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Fractional Tikhonov method for inverse source bi-parabolic: A priori parameter choice rule

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Abstract

In this work, the unknown source function for the bi-parabolic is investigated. This problem is non-well-posed. Applying a Fractional Tikhonov method to construct the regularized solution. After that, we have test the estimation $\|f^\delta_\gamma - f\|_{L_2} \to 0$, then $\delta \to 0$, under a priori rule.

Keywords: ill-posed, regularization method, Tikhonov method, Fractional Tikhonov method
2010 MSC: 26A33, 33E12, 35R11, 44A20

1. Introduction

We examine the problem as follows

$$\begin{cases} u_{tt}(x,t) + 2\Delta u_t(x,t) + \Delta^2 u(x,t) = \varphi(t) f(x), & (x,t) \in \mathcal{D} \times (0,T), \\ u_t(x,0) = 0, & (x) \in \partial\mathcal{D}, t \in (0,T], \\ u|_{\partial\mathcal{D}} = \Delta u|_{\partial\mathcal{D}} = 0, & x \in \mathcal{D}, \\ u(x,0) = 0, & x \in \mathcal{D}, \\ u(x,T) = g(x), & x \in \mathcal{D} \end{cases} \tag{1}$$

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where $\mathcal{D}$ be a bounded domain in $\mathbb{R}^2$ with the sufficiently smooth boundary $\partial\mathcal{D}$. The problem (1) is severely ill-posed in the sense of Hadamard. In this work, we primarily focus on obtaining the source function f from the provided data g and φ. In this case, the data at the final time is noise level. To discover a regularized approach, we must first create approximate solutions f_γ of f and prove that $\lim_{\epsilon\to 0}\left\|f_\gamma^\delta - f\right\|_{L_2} = 0$ in the appropriate norm for both f_γ^δ and f_γ. Applications of the inverse source issue include pollution detection and geophysical prospecting [12]. As far as, there are no solutions for the inverse source problem (1). Applications of bi-parabolic equations can be found in [6, 8]

The outline of this article is as follows. In Section 2, some preliminary results. The ill-posedness of the fractional inverse source problem (1) and the conditional stability are discussed in Theorem 3.1 and 3.2. In Section 4, the convergence rates for the Fractional Tikhonov regularized solution are considered via an a priori parameter choice rule.

2. Preliminaries

Let $\mathcal{K}$ be a real Hilbert space, and let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{K} \to \mathcal{K}$ be a linear, positive-definite, self-adjoint operator with compact inverse on $\mathcal{K}$. $\mathcal{A}$ has an orthonormal basis of eigenvectors $\phi_n \subset \mathcal{K}$ with real eigenvalues $\xi_n \in \mathbb{N}$.

$$\mathcal{A}\phi_n(x) = \xi_n\phi_n(x), \quad n \in \mathbb{N}, \text{ and } 0 < \xi_1 \le \xi_2 \le \cdots \text{ with } \xi_n \to \infty \text{ for } n \to \infty, \tag{2}$$

Definition 2.1. The Hilbert scale space $\mathcal{H}^\tau$, $(\tau > 0)$ defined by

$$\mathcal{H}^\tau := \Big\{ f \in L_2 : \sum_{n=1}^{\infty} \big(1+\xi_n^2\big)^\tau \big\langle f, \phi_n\big\rangle_{L_2}^2 \le \infty \Big\}, \tag{3}$$

with the norm

$$\|f\|_{\mathbb{H}^\tau}^2 = \sum_{n=1}^{\infty} \big(1+\xi_n^2\big)^\tau \big|\big\langle f, \phi_n\big\rangle_{L_2}\big|^2 \le \infty. \tag{4}$$

Lemma 2.2. *Let $\xi_n > \xi_1 > 0$, $\forall n \ge 1$ and $s \in [0, T]$, we have*

$$\int_0^T e^{\xi_n(s-T)}(T-s)ds = \frac{\big(1-(1+T\xi_n)e^{-\xi_n T}\big)}{\xi_n^2}, \tag{5}$$

Lemma 2.3. *See [13] Let $\omega > 0$, it gives*

$$\frac{1}{1+\omega^2} \le \max\Big\{\frac{3}{T^2}, 1\Big\}\frac{\big(1-(1+T\omega)e^{-\omega T}\big)}{\omega^2},$$
$$0 < \frac{\big(1-(1+t\omega)e^{-\omega t}\big)}{\omega^2} < T^2, \quad \forall t \in [0, T]. \tag{6}$$

Lemma 2.4. *[12] For constants $s \ge \xi_1^2$ and $\frac{1}{2} < \alpha < 1$, we have*

$$\mathcal{A}(s) = \frac{s}{\mathcal{C}^{2\alpha} + \gamma s^{2\alpha}} \le \mathcal{C}_1\gamma^{-\frac{1}{2\alpha}}, \tag{7}$$

where $\mathcal{C}_1 = \mathcal{C}_1(\alpha, \mathcal{C}_1) > 0$ are independent on α, s.

