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## Fixed point theorems in new multiplicative metric spaces

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### Abstract

We introduce new concepts of triangular and rectangular multiplicative metric spaces, and establish fixed-point theorems in such spaces. Then, we derive Kannan-type fixed-point theorems in triangular and rectangular multiplicative metric spaces.

*Keywords:* multiplicative metric space, fixed point theorem

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### 1. Introduction

In [2], Bashirov et al. introduced the concept of multiplicative metric on a nonempty set  $X$  as following: A function  $d_m : X \times X \rightarrow \mathbb{R}_+$  is said to be multiplicative metric if for all  $x, y, z \in X$  it satisfies the following axioms:

- (i)  $d_m(x, y) = 1$  if and only if  $x = y$ .
- (ii)  $d_m(x, y) = d_m(y, x)$ .
- (iii)  $d_m(x, y) \leq d_m(x, z)d_m(z, y)$ .

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In this case the pair  $(X, d_m)$  is called multiplicative metric space. It has been noted in [1, 4] that several fixed point theorems in the multiplicative metric space are indeed equivalent to those existing in metric spaces.

In particular, it has been noticed that if  $(X, d_m)$  is a multiplicative metric space, then  $(X, d)$  is a classical metric space where  $d(x, y) = \ln d_m(x, y)$ . To avoid this situation, we then propose new definitions.

**Definition 1.1.** Let  $X$  be a nonempty set,  $d: X \times X \rightarrow \mathbb{R}_+$  be a function and  $c \geq 1$  be a real constant. Then  $d$  is called triangular multiplicative metric if for all  $x, y, z \in X$  such that for all  $z \in X \setminus \{x, y\}$  the following axioms hold true:

$$(t_1) \quad d(x, y) = 0 \text{ if and only if } x = y,$$

$$(t_2) \quad d(x, y) = d(y, x),$$

$$(t_3) \quad d(x, y) \leq c d(x, z)d(z, y).$$

A pair  $(X, d)$  is called triangular multiplicative metric space.

Obviously, the first axiom  $(t_1)$  is different from axiom  $(i)$ , which prevents reducing the multiplicative triangular metric space to a classical metric as in the case of  $d_m$ . Similar to the rectangular metric space of Branciari [3], we define the rectangular multiplicative metric space as following:

**Definition 1.2.** Let  $X$  be a nonempty set,  $d: X \times X \rightarrow \mathbb{R}_+$  be a function and  $c \geq 1$  be a real constants. Then  $d$  is called rectangular multiplicative metric if for all  $x, y \in X$  and for all distinct points  $u, v \in X \setminus \{x, y\}$  the following axioms hold:

$$(r_1) \quad d(x, y) = 0 \text{ if and only if } x = y,$$

$$(r_2) \quad d(x, y) = d(y, x),$$

$$(r_3) \quad d(x, y) \leq c d(x, u)d(u, v)d(v, y).$$

A pair  $(X, d)$  is called rectangular multiplicative metric space.

**Example 1.3.** Let  $X = \mathbb{R}$  and  $d: X \times X \rightarrow [0, +\infty)$  be defined as follows

$$d(x, y) = \begin{cases} e^{|x-y|} & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to show that  $(X, d)$  is a triangular (rectangular) multiplicative metric space for  $c \geq 1$ .

**Example 1.4.** Let  $X = (0, +\infty)$  and  $d: X \times X \rightarrow [0, +\infty)$  be defined as follows

$$d(x, y) = \langle \frac{x}{y} \rangle,$$

where  $\langle \cdot \rangle : (0, +\infty) \rightarrow (0, +\infty)$  is given by

$$\langle a \rangle = \begin{cases} a & \text{if } a > 1, \\ \frac{1}{a} & \text{if } a < 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that all the conditions of the multiplicative rectangular metric are satisfied for all  $c \geq 1$ .

The concepts of convergence and completeness in triangular or rectangular multiplicative metrics are similar to those in metric spaces.

## 2. Main results

In this section, we establish fixed point theorems in triangular and rectangular multiplicative metric spaces.

**Theorem 2.1.** *Let  $(X, d)$  be a complete triangular multiplicative metric space and let  $f : X \rightarrow X$  be a given mapping. Assume there exists  $\lambda \in [0, 1)$  such that*

$$d(fx, fy) \leq \lambda \max\{d(x, fx), d(y, fy)\} \text{ for all } x, y \in X. \tag{1}$$

*Then  $f$  has a unique fixed point  $x_* \in X$  and the sequence  $\{f^n x_0\}$  converges to  $x_*$  for every  $x_0 \in X$ .*

*Proof.* Let  $x_0 \in X$ . Define  $x_n = f^n x_0$  for all  $n \in \mathbb{N}$ . We firstly observe by (1) that the fixed point is unique whenever it exists. Assume now that there exist some  $n, m \in \mathbb{N}$  such that  $n < m$  and  $x_n = x_m$ . Thus, we have

$$d(x_n, x_{n+1}) = d(x_m, x_{m+1}) \leq \lambda \max\{d(x_{m-1}, x_m), d(x_m, x_{m+1})\},$$

that is,

$$d(x_m, x_{m+1}) \leq \lambda d(x_{m-1}, x_m).$$

We deduce that

$$d(x_n, x_{n+1}) = d(x_m, x_{m+1}) \leq \lambda^{m-n} d(x_n, x_{n+1}).$$

which is absurd unless  $d(x_n, x_{n+1}) = 0$ , that is,  $x_n$  is a fixed point of  $f$ .

