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Further discussion on the interpolative contractions

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Abstract

In this article, we shall discuss the notion of the interpolative contractions that take the attention of several authors. The first result in this direction was announced in 2018 in the paper entitled "Revisiting the Kannan Type Contractions via Interpolation." As mentioned in the title, the paper claimed to consider the existing results in different frameworks. In this short note, we underline this fact by indicating certain known relations between classical contractions and interpolative contractions.

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1. Introduction

In the investigative nature of mathematics, not only finding new results and observing new facts, but also extending, improving, and generalizing the existing results are important, worthwhile, and indispensable. Furthermore, obtaining a new solution by using new techniques to address the already-solved problems is valuable. In other words, bringing in new frameworks to the existing solutions and problems or looking at the problems from different perspectives is worthwhile. By reconsidering the old difficulties with new aspects may be superficial at first glance, but they can play important roles in the advances of the qualitative sciences.

If a proposed technique and method are considered meaningless compared to the techniques and methods that already exist to solve a problem, scientific developments cannot be achieved. This is like someone who solves linear equation systems with the elimination method and finds it meaningless and unnecessary to

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solve the mentioned linear system with matrices via the row-reduced echelon form. On the contrary, by using the row-reduced echelon form of matrices, together with developing technology in computer science, software programs lead to a solution in seconds for a system consisting of as many equations with a huge amount of unknowns that we could not even imagine before.

In 2018, along with the motivation mentioned in the previous paragraphs, the notion of interpolative contraction was introduced [5]. As was seen in the paper's title, the author claimed to bring a new aspect to the well-known Kannan-type contraction mappings. The authors propose a new contraction, namely, Kannan-type interpolative contraction, to get better estimations for Kannan-type contractions, see also, [6, 7, 8].

In this short note, we shall consider simple algebraic inequalities to indicate that interpolative contraction inequalities can be dominated by classical contraction inequalities. On the other hand, the equivalences we obtain do not change the fact that the idea of the interpolative contraction brings innovation to the fixed point theory and provides a new perspective.

2. Preliminaries

In 1968, Kannan [3, 4] reported a new type of contraction mapping, which is a partially affirmative answer to one of the first questions regarding the Banach Contraction Mapping Principle (BCMP), below: Is there a discontinuous mapping with a unique fixed point in the complete metric spaces?

Theorem 2.1. *Let (X, d) be a complete metric spaces and $T : X \to X$ be a Kannan contraction mapping, i.e.,*

$$d(Tx, Ty) \leq \lambda \left[d(x, Tx) + d(y, Ty) \right],$$

for all $x, y \in X$, where $\lambda \in \left[0, \frac{1}{2}\right)$. Then T has a unique fixed point.

Another version of the Kannan contraction mapping theorem is the following:

Theorem 2.2. *Let (X, d) be a complete metric spaces and $T : X \to X$ be a Kannan contraction mapping, i.e.,*

$$d(Tx, Ty) \leq \left[ad(x, Tx) + bd(y, Ty) \right],$$

for all $x, y \in X$, where a, b are non-negative real numbers such that $a, b \in [0, 1)$, and $a + b \leq 1$. Then T has a unique fixed point.

Note that this answer is partially affirmative to the question above. Indeed, such mappings are not continuous, yes, but at the same time, the contraction conditions in Kannan's Theorem and Banach's Theorem are not the same, either. The answer is affirmative under the new contraction condition. Now, it is known as the Kannan-type contraction.

In what follows, we recollect the definition of the notion of Kannan-type contraction via interpolation.

Definition 2.3. Let (X, d) be a metric space. We say that the self-mapping $T : X \to X$ is an interpolative Kannan-type contraction if there exist a constant $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \lambda \left[d(x, Tx) \right]^{\alpha} \cdot \left[d(y, Ty) \right]^{1-\alpha}.$$

for all $x, y \in X$.

Theorem 2.4. *Let (X, d) be a complete metric space and T be an interpolative Kannan-type contraction. Then, the mapping T has a fixed point in X.*

In this note, we shall indicate the relations between the interpolative Kannan-type contraction and the standard Kannan-type contraction in the context of complete metric spaces.

3. Main Section

We shall consider a very simple observation based on a trivial algebraic inequality.

Lemma 3.1. *For each $p, q \in [0, \infty)$ and each $\alpha \in (0, 1)$,* $\quad p^\alpha q^{1-\alpha} \leq (p + q)$.

