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Improving Many Metric Fixed Point Theorems

In memory of 65 years with Gyoung

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Abstract

We found that many metric fixed point theorems hold for orbitally complete quasi-metric spaces. In order to show this, we obtain several basic principles extending the Banach contraction principle and new fixed point theorems for the Rus-Hicks-Rhoades maps with a large number of their consequences. Moreover, we improve theorems on mappings with contractive conditions with some auxiliary functions and on mappings with asymptotically regularity. Consequently, a large number of known metric fixed point theorems are extended with almost trivial short proofs.

Keywords: Quasi-metric space, fixed point, RHR contraction principle, orbitally complete, orbitally continuous
2010 MSC: 06A75, 47H10, 54E35, 54H25, 58E30, 65K10

1. Introduction

A Rus-Hicks-Rhoades (RHR) map $f : X \to X$ on a quasi-metric space (X, δ) is the one satisfying

$$\delta(fx, f^2x) \leq \alpha\, \delta(x, fx) \text{ for all } x \in X$$

with a given $\alpha \in [0, 1)$.

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Recently we obtained metric fixed point theorems on the RHR maps with a large number of its consequences; see [1]–[6]. Since then we have found further related literature on the maps. In the present paper, we want to add up further applications of our RHR theorem and related results as a supplement of [6]. Consequently, a large number of metric fixed point theorems for complete metric spaces are known to hold for orbitally complete quasi-metric spaces.

The present article is organized as follows:

In Section 2, we introduce the Rus-Hicks-Rhoades Contraction Principle (Theorem P) as a proper generalization of the Banach Contraction Principle for quasi-metric spaces. Section 3 devotes many historically well-known consequences of Theorem P or the RHR Contraction Principle with short and clear proofs.

The various contractions have been extended with the aid of some auxiliary functions, for example, comparison functions, control functions, gauge functions, simulation functions, etc. In Section 4, we collect fixed point theorems for contractive type maps with some auxiliary functions. We also show that such results can be extended to generalized RHR maps.

Section 5 devotes to the Ćirić-Proinov-Górnicki type maps having the asymptotical regularity. For any quasi-metric fixed point theorem, if we add the asymptotical regularity to the subject map, then its conclusion is definitely strengthened.

In Section 6, we give some comments on a problem raised by Billy E. Rhoades [7] in 1977 with its solution by S.-S. Chang [8] in 1986. Actually, we give a trivial answer to the problem.

Finally, Section 7 devotes some historical remarks on the subject of the present article. We recall our study on the RHR maps and the motivation of the present article. Finally, we suggest some conclusion in order to improve the metric fixed point theory.

2. The Rus-Hicks-Rhoades Contraction Principle

For quasi-metric spaces (X,δ), simply δ is not symmetric. For details on them, we follow Jleli-Samet [9], Alsulami et al. [10] and Park [1]–[6].

Definition. The *orbit* of a selfmap $T : X \to X$ at $x \in X$ is the set $O(x,T) = \{T^n x : n = 0, 1, 2, \dots\}$. The space X is said to be *T-orbitally complete* if every (right)-Cauchy sequence in $O(x,T)$ is convergent in X. A selfmap T of X is said to be *T-orbitally continuous* at $x_0 \in X$ if

$$\lim_{n\to\infty} T^n(x) = x_0 \implies \lim_{n\to\infty} T^{n+1}(x) = T(x_0)$$

for any $x \in X$.

The following in Park [1], [4], [6] is called the Rus-Hicks-Rhoades (RHR) Contraction Principle:

Theorem P. *Let (X,δ) be a quasi-metric space and let $T : X \to X$ be an RHR map; that is,*

$$\delta(T(x), T^2(x)) \le \alpha\, \delta(x, T(x)) \text{ for every } x \in X,$$

where $0 \le \alpha < 1$.

(i) *If X is T-orbitally complete, then, for each $x \in X$, there exists a point $x_0 \in X$ such that*

$$\lim_{n\to\infty} T^n(x) = x_0$$

and

$$\delta(T^n(x), x_0) \le \frac{\alpha^n}{1-\alpha}\delta(x, T(x)), \quad n = 1, 2, \cdots,$$

$$\delta(T^n(x), x_0) \le \frac{\alpha}{1-\alpha}\delta(T^{n-1}(x), T^n(x)), \quad n = 1, 2, \cdots.$$

(ii) *x_0 is a fixed point of T, and, equivalently,*

(iii) $T : X \to X$ *is orbitally continuous at* $x_0 \in X$.

This was proved in [6] by analyzing a typical proof of the Banach Contraction Principle given by Art Kirk ([11], Theorem 2.2).

For the condition: there exists $0 < \alpha < 1$ such that $d(T(x), T^2(x)) \leq \alpha \cdot d(x, T(x))$, for all $x \in X$, we meet the following names: graphic contraction, iterative contraction, weakly contraction, Banach mapping, etc.

Moreover, the following consequence of Theorem P in Park [6] extends the usual Banach Contraction Principle:

Theorem Q. *Let* (X, δ) *be a quasi-metric space and let* $T : X \to X$ *be an improved Banach contraction, that is, for each* $x \in X$, *there exists a* $y \in X$ *such that*

$$\delta(T(x), T(y)) \leq \alpha\, \delta(x, y) \text{ where } 0 \leq \alpha < 1.$$

(i) *If* X *is* T*-orbitally complete, then, for each* $x \in X$, *there exists a point* $x_0 \in X$ *such that*

$$\lim_{n\to\infty} T^n(x) = x_0$$

and

$$\delta(T^n(x), x_0) \leq \frac{\alpha^n}{1-\alpha}\delta(x, T(x)), \quad n = 1, 2, \cdots,$$

$$\delta(T^n(x), x_0) \leq \frac{\alpha}{1-\alpha}\delta(T^{n-1}(x), T^n(x)), \quad n = 1, 2, \cdots.$$

(ii) x_0 *is the unique fixed point of* T *(equivalently,* $T : X \to X$ *is orbitally continuous at* $x_0 \in X$*).*

The Banach Contraction Principle appeared in thousands of publications should be corrected as in Theorem Q.

Some of the most general metric fixed point theorems in 1970's were given in Park [12].

3. Various Examples of RHR maps

We began our study on RHR maps in [1] and [4]. Later we found a large number of examples of RHR maps in [5], [6], where we showed that most of such metric fixed point theorems can be extended or improved. Especially, they include many theorems influenced by Suzuki [13] in 2008 and others.

In this section, we introduce many known consequences of Theorem P or the RHR Contraction Principle with short and clear proofs. Some incorrect theorems are improved.

From now on *the numbers attached to Theorems, Corollaries, Definitions*, etc. are same to the one in the original sources.

Kannan [14] in 1968

The following improves Kannan's fixed point theorem (see Górnicki [60]):

Theorem 1.1. *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be a mapping such that there exists* $K < 1/2$ *satisfying*

$$d(Tx, Ty) \leq K[d(x, Tx) + d(y, Ty)] \quad \text{for all} \quad x, y \in X.$$

Then, T *has a unique fixed point* $v \in X$, *and for any* $x \in X$ *the sequence of iterates* $\{T^n x\}$ *converges to* v *and* $d(T^{n+1}x, v) \leq K \cdot (\frac{K}{1-K})^n \cdot d(x, Tx)$, $n = 0, 1, 2, \ldots$.

Comment. This is a simple consequence of Theorem P and the uniqueness follows from the contractive condition. Therefore the Kannan theorem holds for T-orbitally complete quasi-metric spaces.

Reich [15] in 1971

THEOREM 3. *Let X be a complete metric space with metric d, and let $T : X \to X$ be a function with the following property:*

$$d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, y), \quad x, y \in X,$$

where a, b, c are nonnegative and satisfy $a + b + c < 1$. Then T has a unique fixed point $v \in X$, and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to v.

COMMENT. Note that

$$d(Tx, T^2x) \leq \frac{a+c}{1-b} d(x, Tx), \quad 0 \leq \frac{a+c}{1-b} < 1.$$

Hence Theorem 3 is a simple consequence of Theorem P and it holds for T-orbitally complete quasi-metric spaces.

Hardy and Rogers [16] in 1973

The following theorem is the principal result of this paper [16]:

THEOREM 1. *Let (M, d) be a metric space and T a self-mapping of M satisfying the condition*

(1) *for $x, y \in M$, $d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + ed(y, Tx) + fd(x, y)$,*

where a, b, c, e, f are nonnegative and we set $\alpha = a + b + c + e + f$. Then

(a) *If M is complete and $\alpha < 1$, T has a unique fixed point.*
(b) *If (1) is modified to the condition*

(1′) *$x \neq y$ implies $d(Tx, Ty) < ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + ed(y, Tx) + fd(x, y)$,*

and in this case we assume M is compact, T is continuous and $\alpha = 1$, then T has a unique fixed point.

The conclusion in part (b) is a limiting version of the theorem contained in part (a), and Edelstein obtained this result in 1962 for the case $\alpha = f = 1$.

COMMENTS. Since a metric is symmetric, by exchanging x and y in the Hardy-Rogers contractive condition, we have $c = e$. Moreover, the condition for $y = Tx$ implies

$$d(Tx, T^2x) \leq ad(x, Tx) + bd(Tx, T^2x) + cd(x, T^2x) + ed(Tx, Tx) + fd(x, Tx)$$

and hence

$$d(Tx, T^2x) \leq \frac{a+c+f}{1-b-c} d(x, Tx) \text{ where } 0 \leq \frac{a+c+f}{1-b-c} < 1.$$

Then Theorem 1 follows from Theorem P for orbitally complete metric spaces.

