion should be as in (a). Moreover, the lower semicontinuity there can be replaced by the one *from above*; see Subsection 2.4.

(d) This is a variant of Caristi's theorem and its conclusion should be as in (a). Nonexpansiveness may be redundant.

(e) This is the origin of our study in the present article. Here nonexpansiveness is redundant in view of Theorem H(iv) in [2] and the next section.

Motivated by Jachymski's 2003 theorem, we deduced the following remarkable result in [17]:

Corollary 3.1. *Let $(X, \preccurlyeq)$ be a nonempty partially ordered set and $f : X \to X$. The following statements are equivalent:*

(a) $\mathrm{Per}(f) = \mathrm{Fix}(f) \supset \mathrm{Max}(\preccurlyeq) \neq \emptyset$.

(b) (Zermelo) *There exists an $x_0 \in X$ such that $S(x_0) = \{x \in X : x_0 \preccurlyeq x\}$ has an upper bound and f is progressive with respect to $\preccurlyeq$.*

(c) (Caristi) *There exists a complete metric d and a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$ from above such that f satisfies the Caristi condition*

$$x \preccurlyeq f(x) \Longleftrightarrow d(x, f(x)) \le \varphi(x) - \varphi(f(x))$$

for all $x \in X$.

Note that this theorem implies several of the Caristi type [15] and the Zermelo type theorems [14] in our previous works. Moreover, the Caristi theorem is equivalent to results of Ekeland, Takahashi, Dancs-Hegedš-Medvegyev, Daneš, Penot, and many others; see [2]. Those are also close or far relatives of the Rus-Hicks-Rhoades theorem.

4. Extended Equivalencies of the Nadler Theorem [7] in 1969

Let (X, d) be a metric space and $\mathrm{Cl}(X)$ denote the family of all nonempty closed subsets of X (not necessarily bounded). For $A, B \in \mathrm{Cl}(X)$, set

$$H(A,B) = \max\{\sup\{d(a,B) : a \in A\},\ \sup\{d(b,A) : b \in B\}\},$$

where $d(a,B) = \inf\{d(a,b) : b \in B\}$. Then H is called a generalized Hausdorff metric since it may have infinite values.

Based on our well-known 2023 Metatheorem and the more general form of Nadler's fixed point theorem [7] established by Covitz-Nadler [8] in 1970, we obtained the following ([2], Theorem H):

THEOREM H. *Let X be a complete metric space, $T : X \to \mathrm{Cl}(X)$ a multimap, and $0 < h < 1$. Then the following equivalent statements hold:*

(α) *There exists an element $v \in X$ such that $H(T(v), T(w)) > hd(v, w)$ for any $w \in X \backslash \{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ such that, for any $x \in X\backslash\{f(x)\}$, there exists a $y \in X\backslash\{x\}$ satisfying $d(f(x), f(y)) \le hd(x, y)$, then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(γ) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ satisfying $d(f(x), f^2(x)) \le hd(x, f(x))$ for all $x \in X\backslash\{f(x)\}$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(δ) *Let $\mathfrak{F}$ be a family of multimaps $T : X \multimap X$ such that, for any $x \in X\backslash T(x)$, there exists $y \in X\backslash\{x\}$ satisfying $H(T(x), T(y)) \le hd(x, y)$. Then $\mathfrak{F}$ has a common stationary element $v \in X$, that is, $\{v\} = T(v)$ for all $T \in \mathfrak{F}$.*

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $T : X \to X$ satisfying $H(T(x), T(y)) \le hd(x, y)$ for all $x \in X$ and any $y \in T(x)\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in X$, that is, $\{v\} = T(v)$ for all $T \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that for each $x \in X\backslash Y$ there exists a $z \in X\backslash\{x\}$ satisfying $H(T(x), T(z)) \le hd(x, z)$ for a $T : X \multimap X$, then there exists a $v \in X \cap Y = Y$.*

REMARK 4.1. (1) When $\mathfrak{F}$ is a singleton, (β)-(ϵ) are denoted by (β1)-(ϵ1), respectively, They are also logically equivalent to (α)-(η).

(2) When $\mathfrak{F}$ is a singleton, (γ1) extends Theorems A and B. This is our origin of finding redundantness of the continuity or nonexpansivity there.

(3) Moreover, (δ1) and (ϵ1) extend the well-known theorems of Nadler [7] and Covitz-Nadler [8]. Consequently, these are close relatives of Theorems of Rus [5] and Hicks-Rhoades [3]. This is rather surprising and they are all close relatives of the Banach contraction principle in 1922.

5. Conclusion

In this article, we showed many close relatives of improved versions of fixed point theorems of Rus in 1973 and Hicks-Rhoades in 1979. They are all close relatives of the Banach contraction principle. In fact. by the 2003 Jachymski theorem and our Theorem H, we found so many relatives in Sections 2-4 from Zermelo's implicit theorem in 1904 to our Theorem H in 2022.

This shows the essence of rather surprising nature of the universe of Mathematics. Moreover, almost all of the results mentioned in this article is related to the Banach principle, which is elementary, but occupies very important position among its so many relatives.

In many fields of mathematical sciences, there are plentiful number of theorems concerning maximal points or various fixed points that can be applicable our Metatheorem in [17, 18]. Some of such theorems can be seen in our previous works and the present article. Therefore, our Metatheorem is a machine to expand our knowledge easily. In this article we presented relatively old and well-known examples.

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