Proof. If $p = 0$ or $q = 0$, then the inequality follows trivially. Thus, we presume that $p > 0$ and $q > 0$. In this case, we find

$$p^\alpha q^{1-\alpha} \leq (\max\{p, q\})^\alpha (\max\{p, q\})^{1-\alpha} = \max\{p, q\} \leq p + q.$$

□

Employing the technical lemma above, we can state the following theorem:

Theorem 3.2. *Let the pair (X, d) denote a complete metric space. Then, the Kannan-type interpolative contraction is dominated by the standard Kannan-type contraction.*

Proof. By Lemma 3.1, we have

$$p^\alpha q^{1-\alpha} \leq (p + q),$$

for any $\alpha \in (0, 1)$. Thus, by letting $p = d(x, Tx)$ and $q = d(y, Ty)$, we derive that

$$[d(x, Tx)]^\alpha [d(y, Ty)]^{1-\alpha} \leq d(x, Tx) + d(y, Ty),$$

for all $x, y \in X$ and for all self-mapping T, defined on X. □

Note that in Theorem 2.1, the contraction constant λ varies between 0 and $\frac{1}{2}$. On the other hand, in Theorem 4.5, the contraction constant λ changes between 0 and 1. Notice that for the variant of the standard Kannan-type contraction, the coefficients a, b can be considered as $\lambda := \frac{1}{2}\max\{a, b\}$.

In the following example, we shall underline the fact that the interpolative contractions may help to get better approximations.

Example 3.3. Let X be non-negative real numbers and consider the standard Euclidean metric on it. We shall consider only the following points to illustrate the validity of the main theorem: $8, \frac{1}{8}, x_1 = 32$ and $\frac{1}{32}$. Indeed, these four points are sufficient to examine all distinct possibilities. In addition, we let T be a self-mapping sucht that $Tx = \frac{x}{2}$.

Now, we can see the significance of the interpolation via the following distances:

Table 1: Table of considered points and their values

x	Tx	$d(x, Tx)$
8	4	4
$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$
$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$
32	16	16

Table 2: Table of comparision

(x, y)	$d(x,Tx)^{\alpha}d(y,Ty)^{1-\alpha}$	Comparision	$d(x,Tx)+d(y,Ty)$
(8,8)	4	$<$	8
(8,32)	8	$<$	20
(32,32)	16	$<$	32
(8, 1/8)	1	$<$	5/4
(8,1/32)	1/4	$<$	4+1/64
(32,1/8)	2	$<$	65/4
(32, 1/32)	1/2	$<$	16+1/64
(1/8,1/8)	1/4	$<$	1/2
(1/8,1/32)	1/16	$<$	17/64
(1/32,1/32)	1/64	$<$	1/32

When we take $\alpha = \frac{1}{2}$ for simplicity, we observe the following

As one checks it easily, in any case, the standard Kannan-type contraction inequality dominates the interpolative Kannan-type contraction.

On the other hand, in considering the estimation of the inequalities, the values of the interpolative Kannan-type contraction are very small. This comparison is very useful if we look at it with the aspects of "economical way."

As an immediate consequence, we shall consider Chatterjea-type contractions and interpolative Chatterjea-type contractions. First of all, we recall the standard Chatterjea-type contractions.

Theorem 3.4. *Let (X, d) be a complete metric space and $T : X \to X$ be a Kannan contraction mapping, i.e.,*

$$d(Tx, Ty) \leq \lambda [d(y, Tx) + d(x, Ty)],$$

for all $x, y \in X$, where $\lambda \in \left[0, \frac{1}{2}\right)$. Then T has a unique fixed point.

In what follows, we shall recollect the notion of the interpolative Chatterjea-type contraction:

Definition 3.5. Let (X, d) be a metric space. We say that the self-mapping $T : X \to X$ is an interpolative Chatterjea-type contraction if there exist a constant $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \lambda [d(y, Tx)]^{\alpha} \cdot [d(x, Ty)]^{1-\alpha}.$$

for all $x, y \in X$.

Theorem 3.6. *Let (X, d) be a complete metric space and T be an interpolative Chatterjea-type contraction. Then, the mapping T has a fixed point in X.*

Keeping the technical Lemma 3.1 in mind, we shall state the following theorem:

Theorem 3.7. *Let the pair (X, d) denote a complete metric space. Then, the Chatterjea-type interpolative contraction is dominated by the standard Chatterjea-type contraction.*

Proof. By Lemma 3.1, we have

$$p^{\alpha} q^{1-\alpha} \leq (p + q),$$

for any $\alpha \in (0,1)$. Thus, by letting $p = d(y,Tx)$ and $q = d(x,Ty)$, we derive that

$$[d(y,Tx)]^{\alpha}[d(x,Ty)]^{1-\alpha} \leq d(y,Tx) + d(y,Ty),$$

for all $x, y \in X$ and for all self-mapping T, defined on X. □

In the next section, we shall consider some further consequences of the main theorem. Indeed, it does not need to consider all possibilities. We shall consider only some of the well-known results.