Ćirić [17] in 1974

THEOREM. *Let T be an orbitally continuous self-map on the T-orbitally complete standard metric space (X, d). If there is $k \in [0, 1)$ such that*

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(Tx, y)\} \leq kd(x, y)$$

for all $x, y \in X$, then for each $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

COMMENT. For $y = Tx$, we have

$$\min\{d(Tx, T^2x), d(x, Tx)\} \leq kd(x, Tx) \text{ or } d(Tx, T^2x) \leq kd(x, Tx).$$

Therefore, Theorem P for T-orbitally complete quasi-metric spaces can be applicable.

Ćirić [18] in 1974

Definition. A mapping $T : M \to M$ of a metric space M is said to be a *generalized contraction* iff for every $x, y \in M$ there exist nonnegative numbers q, r, s and t, which may depend on both x and y, such that $\sup\{q + r + s + 2t : x, y \in M\} < 1$ and

$$d(Tx, Ty) \le q \cdot d(x, y) + r \cdot d(x, Tx) + s \cdot d(y, Ty) + t \cdot [d(x, Ty) + d(y, Tx)].$$

This is comparable to the following:

Definition. A mapping $T : M \to M$ is said to be a *quasi-contraction* iff there exists a number $q,\ 0 < q < 1$, such that

$$d(Tx, Ty) \le q \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

holds for every $x, y \in M$.

Now we can state the main result:

Theorem 1. *Let T be a quasi-contraction on a metric space M and let M be T-orbitally complete. Then*

(a) *T has a unique fixed point u in M,*
(b) $\lim T^n x = u$, *and*
(c) $d(T^n x, u) \le (q^n/(1 - q))d(x, Tx)$ *for every $x \in M$.*

Comments. (1) If M is a quasi-metric space and T is a generalized contraction, for $y = Tx$, we have

$$d(Tx, T^2x) \le q \cdot d(x, Tx) + r \cdot d(x, Tx) + s \cdot d(Tx, T^2x) + t \cdot [d(x, T^2x)],$$

$$d(Tx, T^2x) \le \tfrac{q+r+t}{1-s-t} d(x, Tx) \text{ and } 0 \le \tfrac{q+r+t}{1-s-t} < 1.$$

Then Theorem P can be applied for orbitally complete quasi-metric spaces.

(2) If M is a quasi-metric space and $q < 1/2$, then Theorem 1 is a simple consequence of our Theorem P. In fact, for $y = Tx$, we have

$$d(Tx, T^2x) \le q \cdot \max\{d(x, Tx), d(x, T^2x)\} \le q \cdot \{d(x, Tx) + d(Tx, T^2x)\}$$

$$\Longrightarrow d(Tx, T^2x) \le \tfrac{q}{1-q} d(x, Tx), \quad 0 \le \tfrac{q}{1-q} < 1.$$

Then Theorem 1 holds for orbitally complete quasi-metric spaces when $q < 1/2$. In our previous work [6] we stated that T is not an RHR map when $q < 1$.

(3) Theorem 1 holds under the following slightly stronger condition:

$$d(Tx, Ty) \le q \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2} d(x, Ty), d(y, Tx)\}.$$

(4) It may be observed that a quasi-contraction T does not require the continuity for the existence of the fixed point. However, a quasi-contraction T turns out to be orbitally continuous at the fixed point. This follows from Theorem P(iii).

Subrahmanyam [19] in 1974

Let S be a subset of a Banach space. It may be recalled that $f : S \to S$ is said to be a Banach operator of type k on S if there exists a constant $k,\ 0 \le k < 1$ such that

$$||f(x) - f^2(x)|| \le k||x - f(x)||.$$

From the author's previous work,

Corollary 2. *A continuous Banach operator mapping a closed subset of a Banach space into itself has a fixed point.*

Comment. This is one of the origins of the RHR maps and follows from our Theorem P.

Dass and Gupta [20] in 1975

Theorem. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping such that there exists $\alpha, \beta > 0$ with $\alpha + \beta < 1$ satisfying*

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)(1 + d(x, Tx))}{1 + d(x, y)} + \beta d(x, y) \quad \forall x, y \in X.$$

Then T has a unique fixed point.

Comment. For $y = Tx$, the contractive condition reduces to

$$d(Tx, T^2x) \leq \frac{\beta}{1-\alpha} d(x, Tx) \text{ with } 0 < \frac{\beta}{1-\alpha} < 1.$$

Then Theorem P works for T-orbitally complete quasi-metric spaces.

Note that this is an example of extremely artificial one.

Jaggi [21] in 1977

Theorem. *Let T be a continuous self map defined on a complete metric space (X, d). Suppose that T satisfies the following contractive condition:*

$$d(Tx, Ty) \leq \alpha[\frac{d(x, Tx)d(y, Ty)}{d(x, y)}] + \beta(d(x, y)$$

for all $x, y \in X,\ x \neq y$ and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. Then T has a unique fixed point in X.

Comment. For $x \neq y = Tx$, we have

$$d(Tx, T^2x) \leq \alpha d(Tx, T^2x) + \beta(d(x, Tx) \ \text{ or } \ d(Tx, T^2x) \leq \frac{\beta}{1-\alpha}(d(x, Tx).$$

Since $0 \leq \frac{\beta}{1-\alpha} < 1$, we can apply Theorem P for T-orbitally complete quasi-metric spaces.

Billy E. Rhodes [7] in 1977

In this historical article, Billy listed the basic 25 conditions all of them are RHR maps except

$$(6),\ (10),\ (13),\ (22),\ (24),\ \text{and } (25).$$

Each of them implies as follows:

(6), (10), (22): $d(fx, f^2x) < d(x, fx)$
(13): $d(fx, f^2x) < d(x, f^2x)$
(24): Ćirić [18]
(25): $d(fx, f^2x) < \max\{d(x, fx), d(x, f^2x)\},\ x \neq fx$

Therefore, the other 19 conditions are applicable to our Theorem P for f-orbitally complete quasi-metric spaces. Most of such results appears in the present article.

Rhoades [22] in 1978-79

Rhoades proved a fixed point result in Banach spaces for single-valued nonself mapping satisfying the following contraction condition:

$$d(Tx, Ty) \leq \lambda \max\{d(x, y),\ \frac{d(x, Tx)}{2},\ d(y, Ty),\ \frac{d(x, Ty) + d(y, Tx)}{1 + 2\lambda}\}$$

for all x, y, where $0 < \lambda < 1$.

COMMENT. This was given as an example of the class of mappings large enough to include some discontinuous mappings.

For $y = Tx$, the condition reduces to

$$d(Tx, T^2x) \leq \lambda \max\{d(x, Tx), \frac{1}{1 + 2\lambda} d(x, Ty)\}.$$

Case 1. If the maximum is $d(x, Tx)$, then T is an RHR map.

Case 2. Otherwise, we have

$$d(Tx, T^2x) \leq \frac{\lambda}{1 + 2\lambda}[d(x, Tx) + d(Tx, T^2x)]$$

and hence

$$d(Tx, T^2x) \leq \frac{\lambda}{1 + \lambda} d(x, Tx) \leq \lambda d(x, Tx)$$

with $0 < \lambda < 1$. Then T is an RHR map.

Therefore we can apply Theorem P for T-orbitally complete quasi-metric spaces.

Hicks and Rhoades [23] in 1979

Let (X, d) be a complete metric space, T a selfmap of X, and $O(x) := \{x, Tx, T^2x, \dots\}$.

THEOREM HR. [23] *Let $0 < h < 1$. Suppose there exists an x in X such that*

$$d(Ty, T^2y) < hd(y, Ty) \text{ for each } y \in O(x).$$

Then

(i) $\lim_n T^n x = z$ *exists, and*
(ii) $d(T^n x, z) < \frac{h^n}{1-h} d(x, Tx)$.
(iii) *z is a fixed point of T if and only if $G(x) := d(x, Tx)$ is T-orbitally lower semi-continuous at z.*

COMMENTS. Instead of $\overline{O(x)}$, we may assume (X, d) is a T-orbitally complete quasi-metric space. A particular form of Theorem HR was cited Jachymski [24] in 2003, and it was the incorrect base of our [1].

Pachpatte [25] in 1979

THEOREM. *Let T be an orbitally continuous self-map on the T-orbitally complete standard metric space (X, d). Suppose that there exists $k \in [0, 1)$ such that*

$$m(x, y) - n(x, y) \leq kd(x, Tx)d(y, Ty), \text{ for all } x, y \in X,$$

where

$$m(x, y) = \min\{[d(Tx, Ty)]^2, d(x, y)d(Tx, Ty), [d(y, Ty)]^2\},$$

$$n(x, y) = \min\{d(x, Tx)d(y, Ty), d(x, Ty)d(y, Tx)\}$$

with $R(x, y) = \min\{d(x, Tx), d(y, Ty)\} \neq 0$. Then, for each $x_0 \in X$ the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to a fixed point of T.

COMMENTS. For $y = Tx$, we have

$$m(x, Tx) = d(Tx, T^2x) \min\{d(Tx, T^2x), d(x, Tx)\} \text{ and } n(x, Tx) = 0.$$

Hence

$$m(x, Tx) \leq kd(x, Tx)d(Tx, T^2x).$$

If $Tx = T^2x$, then Tx is a fixed point. If $Tx \neq T^2x$, then we have $d(Tx, T^2x) \leq kd(x, Tx)$ and we can apply Theorem P for quasi-metric spaces.

Here $R(x, y) = \min\{d(x, Tx), d(y, Ty)\} \neq 0$ seems to be useless and Theorem is definitely artificial.