We will assume from now on that  $x_n \neq x_m$  for all distinct  $n, m \in \mathbb{N}$ , and we shall show that  $\{x_n\}$  is a Cauchy sequence. By using (1), we get for all  $n \in \mathbb{N}$  that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \lambda \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= \lambda d(x_{n-1}, x_n), \end{aligned}$$

thus, by induction we obtain

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1). \tag{2}$$

Now, by using (1), we have

$$d(x_n, x_m) \leq \lambda \max\{d(x_{n-1}, x_n), d(x_{m-1}, x_m)\}, \text{ for all } n \in \mathbb{N}. \tag{3}$$

Thus, from (2) and (3) we conclude that

$$d(x_n, x_m) \leq \max\{\lambda^n, \lambda^m\} d(x_0, x_1), \text{ for all } n, m \in \mathbb{N}. \tag{4}$$

So,  $\lim_{n, m \rightarrow +\infty} d(x_n, x_m) = 0$ , that is,  $\{x_n\}$  is a Cauchy sequence and since  $X$  is complete, so there exists  $x_* \in X$  such that

$$\lim_{n \rightarrow +\infty} d(x_n, x_*) = 0. \tag{5}$$

If for some  $n$ ,  $x_n = x_*$  or  $x_{n+1} = fx_*$ , we obtain

$$d(x_*, fx_*) = d(x_n, x_{n+1}) \text{ or } d(x_*, fx_*) = d(x_*, x_{n+1})$$

which tend to 0 as  $n$  tends to infinity, thus we deduce that  $x_* = fx_*$ .

Assume next that  $x_n \neq x_*$  and  $x_{n+1} \neq fx_*$  for all  $n \in \mathbb{N}$ , then by using (t3) we have

$$\begin{aligned} d(x_*, fx_*) &\leq c d(x_*, x_n) d(x_n, fx_*) \\ &\leq c^2 d(x_*, x_n) d(x_n, x_{n+1}) d(x_{n+1}, fx_*). \end{aligned}$$

In other hand, from (4) we have

$$\begin{aligned} d(x_{n+1}, fx_*) &\leq \lambda \max\{d(x_n, x_{n+1}), d(x_*, fx_*)\} \\ &\leq \lambda \max\{\lambda^n d(x_0, x_1), d(x_*, fx_*)\}, \end{aligned}$$

so it follows that

$$d(x_*, fx_*) \leq c^2 \lambda^{n+1} d(x_*, x_n) d(x_0, x_1) \max\{\lambda^n d(x_0, x_1), d(x_*, fx_*)\}. \tag{6}$$

If we assume that there exists a subsequence  $\{n(k)\}$  such that

$$\max\{\lambda^{n(k)} d(x_0, x_1), d(x_*, fx_*)\} = d(x_*, fx_*),$$

that is,

$$d(x_*, fx_*) \leq \lambda^{n(k)} d(x_0, x_1),$$

then, if  $k$  tends to infinity, we get that  $fx_* = x_*$ .

Otherwise, if there exists an integer  $N > 0$  such that for all  $n > N$ , we have

$$\max\{\lambda^n d(x_0, x_1), d(x_*, fx_*)\} = \lambda^n d(x_0, x_1),$$

then (6) becomes

$$d(x_*, fx_*) \leq c^2 \lambda^{2n+1} d(x_*, x_n) d(x_0, x_1)^2.$$

thus, if  $n$  tends to infinity, we get by (5) that  $fx_* = x_*$ . □

As an immediate consequence, we obtain a fixed point theorem of Kannan type [5] in triangular multiplicative metric spaces.

**Corollary 2.2.** *Let  $(X, d)$  be a complete triangular multiplicative metric space and let  $f: X \rightarrow X$  be a given mapping. Assume there exists  $\lambda \in [0, \frac{1}{2})$  such that*

$$d(fx, fy) \leq \lambda(d(x, fx) + d(y, fy)) \text{ for all } x, y \in X.$$

*Then  $f$  has a unique fixed point  $x_* \in X$  and the sequence  $\{f^n x_0\}$  converges to  $x_*$  for every  $x_0 \in X$ .*

Next, we present a fixed point theorem in rectangular multiplicative metric space. The proof is similar to that of the previous theorem, but for the sake of completeness, we give a full proof here.