Lemma 2.5. *[12] For the constants $s \ge \big(1+\xi_1^2\big) > 0$ and $\frac{1}{2} < \alpha < 1$, we have*

$$\mathcal{A}_1(s) = \frac{\gamma s^{2\alpha-\tau}}{\mathcal{C}_2^{2\alpha} + \gamma s^{2\alpha}} \le \begin{cases} \mathcal{C}_3\gamma^{\frac{\tau}{2\alpha}}, & 0 < \tau < 2\alpha, \\ \mathcal{C}_4\gamma, & \tau \ge 2\alpha. \end{cases} \tag{8}$$

where $\mathcal{C}_3 = \mathcal{C}_3(\alpha, \tau, \mathcal{C}) > 0, \mathcal{C}_4 = \mathcal{C}_4\left(\alpha, \tau, (1+\xi_1^2)\right) > 0$ are independent on s.

3. Regularization and error estimate for unknown source (1)

Taking the inner product of both sides of (1) with $\phi_n(x)$, it gives

$$\begin{cases} \frac{d^2}{dt^2}\big\langle u(t),\phi_n\big\rangle_{L_2} + 2\xi_n \frac{d}{dt}\big\langle u(t),\phi_n\big\rangle_{L_2} + \xi_n^2\big\langle u(t),\phi_n\big\rangle_{L_2(\Omega)} = \varphi(t)\big\langle f,\phi_n\big\rangle_{L_2}, \quad t\in(0,T),\\ \big\langle u(0),\phi_n\big\rangle_{L_2} = 0,\\ \frac{d}{dt}\big\langle u(0),\phi_n\big\rangle_{L_2} = 0,\\ \big\langle u(T),\phi_n\big\rangle_{L_2} = \big\langle g,\phi_n\big\rangle_{L_2}. \end{cases} \tag{9}$$

We have the following system

$$u_n(t) = \big(1+\xi_n t\big)e^{-\xi_n t}u_n(0) + \Big(\int_0^t e^{\xi_n s}\varphi(s)ds\Big)te^{-\xi_n t}f_n - \Big(\int_0^t e^{\xi_n s}\varphi(s)ds\Big)f_n e^{-\xi_n t}, \tag{10}$$

with $u_n(t) = \big\langle u(.,t),\phi_n\big\rangle$, $f_n = \big\langle f,\phi_n\big\rangle$, $u(x,0) = \big\langle u(.,0),\phi_n\big\rangle = 0$ and $g_n = \big\langle g,\phi_n\big\rangle$. We obtain

$$u_n(t) = t\,\Big(\int_0^t e^{\xi_n(s-t)}\varphi(s)ds\Big)\big\langle f,\phi_n\big\rangle - \Big(\int_0^t e^{\xi_n(s-t)}s\varphi(s)ds\Big)\big\langle f,\phi_n\big\rangle. \tag{11}$$

Letting $t=T$, it gives

$$\big\langle g,\phi_n\big\rangle = T\,\Big(\int_0^T e^{\xi_n(s-T)}\varphi(s)ds\Big)\big\langle f,\phi_n\big\rangle - \Big(\int_0^T e^{\xi_n(s-T)}s\varphi(s)ds\Big)\big\langle f,\phi_n\big\rangle. \tag{12}$$

A simple transformation gives

$$\big\langle f,\phi_n\big\rangle = \frac{\big\langle g,\phi_n\big\rangle}{\int_0^T e^{\xi_n(s-T)}(T-s)\varphi(s)ds}, \text{ this leads to } f(x) = \sum_{n=1}^{\infty}\frac{\big\langle g,\phi_n\big\rangle\phi_n(x)}{\int_0^T e^{\xi_n(s-T)}(T-s)\varphi(s)ds}. \tag{13}$$