Park [26] in 1980

Let f be a selfmap of a topological space X. The orbit $O(x)$ of $x \in X$ under f is defined by $O(x) = \{x, f(x), f^2(x), \dots\}$. A function $G : X \to [0, \infty)$ is said to be f-orbitally lower semicontinuous at a point $p \in X$ if, for every $x_0 \in X$, $x_{n_k} \to p$ implies $G(p) \leq \liminf_k G(x_{n_k})$, where $\{x_{n_k}\}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$, which is defined by $x_{n+1} = f(x_n)$, i.e. $\{x_n\}_{n=1}^{\infty} = O(x_1)$.

We showed that any fixed point therem for maps satisfying one of the many contractive conditions in the list of Rhoades [7] and others follows from the following:

Basic Principle. *Let f be a selfmap of a topological space X, and d a nonnegative, real valued function defined on $X \times X$ such that $d(x, y) = d(y, x)$ and $d(x, y) = 0$ iff $x = y$. If there exists a point $u \in X$ such that $\lim_n d(f^{n+1}(u), f^n(u)) = 0$, and if $\{f^n(u)\}$ has a convergent subsequence with limit $p \in X$, then p is a fixed point of f iff $G(x) = d(x, f(x))$ is f-orbitally lower semicontinuous at p.*

The following extends the Banach contraction principle:

Theorem 2. *Let f be a selfmap of a metric space (X, d). If there exists a point $u \in X$ and a $\lambda \in [0, 1)$ such that $\overline{O(u)}$ is complete and*

$$d(fx, fy) \leq \lambda d(x, y) \text{ holds for any } x,\ y = fx \in O(u),$$

then $\{f^i u\}$ converges to some $\xi \in X$, and

$$d(f^i u, \xi) \leq \frac{\lambda^i}{1 - \lambda} d(u, fu) \text{ for } i \geq 1.$$

Further, if f is orbitally continuous at ξ or if

$$d(fx, fy) \leq \lambda d(x, y) \text{ holds for any } x, y \in \overline{O(u)},$$

then ξ is fixed under f.

Comments. The Basic Principle appears several times in the literature; see also Park and Rhoades [27] in 1980. Theorem 2 is our origin of Theorem P.

Billy E. Rhoades [29] in 2007 stated: These theorems (in Park [26] in 1980) contain as special cases a number of papers involving contractive conditions not covered by my Transactions paper.

Srivastava [28] in 1995

As an example of an application of Theorem P, not previously published, is the following in [28].

Theorem A. [28], [29] *Let T be an orbitally continuous selfmap of a metric space M that is T-orbitally complete. If T satisfies*

$$[d(Tx, Ty)]^3 \leq c_1 d(x, y)[d(y, Ty)]^2 + c_2 d(x, Tx)[d(Tx, Ty)]^2$$

$$+c_3[d(y, T^2x)[d(Tx, T^2x)]^2 + c_4 d(x, Ty)[d(y, Tx)]^2$$

for all $x, y \in M$ and $c_i \in \mathbb{R}$ such that $(c_1 + c_2)/(1 - c_3) = h \in (0, 1)$, then T has a fixed point, which is unique, when $0 < c_4 < 1$.

Comments. Set $y = Tx$ to get

$$[d(Tx, T^2x)]^3 \leq \frac{c_1 + c_2}{1 - c_3} d(x, Tx)[d(Tx, T^2x)]^2.$$

If x is such that $Tx = T^2x$, then Tx is a fixed point of T. If not, then we have $d(Tx, T^2x) \leq hd(x, Tx)$, and condition of Theorem P is satisfied. Therefore, Theorem A holds for a quasi-metric space.

Kirk and Saliga [30] in 2000

Corollary 2.4. *Let M be a complete metric space and suppose $T : M \to M$ satisfies:*

(i) $d(T(x), T^2(x)) \leq \alpha d(x, T(x))$ *for some* $\alpha \in (0, 1)$ *and all* $x \in M$*;*
(ii) $\mu(T(L_c)) \leq k\mu(L_c)$ *for some* $k < 1$ *and all* $c > 0$*;*
(iii) *F is an r.g.i. on M.*

Then the fixed point set $\mathrm{Fix}(T)$ *of T is nonempty and compact. Moreover if* $\{x_n\} \subset M$ *satisfies* $\lim_n d(x_n, T(x_n)) = 0$*, then* $\lim_n \mathrm{dist}(x_n, \mathrm{Fix}(T)) = 0$.

Comment. By Theorem P, (i) is enough for the existence of the unique fixed point $z \in \mathrm{Fix}(T)$ of T for a T-orbitally complete quasi-metric space. The conclusion clearly holds.

Berinde [31], [32] in 2003-04

Definition 1. Let (X, d) be a metric space. A map $T : X \to X$ is called *weak contraction* or (δ, L)-contraction if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx), \text{ for all } x, y \in X.$$

It is easy to see that condition implies the so called Banach orbital condition

$$d(Tx, T^2x) \leq ad(x, Tx), \text{ for all } x \in X,$$

studied by various authors in the context of fixed point theorems, see for example Kasahara, Hicks and Rhoades, Ivanov, Rus, and Tasković.

Comment. This is another origin of our RHR maps.

Suzuki [33] in 2005

Suzuki [33] in 2005 and Kikkawa-Suzuki [34] in 2008 gave a list of various classes of maps related τ-distances on a metric space. It should be noted that all maps in the list are RHR maps whenever τ is a metric. Moreover, our Theorem P can be extended to τ-distances or possible extensions of them.

Enjouji, Nakanishi and Suzuki [35] in 2005

Let $\Delta = \{(\alpha, \beta) : \alpha \geq 0,\ \beta \geq 0,\ \alpha + \beta < 1\} \subset \mathbb{R}^2$.

The authors' Theorem 3.1 can be improved as follows:

Theorem. *Define a function* $\psi : \Delta \to [1/2, 1)$ *arbitrary. Let T be a mapping on a complete metric space* (X, d)*. Assume that there exists* $(\alpha, \beta) \in \Delta$ *such that*

$$\psi(\alpha, \beta)d(x, Tx) \leq d(x, y) \textit{ implies } d(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(y, Ty)$$

for all $x, y \in X$*. Then T has a unique fixed point z. Moreover* $\lim_n T^n x = z$ *holds for every* $x \in X$*.*

Comment. In the original form of Theorem 3.1 in [35], ψ is a particular function. For $y = Tx$, we have

$$d(Tx, T^2x) \leq \frac{\alpha}{1-\beta} d(x, Tx) \text{ with } 0 \leq \frac{\alpha}{1-\beta} < 1.$$

Therefore Theorem 3.1 follows from Theorem P for orbitally complete quasi-metric spaces. The uniqueness of fixed point follows from the original contractive condition.

Suzuki [8] in 2008

Suzuki's well-known 2008 theorem, which is a generalization of the Banach contraction principle, can be extended as follows:

Theorem. *Let (X, d) be a quasi-metric space, let $T : X \to X$ be a map such that X is T-orbitally complete, and let $\theta : [0, 1) \to (1/2, 1]$ be any function. Assume that there exists $r \in [0, 1)$ such that*

$$\theta(r)d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq rd(x, y) \text{ for all } x, y \in X.$$

Then there exists a unique fixed point z of T. Moreover $\lim_n T^n x = z$ for all $x \in X$.

Proof. Since $\theta(r) \leq 1$, $\theta(r)d(x, Tx) \leq d(x, Tx)$ implies $d(Tx, T^2x) \leq rd(x, Tx)$ for all $x \in X$. Then we have the conclusion by Theorem P. □

Comments. According to Theorem P, we can add more conclusion to the above new Theorem. For other results in [8], we can improve or correct. There have been appeared hundreds of articles influenced by the original form of the preceding theorem and they can also be improved by following our method.

Kikkawa and Suzuki [34] in 2008

One of such incorrect theorems in [34] can be corrected as follows:

Theorem. *Let $\theta : [0, 1) \to (1/2, 1]$ be any function. For every $\alpha \in [0, 1)$, there exist a complete metric space (X, d) and a mapping T on X such that T has no fixed points and*

$$\theta(\alpha)d(x, Tx) < d(x, y) \text{ implies } d(Tx, Ty) \leq \alpha \max\{d(x, Tx), d(y, Ty)\}$$

for all $x, y \in X$.

Proof. For $y = Tx$, $\theta(\alpha)d(x, Tx) < d(x, Tx)$ implies T has no fixed point. □

Enjouji, Nakanishi, and Suzuki [35] in 2009

Let $\Delta = \{(\alpha, \beta) : \alpha \geq 0,\ \beta \geq 0, \alpha + \beta < 1\}$. Their Theorem 3.1 with three page proof can be improved as follows:

Theorem. *Let $\psi : \Delta \to (1/2, 1]$ be any function and let T be a mapping on a complete metric space (X, d). Assume that there exists $(\alpha, \beta) \in \Delta$ such that*

$$\psi(\alpha, \beta)d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(y, Ty)$$

for all $x, y \in X$.

Then T has a unique fixed point z. Moreover $\lim_n T^n x = z$ holds for every $x \in X$.

Comment. For $y = Tx$, we have

$$d(Tx, T^2x) \leq \alpha d(x, Tx) + \beta d(Tx, T^2x) \text{ for all } x \in X$$

and hence

$$d(Tx, T^2x) \leq \frac{\alpha}{1-\beta} d(x, Tx) \text{ with } 0 \leq \frac{\alpha}{1-\beta} < 1.$$

Then we can apply Theorem P for T-orbitally complete quasi-metric spaces.