4. Further Consequences

We shall start this section by recalling the very famous fixed point theorem that was proved by Rus, Ćirić and Reich, separately and independently, see [6, 9, 10, 11, 12, 13] and related references therein.

First, we consider the general version of Lemma 3.1 in the following way:

We shall consider a very simple observation based on a trivial algebraic inequality.

Lemma 4.1. *For each $p_1, p_2, \cdots, p_n \in [0,\infty)$ for some fixed $n \in \mathbb{N}$ and each $\alpha_1, ..., \alpha_n \in (0,1)$ with $\sum_{i=1}^{n} \alpha_i < 1$ we have*

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \leq \sum_{i=1}^{n} \alpha_i.$$

The proof of this lemma is obtained by a trivial manipulation of the proof of Lemma 3.1, thus, we skipped the proof.

The following theorem is well-known result of Rus, Reich, Ćirić.

Theorem 4.2. *(See e.g. [9, 10, 11, 12, 13]) Let (X,d) be a complete metric spaces and $T : X \to X$ be a Rus-Reich-Ćirić contraction mapping, i.e.,*

$$d(Tx,Ty) \leq \lambda [d(x,y) + d(x,Tx) + d(y,Ty)], \tag{1}$$

for all $x, y \in X$, where $\lambda \in \left[0, \frac{1}{3}\right)$. Then T has a unique fixed point.

Note that the statement of Rus-Reich-Ćirić Theorem can be given in different but equivalent various ways. As an example, we shall state one of the variants of this theorem as follows:

Theorem 4.3. *Let (X,d) be a complete metric spaces and $T : X \to X$ be a Rus-Reich-Ćirić contraction mapping, i.e.,*

$$d(Tx,Ty) \leq ad(x,y) + bd(x,Tx) + cd(y,Ty), \tag{2}$$

for all $x, y \in X$, where a, b, c are nonnegative real numbers such that $0 \leq a+b+c < 1$. Then T has a unique fixed point.

On what follows, we state the definition of the notion of an interpolative Rus-Reich-Ćirić-type contraction:

Definition 4.4. Let (X,d) be a metric space. We say that the self-mapping $T : X \to X$ is an *interpolative Rus-Reich-Ćirić type contraction*, if there exist a constant $\lambda \in [0,1)$ and $\alpha, \beta \in (0,1)$ such that

$$d(Tx,Ty) \leq \lambda [d(x,y)]^{\beta} \cdot [d(x,Tx)]^{\alpha} \cdot [d(y,Ty)]^{1-\alpha-\beta}. \tag{3}$$

for all distinct $x, y \in X \setminus F_T(X)$.

Theorem 4.5. *Let (X,d) be a complete metric space and T be an interpolative Rus-Reich-Ćirić type contraction. Then T has a fixed point in X.*

The first result of this section is the following theorem.

Theorem 4.6. *Let the pair (X,d) denote a complete metric space. Then, the Rus-Reich-Ćirić-type interpolative contraction is dominated by the standard Rus-Reich-Ćirić-type contraction.*

Proof. On account of Lemma 4.1, we shall choose $p_1 = d(x,y)$ and $\alpha_1 = \alpha$, $\alpha_2 = \beta$ and and $\alpha_3 = 1-\alpha-\beta$, $p_2 = d(x,Tx)$ and $p_3 = d(y,Ty)$, we derive that

$$[d(x,Tx)]^{\alpha} \cdot [d(y,Ty)]^{1-\alpha-\beta} \leq d(x,y) + d(x,Tx) + d(y,Ty),$$

for all $x, y \in X$ and for all self-mapping T, defined on X. □

Next, we shall consider the relation between the standard Hardy-Rogers type contraction and the interpolative Hardy-Rogers-type contraction.

Theorem 4.7. *[2] Let (X,d) be a complete metric space. Let $T: X \to X$ be a given mapping such that*

$$d(Tx,Ty) \leq \lambda \left[d(x,y) + d(x,Tx) + d(y,Ty) + [\frac{1}{2}(d(x,Ty) + d(y,Tx))] \right],$$

for all $x, y \in X$, where $\lambda \in (0,1)$. Then, T has a unique fixed point in X.

Another variant of Hardy-Rogers theorem is the following.

Theorem 4.8. *[2] Let (X,d) be a complete metric space. Let $T: X \to X$ be a given mapping such that*

$$d(Tx,Ty) \leq a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty) + a_4[\frac{1}{2}(d(x,Ty) + d(y,Tx))],$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4 are non-negative reals such that $a_1 + a_2 + a_3 + a_4 < 1$. Then, T has a unique fixed point in X.