Theorem 3.1. *[11] The problem (1) is ill-posed.*

Theorem 3.2. *[11] If $\big\|f\big\|_{\mathcal{H}_{2\tau}} \le \mathcal{M}$ for $\mathcal{M}>0$ then*

$$\big\|f\big\|_{L_2} \le C(r,\mathcal{M})\|g\|_{L_2}^{\frac{r}{r+1}}, \text{ where } C(r,\mathcal{M}) = \frac{\mathcal{M}^{\frac{1}{r+1}}}{|\varphi_1|^{\frac{r}{r+1}}\big|1-(1+T\xi_1)e^{-\lambda_1 T}\big|^{\frac{r}{r+1}}}. \tag{14}$$

4. Fractional Tikhonov method

We have:

$$f_\gamma(x) = \sum_{n=1}^{\infty}\frac{\big|\int_0^T e^{\xi_n(s-T)}(T-s)\varphi(s)ds\big|^{2\alpha-1}}{\gamma+\Big|\int_0^T e^{\xi_n(s-T)}(T-s)\varphi(s)ds\Big|^{2\alpha}}\big\langle g,\phi_n\big\rangle\phi_n(x), \text{ for } \frac{1}{2}<\alpha<2. \tag{15}$$

The observed data g_δ of g with a noise level of δ satisfied

$$\big\|g_\delta - g\big\|_{L_2} \le \delta, \tag{16}$$

then we get

$$f_{\delta,\gamma}(x) = \sum_{n=1}^{\infty} \frac{\Big| \int_0^T e^{\xi_n(s-T)}(T-s)\varphi(s)ds\Big|^{2\alpha-1}}{\gamma + \Big| \int_0^T e^{\xi_n(s-T)}(T-s)\varphi(s)ds\Big|^{2\alpha}} \langle g_\delta, \phi_n\rangle \phi_n(x),, \text{ for } \frac{1}{2} < \alpha < 2. \tag{17}$$

4.1. A priori parameter choice rule

Theorem 4.1. *Suppose the a-priori condition (3.2) and the noise assumption (16) hold, then,*

$$\big\| f_\gamma^\delta - f\big\|_{L_2} \le \begin{cases} \varphi_2^{2\alpha-1}\delta^{\frac{\tau+1}{\tau+2}}\mathcal{M}^{\frac{1}{\tau+2}} + \mathcal{C}_2\delta^{\frac{\tau}{\tau+2}}\mathcal{M}^{\frac{2}{\tau+2}}, & 0<\tau<2\alpha, \\ \varphi_2^{2\alpha-1}\delta^{\frac{2\alpha+1}{2\alpha+2}}\mathcal{M}^{\frac{1}{2\alpha+2}} + \mathcal{C}_2\delta^{\frac{2\alpha}{2\alpha+2}}\mathcal{M}^{\frac{2}{2\alpha+2}}, & \tau \ge 2\alpha. \end{cases} \tag{18}$$

If $\delta \to 0$, then $\big\| f_\gamma^\delta - f\big\|_{L_2} \to 0$.

Proof. It is easy to see that

$$\big\| f_\gamma^\delta - f\big\| \le \big\| f_\gamma^\delta - f_\gamma\big\| + \big\| f - f_\gamma\big\| = \mathcal{I}_1 + \mathcal{I}_2. \tag{19}$$

Estimate of $\mathcal{I}_1$, thank to Lemma 2.2, 2.3 and (16), we have

$$\begin{aligned} \mathcal{I}_1 = \big\| f_\gamma^\delta - f_\gamma\big\|_{L_2} &= \Big\| \sum_{n=1}^{\infty} \frac{\big| \int_0^T e^{\xi_n(s-T)}(T-s)\varphi(s)ds\big|^{2\alpha-1}}{\gamma + \Big| \int_0^T e^{\xi_n(s-T)}(T-s)\varphi(s)ds\Big|^{2\alpha}} \langle g_\delta - g, \phi_n\rangle \phi_n(x)\Big\|_{L_2} \\ &\le \delta \sup_{n\in\mathbb{N}} \frac{\varphi_2^{2\alpha-1}{\xi_n^2}^{1-2\alpha}}{\gamma + \varphi_1^{2\alpha}\Big[\max\{\frac{3}{T^2},1\}\Big]^{-2\alpha}\big(\frac{1}{1+\xi_n^2}\big)^{2\alpha}} \le \delta \sup_{n\in\mathbb{N}} \frac{\varphi_2^{2\alpha-1}\xi_n^2}{\gamma\xi_n^{4\alpha} + |\varphi_1|^{2\alpha}\Big[\max\{\frac{3}{T^2},1\}\Big]^{-2\alpha}} \\ &\le \delta\ \varphi_2^{2\alpha-1} \sup_{n\in\mathbb{N}} \frac{\xi_n^2}{\underline{C}^{2\alpha} + \gamma\xi_n^{4\alpha}} \le \varphi_2^{2\alpha-1}\delta[\gamma]^{-\frac{1}{2\alpha}}. \end{aligned} \tag{20}$$