Nakanishi and Suzuki [36] in 2010

Their main Theorem 3.1 with a lengthy proof can be improved as follows:

Theorem. *Define any function φ from $\Delta = [0,1)^2$ into $(1/2, 1]$. Let T be a mapping on a complete metric space (X, d). Assume that there exists $(\alpha, \beta) \in \Delta$ such that*

$$\varphi(\alpha, \beta)d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq \max\{\alpha d(x, Tx), \beta d(y, Ty)\}$$

for all $x, y \in X$. Then T has a unique fixed point z. Moreover $\lim_n T^n x = z$ holds for every $x \in X$.

Comment. For $y = Tx$, we have

$$d(Tx, T^2x) \leq \max\{\alpha d(x, Tx), \beta d(Tx, T^2x)\} = \alpha d(x, Tx).$$

Then we can apply Theorem P for T-orbitally complete quasi-metric spaces.

Altun and Erduran [37] in 2011

Abstract: We present a fixed-point theorem for a single-valued map in a complete metric space using implicit relation, which is a generalization of several previously stated results including that of Suzuki in 2008.

Let Ψ be the set of all continuous functions $F : [0, \infty)^6 \to \mathbb{R}$ satisfying the following conditions:

F1: $F(t_1, ..., t_6)$ is nonincreasing in variables $t_2, ..., t_6$,
F2: there exists $r \in [0, 1)$, such that

$$F(u, v, v, u, u + v, 0) \leq 0 \text{ or } F(u, v, 0, u + v, u, v) \leq 0 \text{ or } F(u, v, v, v, v, v) \leq 0$$

implies $u \leq rv$,
F3: $F(u, 0, 0, u, u, 0) > 0$, for all $u > 0$.

Then five examples of such classes are given.

Theorem 3.1. *Let (X, d) be a complete metric space, and let T be a mapping on X. Define a nonincreasing function θ from $[0, 1)$ into $(1/2, 1]$ as in Suzuki's theorem in 2008. Assume that there exists $F \in \Psi$, such that $\theta(r)d(x, Tx) \leq d(x, y)$ implies*

$$F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) \leq 0,$$

for all $x, y \in X$, then T has a unique fixed point z and $\lim_n T^n x = z$ holds for every $x \in X$.

Comments. The first part of proof runs as follows:

Since $\theta(r) \leq 1$, $\theta(r)d(x, Tx) \leq d(x, Tx)$ holds for every $x \in X$, by hypotheses, we have

$$F(d(Tx, T^2x), d(x, Tx), d(x, Tx), d(Tx, T^2x), d(x, T^2x), 0) \leq 0,$$

and so from (F1),

$$F(d(Tx, T^2x), d(x, Tx), d(x, Tx), d(Tx, T^2x), d(x, Tx) + d(Tx, T^2x), 0) \leq 0.$$

By (F2), we have $d(Tx, T^2x) \leq rd(x, Tx)$, that is, T is an RHR map and applicable our Theorem P for quasi-metric spaces.

The original proof was five page long. The uniqueness of the fixed point was also given.
In Theorem 3.1, $\theta : [0, 1) \to (1/2, 1]$ can be any function.

Karapinar [38] in 2012

Theorem. *Let $T : X \to X$ be an orbitally continuous self-map on the T-orbitally complete metric space (X, d). Suppose there exist real numbers a_1, a_2, a_3, a_4, a_5 satisfying the conditions*

$$0 \leq \frac{a_4 - a_2}{a_1 + a_2} < 1, \ a_1 + a_2 > 0, \quad \text{and } 0 \leq a_3 - a_5,$$

such that

$$E(x,y) \leq a_4 d(x,y) + a_5 d(x,T^2x)$$

where

$$E(x,y) := a_1 d(Tx,Ty) + a_2[d(x,Tx) + d(y,Ty)] + a_3[d(y,Tx) + d(x,Ty)]$$

for all $x,\ y \in X$. Then, T has at least one fixed point.

Comment. For $y = Tx$, we have

$$E(x,y) := a_1 d(Tx,T^2x) + a_2[d(x,Tx) + d(Tx,T^2x)] + a_3 d(x,T^2x) \leq a_4 d(x,Tx) + a_5 d(x,T^2x),$$

$$(a_1 + a_2)d(Tx,T^2x) \leq (a_4 - a_2)d(x,Tx) + (a_5 - a_3)d(x,T^2x) \leq (a_4 - a_2)d(x,Tx),$$

$$d(Tx,T^2x) \leq \frac{a_4 - a_2}{a_1 + a_2} d(x,Tx) \text{ with } 0 \leq \frac{a_4 - a_2}{a_1 + a_2} < 1.$$

Therefore Theorem P for T-orbitally complete quasi-metric spaces can be applicable.

Alsulami et al. [10] in 2014

Corollary 30. *Let (X,q) be a complete quasi-metric space and let $T : X \to X$ be a mapping such that*

$$q(Tx,Ty) \leq \lambda q(x,y), \quad \forall x,y \in X,$$

where $\lambda \in [0,1)$. Then T has a unique fixed point in X.

Comment. Corollary 30 follows from our Theorem Q for T-orbitally complete (X,q).

Chandok, Narang, and Taoudi [39] in 2014

Definition 2.1. Suppose $(X,\leq)$ is a partially ordered set and $T : X \to X$. T is said to be *monotone nondecreasing* if for all $x,y \in X$, $x \leq y$ implies $Tx \leq Ty$.

Theorem 2.2. *Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Suppose that T is a continuous self-mapping on X, T is monotone nondecreasing mapping and*

$$d(Tx,Ty) \leq \alpha \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta(d(x,y)) + \gamma(d(x,Tx) + d(y,Ty)) + \delta(d(x,Ty) + d(y,Tx))$$

for all $x,y \in X,\ x \geq y,\ x \neq y$ and for some $\alpha,\beta,\gamma,\delta \in [0,1)$ with $\alpha+\beta+2\gamma+2\delta < 1$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Comments. For a quasi-metric space without considering the order, $x \neq y = Tx$ implies

$$d(Tx,T^2x) \leq \frac{\beta+\gamma+\delta}{1-\alpha-\gamma-\delta} d(x,Tx) \text{ with } 0 \leq \frac{\beta+\gamma+\delta}{1-\alpha-\gamma-\delta} < 1.$$

Hence Theorem P implies the existence of a fixed point for a T-orbitally complete space.

Kumam, Dung, and Sitthithakerngkiet [40] in 2015

Definition 2.4. Let $T : X \to X$ be a mapping on metric space X. The mapping T is said to be a *generalized quasi-contraction* iff there exists $q \in [0,1)$ such that for all $x,y \in X$,

$$d(Tx,Ty) \leq q \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), d(T^2x,x),$$

$$d(T^2x,Tx), d(T^2x,y), d(T^2x,Ty)\}.$$

We state and prove the new fixed point theorems which are general cases of the Ćirić fixed point theorem. ¢¥

THEOREM 3.1. *Let (X, d) be a metric space. Suppose that $T : X \to X$ is a generalized quasi-contraction and X is T-orbitally complete. Then we have*

1. *T has a unique fixed point x^* in X.*
2. *$\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$.*
3. *$d(T^n x, x^*) \leq \frac{q^n}{1-q} d(x, Tx)$ for all $x \in X$ and $n \in \mathbb{N}$.*

COMMENTS. We will give a simple form of Theorem 3.1 as in Kannan's and Ćirić's theorems.

By putting $y = Tx$, the generalized quasi-contraction becomes

$$d(Tx, T^2x) \leq q \max\{d(x, Tx), d(x, T^2x)\}.$$

Case 1. If $d(x, Tx) \geq d(x, T^2x)$ for all $x \in X$, then $d(Tx, T^2x) \leq q\, d(x, Tx)$ and hence Theorem P works

Case 2. If $d(x, Tx) \leq d(x, T^2x)$ for some $x \in X$, then

$$d(Tx, T^2x) \leq q\, d(x, T^2x) \leq q[d(x, Tx) + d(Tx, T^2x)]$$

implies

$$d(Tx, T^2x) \leq \frac{q}{1-q} d(x, Tx).$$

If $0 \leq \frac{q}{1-q} < 1$ or $0 \leq q < \frac{1}{2}$, then T is an RHR map.

Therefore, Theorem P can be applicable to Theorem 3.1 when $0 \leq q < \frac{1}{2}$.

Nalawade and Dolhare [41] in 2016

THEOREM 3.2. *Let (X, d) be a complete metric space and let T be a mapping on X . For a non-increasing function $\theta : [0, 1) \to (1/2, 1]$, $\alpha \in [0, 1/2)$, and $r = \frac{\alpha}{1-\alpha}$, assume that $\theta(r)d(x, Tx) \leq d(x, y)$ implies*

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty) \ for\ all\ x, y \in X.$$

Then T has a unique fixed point z and $\lim_n T^n x \to z$ holds for every $x \in X$.

COMMENTS. In the original form of Theorem 3.2, $\theta : [0, 1) \to (1/2, 1]$ was given by Suzuki in 2008.

For $y = Tx$, we have

$$d(Tx, T^2x) \leq \frac{\alpha}{1-\alpha} d(x, Tx) \text{ with } 0 \leq \frac{\alpha}{1-\alpha} < 1.$$

Therefore Theorem 3.2 follows from Theorem P for quasi-metric spaces. The uniqueness of fixed point follows from the original contractive condition.

I.A. Rus [42] in 2018

By definition, f is a *weakly Picard operator* if the sequence $\{f^n(x)\}_{n\in\mathbb{N}}$ converges for all $x \in X$ at its limit (which may depend on x) is a fixed point of f. If f is a weakly Picard operator, then we consider the operator $f^\infty : X \to X$, defined by $f^\infty(x) := \lim_{n\to\infty} f^n(x)$.

THEOREM 2.3. [43] *Let (X, d) be a complete metric space and $f : X \to X$ be an operator. We suppose that:*

(1) *There exists $\alpha \in]0, 1[$ such that,*

$$d(f^2(x), f(x)) \leq \alpha d(x, f(x)),\ for\ all\ x \in X,$$

i.e., f is a graphic contraction.