We shall recall the notion of *interpolative Hardy-Rogers-type contractions.*

Definition 4.9. Let (X,d) be a metric space. We say that the self-mapping $T: X \to X$ is an *interpolative Hardy-Rogers type contraction*, if there exist $\lambda \in [0,1)$ and $\alpha, \beta, \gamma \in (0,1)$ with $\alpha + \beta + \gamma < 1$ such that

$$d(Tx,Ty) \leq \lambda [d(x,y)]^{\beta} \cdot [d(x,Tx)]^{\alpha} \cdot [d(y,Ty)]^{\gamma} \cdot \left[\frac{1}{2}(d(x,Ty) + d(y,Tx)) \right]^{1-\alpha-\beta-\gamma}, \tag{4}$$

for all $x, y \in X$ such that $x \neq Tx$.

Theorem 4.10. *[7] Let (X,d) be a complete metric space and T be an interpolative Hardy-Rogers-type contraction. Then, T has a fixed point in X.*

The second result of this section is the following theorem.

Theorem 4.11. *Let the pair (X,d) denote a complete metric space. Then, the Hardy-Rogers -type interpolative contraction is dominated by the standard Hardy-Rogers -type contraction.*

Proof. On account of Lemma 4.1, we shall consider $p_1 = d(x,y)$ $p_2 = d(x,Tx)$, $p_3 = d(y,Ty)$, and $p_4 = \frac{1}{2}(d(x,Ty) + d(y,Tx))$, together $\alpha_1 = \beta$, $\alpha_2 = \alpha$ and $\alpha_3 = \gamma$ and $\alpha_4 = 1-\alpha-\beta-\gamma$, we find that

$$[d(x,y)]^{\beta} \cdot [d(x,Tx)]^{\alpha} \cdot [d(y,Ty)]^{\gamma} \cdot \left[\tfrac{1}{2}(d(x,Ty) + d(y,Tx))\right]^{1-\alpha-\beta-\gamma}$$

$$\leq d(x,y) + d(x,Tx) + d(y,Ty) + \tfrac{1}{2}(d(x,Ty) + d(y,Tx)),$$

for all $x, y \in X$ and for all self-mapping T, defined on X. □

5. Discussions

In [5], the aim of the paper was mentioned as to involve the idea of interpolation theory into the fixed point theory. Indeed, the interpolation theory is a very rich and interesting topic in mathematics. We shall rewrite the obtained inequality to better understand the main result of this paper:

$$[d(y,Tx)]^{\alpha}[d(x,Ty)]^{1-\alpha} \leq d(y,Tx)+d(y,Ty). \tag{5}$$

More precisely, we observe that

$$d(Tx,Ty) \leq \lambda [d(y,Tx)]^{\alpha}[d(x,Ty)]^{1-\alpha} \leq \lambda d(y,Tx)+d(y,Ty), \tag{6}$$

for some $\lambda \in (0,1)$.

Regarding Example 3.3 and the inequality above, we conclude that the estimation that is derived from the interpolative component is finer than the standard component. For the convenient iteration process, the estimation of "interpolative components" brings the desired results faster. Roughly speaking, it is more economical than the standard setting. As it was claimed in the first paper in this direction [5], it proposed to revisit the existing results to get finer results.

Here, we can underline also that usage of some auxiliary functions do not change our conclusion. For instance, let us consider c-comparision function which has been used very often in many articles deals with the fixed point theory.

Let Φ be the family of functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(ψ_1) φ is nondecreasing;

(ψ_2) $\sum\limits_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$, where φ^n is the n^{th} iterate of φ.

These functions are known in the literature as *(c)-comparison functions*. One can easily deduces that if ψ is a (c)-comparison function, then

$$\varphi(t) < t \quad \text{for all } t > 0. \tag{7}$$

Examples of (c)-comparison functions are $\varphi_k(t) = kt$ for all $t \geq 0$, where $k \in [0,1)$.

Since such auxiliary functions are increasing or non-decreasing, the inequalities in (82) turn into

$$\varphi\left([d(y,Tx)]^{\alpha}[d(x,Ty)]^{1-\alpha}\right) \leq \varphi\left(d(y,Tx)+d(y,Ty)\right). \tag{8}$$

Hence, the main idea of this paper is also valid for the contraction inequalities via auxiliary functions.

Following the pioneer result [5], a number of papers appeared in this trend, in the setting of distinct structures, likes, partial metric spaces, b-metric spaces, dislocated metric spaces, fuzzy metric spaces, quasi metric spaces, and so on.

The observations of this paper are also valid in these different structures. The reason is trivial since we focus only the estimations of positive real numbers. In other words, in all these structures, we deal with distinct distance functions with a range of non-negative real numbers. The main theorem of the paper is based on Lemma 4.1 which is a simple observation on the non-negative real numbers. Naturally, the inequality 8 is still valid if we replace the standard metric d with another distance function, like quasi-metric, partial metric, b-metric, and so on.

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