whereby $\underline{\mathcal{C}} = |\varphi_1|\big[\max\{\frac{3}{T^2},1\}\big]^{-1}$. Next, estimate of $\mathcal{I}_2$, thank to the Lemma 2.5, one has

$$\begin{aligned} \mathcal{I}_2 &= \big\| f - f_\gamma\big\|_{L_2} \\ &= \Big\| \sum_{n=1}^{\infty}\Big(\frac{\big| \int_0^T e^{\xi_n(s-T)}(T-s)\varphi(s)ds\big|^{2\alpha-1}}{\big| \int_0^T e^{\xi_n(s-T)}(T-s)\varphi(s)ds\big|^{2\alpha} + \gamma}\langle g, \phi_n\rangle - \frac{1}{\big| \int_0^T e^{\xi_n(s-T)}(T-s)\varphi(s)ds\big|}\langle g, \phi_n\rangle\Big)\phi_n(x)\Big\|_{L_2} \\ &= \Big\| \sum_{n=1}^{\infty} \frac{\gamma(1+\xi_n^2)^{-\tau}}{\Big| \int_0^T e^{\xi_n(s-T)}(T-s)\varphi(s)ds\Big|^{2\alpha} + \gamma} \cdot \frac{(1+\xi_n^2)^{\tau}\langle g, \phi_n\rangle}{\Big| \int_0^T e^{\xi_n(s-T)}(T-s)\varphi(s)ds\Big|}\phi_n(x)\Big\|_{L_2} \\ &\le \mathcal{M} \sup_{n\in\mathbb{N}} \frac{\gamma(1+\xi_n^2)^{-\tau}}{\Big|\gamma + \varphi_1^{2\alpha}\Big[\max\{\frac{3}{T^2},1\}\Big]^{-2\alpha}\big(\frac{1}{1+\xi_n^2}\big)^{2\alpha}\Big| + \gamma} \\ &\le \mathcal{M} \sup_{n\in\mathbb{N}} \frac{\gamma(1+\xi_n^2)^{2\alpha-\tau}}{\Big|\varphi_1\Big[\max\{\frac{3}{T^2},1\}\Big]^{-1}\Big|^{2\alpha} + \gamma(1+\xi_n^2)^{2\alpha}} \le \begin{cases} \mathcal{C}_2\mathcal{M}\gamma^{\frac{\tau}{2\alpha}}, & 0<\tau<2\alpha, \\ \mathcal{C}_3\mathcal{M}\gamma, & \tau\ge 2\alpha. \end{cases} \end{aligned} \tag{21}$$

From (20) and (21), we obtain

$$\left\|f_{\gamma}^{\delta}-f\right\|_{L_2} \leq \varphi_2^{2\alpha-1}\delta\gamma^{-\frac{1}{2\alpha}} + \begin{cases}\mathcal{C}_2\mathcal{M}\gamma^{\frac{\tau}{2\alpha}}, & 0<\tau<2\alpha,\\ \mathcal{C}_3\mathcal{M}\gamma, & \tau \geq 2\alpha.\end{cases} \tag{22}$$

γ is chosen as follows

$$\gamma = \begin{cases}\left(\frac{\delta}{\mathcal{M}}\right)^{\frac{2\alpha}{\tau+2}}, & 0<\tau<2\alpha,\\ \left(\frac{\delta}{\mathcal{M}}\right)^{\frac{2\alpha}{2\alpha+1}}, & \tau\geq 2\alpha.\end{cases} \tag{23}$$

Then we have the following result

$$\left\|f_{\gamma}^{\delta}-f\right\|_{L_2} \leq \begin{cases}\varphi_2^{2\alpha-1}\delta^{\frac{\tau+1}{\tau+2}}\mathcal{M}^{\frac{1}{\tau+2}} + \mathcal{C}_2\delta^{\frac{\tau}{\tau+2}}\mathcal{M}^{\frac{2}{\tau+2}}, & 0<\tau<2\alpha,\\ \varphi_2^{2\alpha-1}\delta^{\frac{2\alpha+1}{2\alpha+2}}\mathcal{M}^{\frac{1}{2\alpha+2}} + \mathcal{C}_2\delta^{\frac{2\alpha}{2\alpha+2}}\mathcal{M}^{\frac{2}{2\alpha+2}}, & \tau\geq 2\alpha.\end{cases} \tag{24}$$

The proof is completed. □

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