(2) $\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x))$, *for all $x \in X$.*

Then we have

(i) *f is a weakly Picard operator.*

(ii) $d(x, f^\infty(x)) \leq \frac{\alpha}{1-\alpha} d(x, f(x))$, *for all* $x \in X$.

(iii) *For* $x^* \in \mathrm{Fix}(f)$, *let* $X_{x^*} := \{x \in X : f^n(x) \to x^* \text{ as } n \to \infty\}$. *Let* $\{y_n\}_{n\in\mathbb{N}}$ *be a sequence in* X_{x^*} *such that*

$$d(y_n, f(y_n)) \to 0 \text{ as } n \to \infty.$$

Then, $y_n \to x^*$ *as* $n \to \infty$.

(iv) *Let* $\{y_n\}_{n\in\mathbb{N}}$ *be a sequence in* X_{x^*}, $x^* \in \mathrm{Fix}(f)$. *If* $\alpha < 1/3$ *and*

$$d(y_{n+1}, f(y_n)) \to 0 \text{ as } n \to \infty,$$

then, $y_n \to x^*$ *as* $n \to \infty$.

This result suggests the following problem:

Problem 2.4. [43] Which metric conditions imposed on an operator f imply a similar conclusion as that in Theorem 2.3?

Comments. We get the above from [42] in 2018. Theorem P can be an answer to Problem 2.4.

We obtained a similar theorem of Park [26] in 1980 which was our first result on the RHR maps.

Miñana and Valero [44] in 2019

It is stated in [44]: Taking into account the exposed facts about G-metric spaces and quasi-metric spaces, we are able to show that most fixed point results obtained in G-metric spaces can be deduced from a fixed point result stated in quasi-metric spaces obtained by Park in [26]. To this end, let us recall such a result:

Theorem 5.15. [44] *Let* (X, τ) *be a topological space, let* $\delta : X \times X \to [0, \infty)$ *be a continuous mapping, such that* $\delta(x, y) = 0 \iff x = y$, *and let* $f : X \to X$ *be a mapping. Suppose that there exist* x, $x_0 \in X$, *such that the following conditions hold:*

(1) $\lim_{n\to\infty} \delta(f^n(x_0), f^{n+1}(x_0)) = 0$;

(2) $(f^n(x_0))_{n\in\mathbb{N}}$ *converges to* x *with respect to* δ;

(3) f *is orbitally continuous at* x *with respect to* δ.

Then $x \in Fix(f) = \{y \in X : f(y) = y\}$.

Comments. This is a variant of our Basic Principle [26] in 1980. Theorem 5.15 suggests that all results in this article might hold for more general settings.

Berinde and Păcurar [45] in 2022

The RHR maps are called a graphic contraction (orbital contraction) and gave its examples as Banach contraction, Kannan mapping, Ćirić-Reich-Rus contraction, Bianchini mapping, Chatterjea mapping, Zamfirescu mapping, Hardy-Rogers contraction, and the Berinde almost contraction.

In this paper the aim is to emphasize, by means of several examples, how one can simplify and unify the proofs of some classical fixed point theorems emerging from Banach contraction principle, such as Kannan fixed point theorem, Chatterjea fixed point theorem, Bianchini fixed point theorem, and Zamfirescu fixed point theorem, using a technique based on the concepts of graphic contractions and approximate fixed point sequence.

Chandra, Joshi and Joshi [46] in 2022

Let (X, d) be a metric space, and $T : X \to X$. Then for all $x, y \in X$, we denote

$$m(Tx, Ty) = ad(x, y) + b \max\{d(x, Tx), d(y, Ty)\} + c[d(x, Ty) + d(y, Tx)],$$

where a, b and c are non-negative reals such that $a + b + 2c = r$ with $r \in [0, 1)$. Now, we consider the following generalized contractive condition

$$\theta(r) \min\{d(x, Tx), d(x, Ty)\} \leq d(x, y) \text{ implies } d(Tx, Ty) \leq m(Tx, Ty).$$

THEOREM 4. *Let (X, d) be a complete metric space, and $T : X \to X$. Assume that there exists $r \in [0, 1)$ such that the preceding generalized contractive condition is satisfied for each $x, y \in X$, where $\theta : [0, 1) \to (1/2, 1]$ is the function defined by Suzuki in 2008. Then T has a unique fixed point $z \in X$. Moreover, $\lim_{n\to\infty} T^n x = z$ for all $x \in X$.*

COMMENTS. As we have seen several times, Suzuki's function can be replaced by any function $\theta : [0, 1) \to (1/2, 1]$. For $y = Tx$, the contractive condition reduces to

$$d(Tx, T^2x) \leq rd(x, Tx), \ \forall x \in X.$$

Hence Theorem P for T-orbitally complete quasi-metric spaces is applicable to T.

The original proof of Theorem 4 was almost four page long. Moreover, several related results are added.

FINAL REMARK FOR SECTION 3. In this section, we showed that most of metric fixed point theorems holds for quasi-metric spaces. Even our Basic Principle in 1980 and the main theorem of Miñana and Valero [44] in 2019 state that they can be deduced from the one for particular topological spaces.

However, from now on, we review the original forms of known theorems without checking whether the metric spaces can be extended or not.

4. Extended RHR maps with auxiliary functions

The various contractive type conditions have been extended with the aid of some auxiliary functions; for example, comparison functions, control functions, gauge functions, simulation functions, etc.

There are many generalizations of the RHR maps even in the realm of single-valued maps. From the 1980 paper of Park and Rhoades [27], we deduced a new Theorem R. In this section, we collect old and new results based on some auxiliary functions.

Rakotch [47] in 1962 and Browder [48] in 1968

A selfmap T on a complete metric space (X, d) is is called a *Rakotch contraction* if there exists a (not necessarily strictly) decreasing function α from $\mathbb{R}^+$, the set of all nonnegative reals, into $[0, 1]$ such that $\alpha(t) < 1$ for all $t > 0$, and

$$d(Tx, Ty) \leq \alpha(d(x, y))d(x, y) \text{ for all } x, y \in X.$$

Then T has a fixed point.

Subsequently, Rakotch's result was generalized by Felix Browder [48] in 1968 who introduced the following contractive condition: Given a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for all $t > 0$, we call T *ϕ-contractive* if

$$d(Tx, Ty) \leq \phi(d(x, y)) \ for \ all \ x, y \in X.$$

THEOREM 1. (Browder) *Let (X, d) be complete and T be ϕ-contractive, where ϕ is increasing and right continuous. Then T has a fixed point.*

Every such mapping T is called a *Browder contraction.*

Boyd and Wong [49] in 1969

Let (X, ρ) be a metric space. We shall denote the range of ρ by P, and the closure of P by $\overline{P}$, so $P = \{\rho(x, y) : x, y \in X\}$.

THEOREM 1. *Let X be a complete metric space, and let $T : X \to X$ satisfy*

$$\rho(Tx, Ty) \le M\psi(\rho(x, y))$$

for some constant $M < \infty$, where $\psi : \overline{P} \to [0, \infty)$ is upper semicontinuous from the right on $\overline{P}$, and satisfies $\psi(t) < t$ for all $t \in \overline{P}\backslash\{0\}$. Then, T has a unique fixed point x_0, and $T^n x \to x_0$ for each $x \in X$.

COMMENT. A slightly different version of this follows from Theorem R of Park and Rhoades [9] in 1980 later.

Geraghty [50] in 1973

Let Ψ denote the class of all real functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the condition

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0, \text{ as } n \to \infty.$$

In order to generalize the Banach Contraction Principle, Geraghty proved the following result:

THEOREM. [50] *Let (X, d) be a complete metric space and let $f : X \to X$ be a self-map. Suppose that there exists $\beta \in \Psi$ such that*

$$d(fx, fy) \le \beta(d(x, y))d(x, y)$$

holds for all $x, y \in X$. Then f has a unique fixed point $z \in X$ and for each $x \in X$ the Picard sequence $\{f^n(x)\}$ converges to z.

COMMENT. Theorem can be extended by taking the condition

$$d(fx, f^2x) \le \beta(d(x, fx))d(x, fx).$$

Kasahara [51] in 1979

Let f be a self-map of a metric space (X, d). Given $x \in X$, let $O(x) = \{f^n x : n \in \mathbb{N}\}$ and $\overline{O(x)}$ be its closure. A point $x \in X$ is said to be *regular* for f if diam $O(x) < \infty$. Given $x, y \in X$, let

$$m(x, y) = \max\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\},$$

and

$$\delta(x, y) = \text{diam}\{O(x) \cup O(y)\}.$$

The Browder contraction was extended as follows:

There exists a non-decreasing right continuous function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) < t$ for $t > 0$ and, for any $x, y \in X$,

(Dd) $d(fx, fy) \le \phi(d(x, y))$ (Browder [48])
(Dm) $d(fx, fy) \le \phi(m(x, y))$ (Daneš)
(Dδ) $d(fx, fy) \le \phi(\delta(x, y))$ if x, y are regular (Kasahara [51])

For details, see Park [12] in 1980.

COMMENT. For $y = Tx$, (Dδ) can be extended to

$$d(fx, f^2x) \le \phi(\delta(x, fx)) \text{ if } x \text{ is regular.}$$

Park [26] in 1980

Now we have our main result, which is motivated by Husain and Sehgal (1977). Let $\mathbb{R}_+$ be the set of nonnegative real numbers.

Theorem 4. *Let f be a selfmap of a metric space (X, d) satisfying the following conditions:*

(i) *there is a $u \in X$ such that $O(u)$ has a cluster point $\xi \in X$ and* $\operatorname{diam} O(u) < \infty$.