**Theorem 2.3.** *Let  $(X, d)$  be a complete rectangular multiplicative metric space and let  $f: X \rightarrow X$  be a given mapping. Assume there exists  $\lambda \in [0, \frac{1}{2})$  such that*

$$d(fx, fy) \leq \lambda \max\{d(x, fx), d(y, fy)\} \text{ for all } x, y \in X. \tag{7}$$

*Then  $f$  has a unique fixed point  $x_* \in X$  and the sequence  $\{f^n x_0\}$  converges to  $x_*$  for every  $x_0 \in X$ .*

*Proof.* Let  $x_0 \in X$ . Define  $x_n = f^n x_0$  for all  $n \in \mathbb{N}$ . Clearly by (7) the fixed point is unique whenever it exists. Assume now that there exists some  $n, m \in \mathbb{N}$  such that  $n < m$  and  $x_n = x_m$ . Thus, we have

$$\begin{aligned} d(x_n, x_{n+1}) = d(x_m, x_{m+1}) &\leq \lambda \max\{d(x_m, x_{m+1}), d(x_{m-1}, x_m)\} \\ &= \lambda d(x_{m-1}, x_m), \end{aligned}$$

We deduce that

$$d(x_n, x_{n+1}) = d(x_m, x_{m+1}) \leq \lambda^{m-n} d(x_n, x_{n+1}).$$

which is absurd unless  $d(x_{n+1}, x_n) = 0$ , that is,  $x_n$  is a fixed point of  $f$ .

We will assume from now on that  $x_n \neq x_m$  for all distinct  $n, m \in \mathbb{N}$ , and we shall show that  $\{x_n\}$  is a Cauchy sequence. Observe first that from (7), for all  $n \in \mathbb{N}$ , we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \lambda \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= \lambda d(x_{n-1}, x_n), \end{aligned}$$

thus, we obtain

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1) \text{ for all } n \in \mathbb{N}, \tag{8}$$

Now, by using condition (7), we have

$$d(x_n, x_m) \leq \lambda \max\{d(x_{n-1}, x_n), d(x_{m-1}, x_m)\} \text{ for all } n, m \in \mathbb{N}. \tag{9}$$

Combining (8) and (9), we get

$$d(x_n, x_m) \leq \max\{\lambda^n, \lambda^m\} d(x_0, x_1) \text{ for all } n, m \in \mathbb{N}.$$

So,  $\lim_{n,m \rightarrow +\infty} d(x_n, x_m) = 0$ , that is,  $\{x_n\}$  is a Cauchy sequence and since  $X$  is complete there exists  $x_* \in X$  such that

$$\lim_{n \rightarrow +\infty} d(x_n, x_*) = 0. \tag{10}$$

If for some  $n$ ,  $x_n = x_*$  or  $x_{n+1} = fx_*$ , we obtain

$$d(x_*, fx_*) = d(x_n, x_{n+1}) \text{ or } d(x_*, fx_*) = d(x_*, x_{n+1})$$

which tend to 0 as  $n$  tends to infinity, thus we deduce that  $x_* = fx_*$ .

Assume next that  $x_n \neq x_*$  and  $x_{n+1} \neq fx_*$  for all  $n \in \mathbb{N}$ , then by using (r3) we have

$$d(x_*, fx_*) \leq c d(x_*, x_n) d(x_n, x_{n+1}) d(x_{n+1}, fx_*).$$

Now, from (8) we have

$$\begin{aligned} d(x_{n+1}, fx_*) &\leq \lambda \max\{d(x_n, x_{n+1}), d(x_*, fx_*)\} \\ &= \lambda \max\{\lambda^n d(x_0, x_1), d(x_*, fx_*)\}. \end{aligned}$$

Thus, it follows that

$$d(x_*, fx_*) \leq c \lambda^{n+1} d(x_0, x_1) d(x_*, x_n) \max\{\lambda^n d(x_0, x_1), d(x_*, fx_*)\}. \tag{11}$$

If we assume that there exists a subsequence  $\{n(k)\}$  such that

$$\max\{\lambda^{n(k)} d(x_0, x_1), d(x_*, fx_*)\} = d(x_*, fx_*),$$

that is,

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thus, if  $k$  tends to infinity, we get that  $fx_* = x_*$ .

Otherwise, if there exists an integer  $N > 0$  such that for all  $n > N$ , we have

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then (11) becomes

$$d(x_*, fx_*) \leq c \lambda^{2n+1} d(x_*, x_n) d(x_0, x_1)^2.$$

thus, if  $n$  tends to infinity, we get by (10) that  $fx_* = x_*$ . □

As an immediate consequence, we obtain a fixed point theorem of Kannan type in rectangular multiplicative metric spaces..

**Corollary 2.4.** *Let  $(X, d)$  be a complete rectangular multiplicative metric space and let  $f: X \rightarrow X$  be a given mapping. Assume there exists  $\lambda \in [0, \frac{1}{2})$  such that*

$$d(fx, fy) \leq \lambda(d(x, fx) + d(y, fy)) \text{ for all } x, y \in X.$$

*Then  $f$  has a unique fixed point  $x_* \in X$  and the sequence  $\{f^n x_0\}$  converges to  $x_*$  for every  $x_0 \in X$ .*

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