(ii) *there is an upper semicontinuous map $\phi : \mathbb{R}^5_+ \to \mathbb{R}_+$ which is nondecreasing in each coordinate variable and satisfies the condition $\phi(t, t, t, t, t) < t$ for any $t > 0$ and the inequality*

$$d(fx, fy) \leq \phi(d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)) \text{ for all } x, y \in \overline{O(u)}.$$

Then $\xi \in X$ is a fixed point of f and $f^i u \to \xi$.

Comment. For $y = fx$, Theorem 4 can be extended. Since then there have been appeared many Meir-Keeler type contractive conditions.

Park [12] in 1980

Bisht [52] in 2023 commented: Later in 1980, Park [12] augmented Rhoades' comparative study of various contractive definitions by including Meir-Keeler and Matkowski type of contractive definitions.

Billy Rhoades [29] in 2007 stated: In 1980 Sehie Park [12] constructed a table of contractive conditions of Meir-Keeler type, which extended the list in my Transactions paper.

A point u in a metric space X is said to be regular if $\operatorname{diam}(O(x))$ is finite. The most general fixed point theorem he obtained is the following.

Theorem 2.3. *Let T be a selfmap of a metric space (X, d). Suppose there exists a regular point $u \in X$ such that*

(i) *$O(u)$ has a regular cluster point $p \in X$ and*

(ii) *the following condition holds on $O(u) \cup O(p)$:*

Given an $\varepsilon > 0$ there exists a $\delta > 0$ such that, for any $x, y \in X$,

$$\varepsilon \leq m(x, y) < \varepsilon + \delta_0 \text{ implies } d(Tx, Ty) \leq \varepsilon_0,$$

where

$$m(x, y) := \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Then T has a unique fixed point $p \in \overline{O(u)}$ and $T^n u \to p$.

Comment. For $y = Tx$, Theorem 2.3 can be extended.

Park and Rhoades [9] in 1980

In this paper we established several fixed point theorems involving hypotheses weak enough to include a number of known theorems as special cases.

Theorem 2. *Let f be a selfmap of a metric space (X, d) satisfying:*

(1) *$\delta(O(x)) < \infty$ for each $x \in X$, where δ denotes the diameter.*

(2) *There exists a $u \in X$ such that $O(u)$ has a cluster point $p \in X$.*

(3) *There exists a function $\varphi : [0, \infty) \to [0, \infty)$ which is nondecreasing, continuous from the right and satisfies $\varphi(t) < t$ for each $t > 0$ and the inequality*

$$d(f(x), f^2(y)) \leq \varphi(\delta(O(x) \cup O(f(y)))) \quad \text{for each } x, y \in X.$$

Then p is the unique fixed point of f and $\lim_n f^n(u) = p$.

These results extend works of Pal-Maiti, Park, Hegedüs and Daneš. A 2-metric space version of Theorem 2 is added in [9].

Moreover, Theorems of Rus and Hicks-Rhoades are consequences of Theorem 2.

Comments. In some works of Park (e.g., [1] in 2023), the contractive condition in Theorem 2 is stated as follows:

$$d(f(x), f^2(x)) \leq \varphi(\delta(O(x) \cup O(f(y)))) \ \ \textit{for each } x, y \in X.$$

Then Theorem 2 should be corrected as follows:

Theorem R. *Let f be a selfmap of a quasi-metric space (X, d) satisfying:*

(1) *$\delta(O(x)) < \infty$ for each $x \in X$, where δ denotes the diameter.*

(2) *There exists a $u \in X$ such that $O(u)$ has a cluster point $p \in X$.*

(3) *There exists a function $\varphi : [0, \infty) \to [0, \infty)$ which is nondecreasing, continuous from the right and satisfies $\varphi(t) < t$ for each $t > 0$ and the inequality*

$$d(f(x), f^2(x)) \leq \varphi(\delta(O(x)) \ \ \textit{for each } x \in X.$$

Then p is the unique fixed point of f and $\lim_n f^n(u) = p$.

Tasković [53] in 1980

Theorem 2. [53] *Let T be a mapping of a metric space (X, ρ) into itself and let X be T-orbitally complete. Suppose that there exists a function $\varphi : \mathbb{R}^0_+ \to \mathbb{R}^0_+ := [0, +\infty)$ satisfying*

$$[\forall t \in \mathbb{R}_+ := (0, +\infty)] \ [\varphi(t) < t \textit{ and } \limsup_{z \to t+0} \varphi(z) < t]$$

such that

$$\rho[Tx, Ty] \leq \varphi(\mathrm{diam}\{x, y, Tx, Ty, T^2x, T^2y, \dots\})$$

and $\mathrm{diam} O(x) \in \mathbb{R}^0_+$ *for all $x, y \in X$.*

Then T has a unique fixed point $\zeta \in X$ and $\{T^n(a)\}_{n \in \mathbb{N}}$ converges to ζ for every $a \in X$.

Comment. Tasković claimed in 2009 that his theorem implies theorems of Walter [54] and Kirk and Saliga [30] below. Note that our Theorem R can be applied to Theorem 2 of Tasković.

Walter [54] in 1981

Let ϕ denotes a contractive gauge function on a metric space M. This means $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, nondecreasing, and satisfies $\phi(s) < s$ for $s > 0$. Also, $O(x) = \{x, T(x), T^2(x), \cdots\}$ and $O(x, y) = O(x) \cup O(y)$ for $x, y \in M$.

Theorem 4.2. [54] *Let M be a complete metric space and suppose $T : M \to M$ has bounded orbits and satisfies the following condition. For each $x, y \in M$,*

$$d(T(x), T(y)) \leq \phi(\mathrm{diam}(O(x, y))).$$

Then T has a unique fixed point $z \in M$ and $\lim_{k \to \infty} T^k(x) = z$ for each $x \in M$.

Comment. Walter's Theorem 4.2 is a consequence of our Theorem R.

Rhoades [55] in 1997

From text: In this paper we trace the history of some general fixed point principles, from single maps, to pairs of maps, and to more than two maps. In most cases we also provide an application of the procedure.

In applying Theorem HR in 1979 to specific situations, it is often the case that the contractive definition is strong enough that condition (iii) there is not needed. Applications of Theorem HR to some other contractive definitions appear in Rhoades [7].

Sehie Park also established some general fixed point theorems based on this observation. Theorem 2 of Park [26] strengthens Theorem HR by replacing the completeness of X with the completeness of $O(x)$.

Theorem P1. [26, Theorem 1] *Let T be a selfmap of a metric space (X,d). If*

(i) *there exist a point $u \in X$ such that the orbit $O(u)$ has a cluster point $z \in X$,*
(ii) *T is orbitally continuous at z and Tz, and*
(iii) *T satisfies $d(Tx,Ty) < d(x,y)$ for all $x, y = Tx \in \overline{O(u)}$, $x \neq y$,*

then z is a fixed point of T.

Some applications of Theorem Pl appear in [26] and [7]. A modest extension of Theorem HR is the following:

Proposition 1. *Let T be a selfmap of a metric space (X,d). Let $0 < h < 1$. Suppose there exists a point $x \in X$ such that*

(A) *$x_n := T^n x$ has a convergent subsequence with limit $z \in X$, and, for this x,*
(B) *$d(Ty, T^2y) < hd(y, Ty)$ for each $y \in O(x)$.*
Then, for this x,
(i) $\lim T^n x := z$,
(ii) *$d(T^n x, z) \leq \frac{h^n}{1-h} d(x, Tx)$, and*
(iii) *z is a fixed point of T in X if and only if $G(x) := d(x, Tx)$ is lower semicontinuous at z.*

Park [26] established the following general principle [26, Theorem 3.1] for a pair of maps.

Theorem P2. *Let S and T be selfmaps of a metric space (X,d). If there exists a sequence $\{x_i\} \subset X$, where $x_{2i+i} := Sx_{2i}, x_{2i+2} := Tx_{2i+1}$, such that $\overline{\{x_i\}}$ is complete, and if there exists a $\lambda \in [0,1)$ such that*

(1) *$d(Sx, Ty) \leq \lambda d(x,y)$ for each distinct $x, y \in \overline{\{x_i\}}$*

satisfying either $x = Ty$ or $y = Sx$, then either:

(i) *S or T has a fixed point in $\{x_i\}$, or*
(ii) *$\{x_i\}$ converges to some $z \in X$ and*

$$d(x_i, z) \leq \frac{\lambda^i}{1-\lambda} d(x_0 . x_1) \ for \ i > 0.$$

Further, if either S or T is continuous at z and (1) holds for any distinct $x, y \in \overline{\{x_i\}}$, then z is a common fixed point of S and T.

Comment. Then 12 Corollaries of other authors added as examples. In this paper, more sophisticated theorems with known examples were added.

Kirk and Saliga [30] in 2000

Theorem 4.3. *Let M be a complete metric space and suppose $T : M \to M$ has bounded orbits and satisfies: there exists $\alpha < 1$ such that for each $x, y \in M$,*

$$d(T(x), T(y)) \leq \alpha \operatorname{diam}(O(x,y)) \ \forall \ x, y \in M.$$

Suppose $\{x_n\} \subset M$ satisfies $\lim_n d(x_n, T(x_n)) = 0$. Then T has a unique fixed point $z \in M$, and $\lim_{c \to 0+} \operatorname{diam}(L_c) = 0$. Moreover, $\lim_n d(x_n, T(x_n)) = 0$ if and only if $\lim_n x_n = z$.

Here, $O(x,y) = O(x) \cup O(y)$, where $O(x) := \{x, Tx, T^2x, \dots\}$ for all $x, y \in M$, and $L_c := \{x \in M : F(x) \leq c\}$ for all $c \geq 0$, where $F(x) := d(x, Tx)$ for every $x \in M$. It was noticed in [ks] that F is an r.g.i. on M. We recall that a function $G : M \to \mathbb{R}$ is said to be a *regular-global-inf* (r.g.i.) at $x \in M$ if $G(x) > \inf_M(G)$ implies there exist $\varepsilon > 0$ such that $\varepsilon < G(x) - \inf_M(G)$ and a neighborhood N_x of x such that $G(y) > G(x) - \varepsilon$ for each $y \in N_x$. If this condition holds for each $x \in M$, then G is said to be an r.g.i. on M.

5. Concluding remarks 2. By taking $y = T(x)$ in Theorem 4.3 one has

$$d(T(x), T^2(x)) \le \alpha \mathrm{diam}(O(x, T(x))) = \alpha \mathrm{diam}(O(x)) \; for \; all \; x \in M,$$

and this quickly leads to

$$\mathrm{diam}(O(T(x))) \le \alpha \mathrm{diam}(O(x)) \; for \; all \; x \in M.$$

This can be rewritten as

$$\mathrm{diam}(O(x)) \le (1-\alpha)^{-1}[\mathrm{diam}(O(x)) - \mathrm{diam}(O(T(x)))] \; for \; all \; x \in M.$$

Since $d(x, T(x)) \le \mathrm{diam}(O(x))$, if the mapping $\varphi : M \to \mathbb{R}$ defined by setting $\varphi(x) = \mathrm{diam}(O(x))$ is lower semicontinuous then this condition, which is much weaker than the one in Theorem 4.3, assures that T has at least one fixed point by Caristi's Theorem.

COMMENT. Here the reader can find another generalization of RHR maps.

Akkouch [56] in 2002

We prove that the conclusion of a result of Kirk and Saliga ([30], Theorem 4.3, p.149) remain valid for a wide class of contractive gauge functions.

A natural question was addressed by Kirk and Saliga in [30]. Does the conclusion of their Theorem 4.3 remain valid under the weaker assumption of Theorem 4.2 of Walter [54]. A first answer is given by the following. Let Φ_1 be the set of contractive gauge functions ϕ satisfying, $\phi(s) < \alpha s, \forall s > 0$ for some given $\alpha \in [0; 1[$. Then the conclusions of Theorem 4.3 are true for functions ϕ in the class Φ_1. The aim of this paper is to bring a partial response to the question addressed above by introducing a wide class of contractive gauge functions for which the answer of this question is yes but their case could not be directly handled with Theorem 4.3.

MAIN RESULT. Let Φ be the set of continuous functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that ϕ is nondecreasing on $\mathbb{R}^+$ and such that the mapping $x \mapsto x - \phi(x)$ from $[0, +\infty[$ onto $[0, +\infty[$ is strictly increasing. We notice that each element ϕ in Φ is a gauge function and that Φ_1 is strictly contained in Φ. Indeed, we can give examples of elements in $\Phi \backslash \Phi_1$.

The following theorem is the main result of this short communication.

THEOREM C. [56] *Let (M, d) be a complete metric space and suppose $T : M \to M$ has bounded orbits and satisfies the following condition:*

$$d(Tx, Ty) \le \phi(\mathrm{diam}(O(x, y))) \; for \; all \; x, y \in M,$$

where $\phi \in \Phi$. Then

(1) *T has a unique fixed point $z \in M$, and $\lim_{k \to +\infty} T^k(x) = z$ for each $x \in M$.*
(2) $\lim_{c \to 0^+} \mathrm{diam}(L_c) = 0$.
(3) *For each sequence $\{x_n\} \subset M$; $\lim_n d(x_n, Tx_n) = 0$ if and only if $\lim_n x_n = z$.*
(4) *The map $F : x \mapsto d(x, Tx)$ is an r.g.i. on M.*

COMMENTS. Note that (1) is a consequence of Walter's Theorem 4.2. Moreover, Theorem C can be extended like our Theorem R.

Berinde [31] in 2003

A map $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called *comparison function* if it satisfies:
(i) φ is monotone increasing, i.e., $t_1 < t_2 \Rightarrow \varphi(t_1) \le \varphi(t_2)$
(ii) the sequence $\{\varphi^n(t)\}_{n=0}^{\infty}$ converges to zero, for all $t \in R_+$.
If φ satisfies (i) and

(iii) $\Sigma_{k=0}^{\infty}(t)$ converges for all $t \in \mathbb{R}_+$, then φ is said to be a (**c**)-comparison function.

Definition 2. Let (X, d) be a metric space. A self operator $T : X \to X$ is called a *weak φ-contraction* or (φ, L)-*weak contraction*, provided that there exist a comparison function φ and some $L \geq 0$, such that

$$d(Tx, Ty) \leq \varphi(d(x, y)) + Ld(y, Tx) \text{ for all } x, y \in X.$$

Comment. For $y = Tx$, this condition reduces to

$$d(Tx, T^x) \leq \varphi(d(x, Tx)) \text{ for all } x \in X.$$

Proinov [57] in 2007

From Abstract: In this paper, we present the following generalization of the Banach contraction principle. Let $T : D \subset X \to X$ be a continuous operator on a complete metric space (X, d) satisfying

$$d(Tx, T^2x) \leq \varphi(d(x, Tx)) \text{ for all } x \in D,\ Tx \in D \text{ with } d(x, Tx) \in J,$$

where $J \subset \mathbb{R}_+$ is an interval containing 0, $\varphi : J \to J$ is a gauge function of order $r \geq 1$ on J in the sense that $\varphi(0) = 0$, $\varphi(t)/t^r$ is non-decreasing on $J\backslash\{0\}$ and $\varphi(t) < t$ on $J\backslash\{0\}$.

Throughout the paper J denotes an interval on $\mathbb{R}_+$ containing 0, that is an interval of the form $[0, R]$, $[0, R)$ or $[0, \infty)$. Let us introduce the following definition.

Definition 1.1. (Gauge Functions of High Order). Let $r \geq 1$. A function $\varphi : J \to J$ is called a *gauge function of order* r on J if it satisfies the following conditions:
1. $\varphi(\lambda t) \leq \lambda^r \varphi(t)$ for all $\lambda \in (0, 1)$ and $t \in J$;
2. $\varphi(t) < t$ for all $t \in J\backslash\{0\}$.

It is easy to see that the first condition of Definition 1.1 is equivalent to the following one:

$$\varphi(0) = 0 \text{ and } \varphi(t)/t^r \text{ is nondecreasing on } J\backslash\{0\}.$$

Comment. This gives another generalization of the RHR map.

Alsulami et al. [10] in 2014

Corollary 35. *Let (X, q) be a complete quasi-metric space and let $T : X \to X$ be a mapping. Suppose that, for every $x, y \in X$,*

$$q(Tx, Ty) \leq \eta(q(x, y)) \text{ for all } x, y \in X,$$

where $\eta : [0, +\infty) \to [0, +\infty)$ is an upper semicontinuous mapping such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$. Then T has a unique fixed point.

Comment. Corollary 35 can be extended for

$$q(Tx, T^2x) \leq \eta(q(x, Tx)) \text{ for all } x \in X.$$

5. On the Ćirić-Proinov-Górnicki type maps with asymptotical regularity

Ćirić [58] in 1971 introduced the notion of asymptotical regularity.

Definition. A mapping T of a metric space (X, d) into itself is said to be asymptotically regular if $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$ for all $x \in X$.

For a metric fixed point theorem, if we add the asymptotical regularity to the map, then the conclusion is definitely strengthened. Conversely, some theorems on asymptotically regular maps may have more restrictive conclusions without assuming the asymptotical regularity.

In 2006, Proinov [59] obtained some metric fixed point results under the asymptotic regularity of mappings. Since then a number of followers made works influenced by Proinov and others.

The Ćirić-Proinov-Górnicki type mapping is a generic name of such maps generalizing the works of Proinov, Górnicki and others.

We borrow the following from Alsulami et al. [10] in 2014:

Definition 25. We will say that a self-mapping $T : X \to X$ on a quasi-metric space (X, q) is

(i) asymptotically right-regular at a point $x \in X$ if $\lim_{n\to\infty} q(T^n x, T^{n+1}x) = 0$;
(ii) asymptotically left-regular at a point $x \in X$ if $\lim_{n\to\infty} q(T^{n+1}x, T^n x) = 0$;
(iii) asymptotically regular if it is both asymptotically right-regular and asymptotically left-regular.

In this section, we show that many theorems on such Ćirić-Proinov-Górnicki type mappings can be modified by our Theorem P without assuming the asymptotic regularity.

Proinov [59] in 2006

In this paper, Proinov [59] unified the Boyd-Wong, Jachymski, Matkowski, and Meir-Keeler type contractions.

For a metric space (X, d) and a map $T : X \to X$, let

$$D(x, y) = d(x, y) + \gamma \cdot [d(x, Tx) + d(y, Ty)] \text{ where } \gamma > 0,$$

for $x, y \in X$.

Φ_1 is a class of all functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying: for any $\varepsilon > 0$ there exists $\delta > \varepsilon$ such that $\varepsilon < t < \delta$ implies $\varphi(t) \leqslant \varepsilon$.

Theorem 4.1. [59] *Let T be a continuous and asymptotically regular selfmapping on a complete metric space (X, d) satisfying the following conditions:*
(i) *There exists $\varphi \in \Phi_1$ such that $d(Tx, Ty) \leqslant \varphi(D(x, y))$ for all $x, y \in X$;*
(ii) *$d(Tx, Ty) < D(x, y)$ for all $x, y \in X$ with $x \neq y$.*
Then T has a contractive fixed point.

Corollary 4.6. [59] *Let T be a selfmapping on a complete metric space (X, d). Suppose that for all $x, y \in X$ the following conditions hold:*

(i) $d(Tx, Ty) \leq \mu[d(x, y) + d(x, Tx) + d(y, Ty)]$ *where* $0 < \mu < 1$;
(ii) $d(Tx, T^2x) \leq d(x, Tx)$;
(iii) $d(T^k x, T^{k+1}x) \leq \lambda\, d(x, Tx)$ *where* $k \in \mathbb{N}$ *and* $0 < \lambda < 1$.

Then T has a contractive fixed point $z \in X$.

Comments. (1) In Theorem 4.1, (i) implies (ii).
(2) In Corollary 4.6, (i) implies

$$(1 - \mu)d(Tx, T^2x) \leq 2\mu\, d(x, Tx).$$

Then Theorem P works for $0 \leq \mu < \frac{1}{3}$ or $0 \leq \frac{2\mu}{1-\mu} < 1$.
(3) Theorem P works for (iii) with $k = 1$ in Corollary 4.6.

Górnicki [60] in 2017

Theorem 3.1. [60] *If (X, d) is a complete metric space and $T : X \to X$ is an asymptotically regular mapping such that there exists $K < 1$ satisfying*

$$d(Tx, Ty) \leq K[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X.$$

Then, T has a unique fixed point $v \in X$.

Theorem 3.3. [60] *If (X, d) is a complete metric space and $T : X \to X$ is an asymptotically regular mapping such that there exists $M < 1$ satisfying*

$$d(Tx, Ty) \leq M[d(x, Tx) + d(y, Ty) + d(x, y)] \text{ for all } x, y \in X.$$

Then, T has a unique fixed point $v \in X$.

Comments. (1) Theorem 3.1 is a typical example for an asymptotically regular map. In fact, the Kannan theorem is Theorem 3.1 for $0 \leq K < 1/2$ without assuming asymptotical regularity and a consequence of Theorem P for orbitally complete quasi-metric spaces.

(2) Theorem 3.3 follows from Theorem P for $0 < M < 1/3$ without assuming asymptotical regularity.

Górnicki [61] in 2019

Górnicki [61] studied a new class of contractive mappings and proved a fixed point theorem for such mappings over metric spaces with assumption of continuity, which is given as follows. It is to be noted that asymptotic regularity has been assumed for these mappings in all over the metric space.

Theorem 2.6. [61] *If (X, d) is a complete metric space and $T : X \to X$ is a continuous asymptotically regular mapping and if there exists $M \in [0, 1)$ and $K \geq 0$ satisfying*

$$d(Tx, Ty) \leq M \cdot d(x, y) + K \cdot \{d(x, Tx) + d(y, Ty)\} \text{ for all } x, y \in X,$$

then T has a unique fixed point $p \in X$ and $T^n x \to p$ for each $x \in X$.

Bisht [62] replaced the assumption of continuity in Theorem 2.6 by a weaker version of continuity condition like orbital continuity or k-continuity (For definitions of orbital continuity and k-continuity one may refer [62]).

Panja et al. [63] have generalized the contractive condition (4) and have introduced a new type of contractive mapping called Ćirić-Proinov-Górnicki type mapping.

Comments. (1) In Theorem 2.6, if $0 \leq K < \frac{1}{2}$ such that $M + K < 1 - K$, then

$$d(Tx, T^2x) \leq \frac{M + K}{1 - K} d(x, Tx).$$

Therefore, our Theorem P works without assuming the continuity and asymptotical regularity of the mapping.

(2) Górnicki [61] stated in Section 3: *A Kannan type mapping $T : X \to X$ such that*

$$d(Tx, Ty) < d(x, Tx) + d(y, Ty) \text{ for all } x, y \in X \text{ with } x \neq y,$$

and asymptotically regular may not have a fixed point.

For $x \neq y = Tx$, the condition implies $0 < d(x, Tx)$ and hence the statement hold without assuming the asymptotical regularity.

Bisht [62] in 2019

Abstract. In this note, we show that the main result (Theorem 2.6) due to Górnicki [61] is still valid if we replace the assumption of continuity of the mapping by some weaker versions of continuity conditions. As a by-product, we provide few more new answers to the open question of Rhoades [64] in 1988.

Definition 1.2. [65] A self-mapping T of a metric space (X, d) is called *k-continuous*, $k = 1, 2, 3, \cdots$, if $T^k x_n \to Tz$, whenever $\{x_n\}$ is a sequence in X such that $T^{k-1}x_n \to z$.

Theorem 2.1. *If (X, d) is a complete metric space and $T : X \to X$ is an asymptotically regular mapping and if there exists $0 \leq M < 1$ and $0 \leq K < +\infty$ satisfying*

$$d(Tx, Ty) \leq Md(x, y) + K\{d(x, Tx) + d(y, Ty)\}$$

for all $x, y \in X$, then T has a unique fixed point $p \in X$ provided T is either k-continuous for $k \leq 1$ or orbitally continuous.

Comments. As for previous Theorem 2.6 of Górnicki, if $0 \leq K < \frac{1}{2}$ such that $M + K < 1 - K$, then our Theorem P works for Theorem 2.1 without assuming the continuity and asymptotical regularity of the map.

Final Remarks for Section 5. Recently there have appeared some complicated contractive type conditions. They might be extensions of the Banach contraction, but seem to be not applicable Theorem P. Such examples are Proinov [66] in 2020, Pant and Khantwahl [67] in 2023, Roy [68] in 2024, and some others.

6. A Rhoades Problem in 1977

From S.-S. Chang [8] in 1986: "Let (X, d) be a complete metric space and T a mapping from X into X. T is called a Rhoades mapping if for any $x, y \in A$, $x \neq y$,

(1.1) $d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$.

Concerning the existence of a fixed point for such mappings T, Rhoades [7] pointed out that the condition

A_1. T is continuous and $\{T^n x_0\}_{n=0}^{\infty}$ has a cluster point for some $x_0 \in X$, is needed in order to ensure that every such T possesses a fixed point.

The following open questions were raised by Rhoades [7] (also see [64]).

(1) If T is a Rhoades mapping satisfying condition A_1, does T possess a fixed point?

(2) If the answer to (1) is no, then what additional hypotheses on T or X are needed to guarantee the existence of a fixed point?

Later, (1) was answered in the negative, see [8]. For (2) a partial answer is that the condition

A_2. T is continuous and X is compact, is sufficient. (See [12, Theorem 1].)

The purpose of this paper is to introduce the concept of a C-mapping and to establish necessary and sufficient conditions for such mappings to possess a fixed point. Using these results Rhoades' open questions are answered."

We want to add up a trivial negative answer to (1) for a *quasi-metric space*: Let T be a Rhoades map. Then, for each $x \in X$, we have

$$d(Tx, T^2x) < \max\{d(x, Tx), d(x, T^2x)\} \leq \max\{d(x, Tx), d(x, Tx) + d(Tx, T^2x)\},$$

and hence

$$d(Tx, T^2x) < d(x, Tx) + d(Tx, T^2x) \text{ or } 0 < d(x, Tx).$$

Hence T has no fixed point.

7. Historical Remarks and Conclusion

We began to study on our RHR maps in [1] in 2022. Then we knew that kind of maps only in Rus [69], Park [26], and Jachymski [24]. That is why we named RHR maps without knowing other names due to several authors.

In [1] we found that the RHR theorem is logically equivalent to several propositions on fixed points, common fixed points, stationary points, maximal elements of various multimaps by our well-known 2023 Metatheorem. So we could find many "relatives" of RHR maps.

Since then we have tried to collect the literature on RHR maps, for example, Hicks-Rhoades [23] in 1979. Moreover, in our previous works [2], [3], [4], we noticed the importance of quasi-metric spaces.

In [2] and [3], we obtained the original forms of Theorems P and Q for quasi-metric spaces and showed that many known metric fixed theorems hold without the symmetry.

In [4], we collected almost all things we know about RHR maps and their examples. Moreover, we derive new classes of generalized RHR maps and fixed point theorems on them. Consequently, many of the known results in metric fixed point theory are improved and reproved in an easy way.

Later we found some literature where the RHR maps were already called graphic contraction, iterative contraction, weakly contraction, orbital contraction or Banach mapping; see Berinde et al. [70] in 2023.

Moreover, it is recently known that well-known metric fixed point theorems related to the RHR maps hold for quasi-metric spaces (without assuming the symmetry) as in the present article.

In [5], we showed that the so-called Suzuki type maps are RHR maps and the proofs of results of Suzuki and his colleagues can be simplified within a few line proofs based on our recent works on quasi-metric spaces.

Our [6], we trace the history of the RHR maps in ordered fixed point theory. Especially, from our well-known Metatheorem, we show that the metric completeness is equivalent to generalized forms of the RHR theorem, the Nadler fixed point theorem and some others. Moreover, the so-called Suzuki type maps are RHR maps and proofs of related theorems are simplified within few lines.

Consequently, our recent study on RHR maps improves the metric fixed point theory in several aspects as follows:

(1) From the beginning, the Banach contraction principle in thousands of publications is inaccurately stated.

(2) Many contractive type conditions extending Banach contractions can be unified to BHR maps and their extensions. All possible such conditions can be destroyed.

(3) There are hundreds of characterizations of metric completeness, see Cobzaş [71] or Park [72]. Many of them are related to a proper generalizations of the Banach contraction, that is, the RHB theorem. We can destroy such characterizations.

(4) There are thousands of artificial spaces extending quasi-metric spaces for metric fixed point theory. We doubt their actual necessity for human life.

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