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Fixed point theorems in new multiplicative metric spaces

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Abstract

We introduce new concepts of triangular and rectangular multiplicative metric spaces, and establish fixed-point theorems in such spaces. Then, we derive Kannan-type fixed-point theorems in triangular and rectangular multiplicative metric spaces.

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1. Introduction

In [2], Bashirov et al. introduced the concept of multiplicative metric on a nonempty set X as following: A function $d_m\colon X\times X\to\mathbb{R}_+$ is said to be multiplicative metric if for all $x,y,z\in X$ it satisfies the following axioms:

(i) $d_m(x,y)=1$ if and only if $x=y$.

(ii) $d_m(x,y)=d_m(y,x)$.

(iii) $d_m(x,y)\le d_m(x,z)d_m(z,y)$.

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In this case the pair (X, d_m) is called multiplicative metric space. It has been noted in [1, 4] that several fixed point theorems in the multiplicative metric space are indeed equivalent to those existing in metric spaces.

In particular, it has been noticed that if (X, d_m) is a multiplicative metric space, then (X, d) is a classical metric space where $d(x, y) = \ln d_m(x, y)$. To avoid this situation, we then propose new definitions.

Definition 1.1. Let X be a nonempty set, $d\colon X \times X \to \mathbb{R}_+$ be a function and $c \geq 1$ be a real constant. Then d is called triangular multiplicative metric if for all $x, y, z \in X$ such that for all $z \in X \setminus \{x, y\}$ the following axioms hold true:

(t_1) $d(x, y) = 0$ if and only if $x = y$,

(t_2) $d(x, y) = d(y, x)$,

(t_3) $d(x, y) \leq c\, d(x, z)d(z, y)$.

A pair (X, d) is called triangular multiplicative metric space.

Obviously, the first axiom (t_1) is different from axiom (i), which prevents reducing the multiplicative triangular metric space to a classical metric as in the case of d_m. Similar to the rectangular metric space of Branciari [3], we define the rectangular multiplicative metric space as following:

Definition 1.2. Let X be a nonempty set, $d\colon X \times X \to \mathbb{R}_+$ be a function and $c \geq 1$ be a real constants. Then d is called rectangular multiplicative metric if for all $x, y \in X$ and for all distinct points $u, v \in X \setminus \{x, y\}$ the following axioms hold:

(r_1) $d(x, y) = 0$ if and only if $x = y$,

(r_2) $d(x, y) = d(y, x)$,

(r_3) $d(x, y) \leq c\, d(x, u)d(u, v)d(v, y)$.

A pair (X, d) is called rectangular multiplicative metric space.

Example 1.3. Let $X = \mathbb{R}$ and $d\colon X \times X \to [0, +\infty)$ be defined as follows

$$d(x, y) = \begin{cases} e^{|x-y|} & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to show that (X, d) is a triangular (rectangular) multiplicative metric space for $c \geq 1$.

Example 1.4. Let $X = (0, +\infty)$ and $d\colon X \times X \to [0, +\infty)$ be defined as follows

$$d(x, y) = \langle \tfrac{x}{y} \rangle,$$

where $\langle \cdot \rangle : (0, +\infty) \to (0, +\infty)$ is given by

$$\langle a \rangle = \begin{cases} a & \text{if } a > 1, \\ \frac{1}{a} & \text{if } a < 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that all the conditions of the multiplicative rectangular metric are satisfied for all $c \geq 1$.

The concepts of convergence and completeness in triangular or rectangular multiplicative metrics are similar to those in metric spaces.

2. Main results

In this section, we establish fixed point theorems in triangular and rectangular multiplicative metric spaces.

Theorem 2.1. *Let (X, d) be a complete triangular multiplicative metric space and let $f : X \to X$ be a given mapping. Assume there exists $\lambda \in [0, 1)$ such that*

$$d(fx, fy) \leq \lambda \max\{d(x, fx), d(y, fy)\} \text{ for all } x, y \in X. \tag{1}$$

Then f has a unique fixed point $x_* \in X$ and the sequence $\{f^n x_0\}$ converges to x_* for every $x_0 \in X$.

Proof. Let $x_0 \in X$. Define $x_n = f^n x_0$ for all $n \in \mathbb{N}$. We firstly observe by (1) that the fixed point is unique whenever it exists. Assume now that there exist some $n, m \in \mathbb{N}$ such that $n < m$ and $x_n = x_m$. Thus, we have

$$d(x_n, x_{n+1}) = d(x_m, x_{m+1}) \leq \lambda \max\{d(x_{m-1}, x_m), d(x_m, x_{m+1})\},$$

that is,

$$d(x_m, x_{m+1}) \leq h\, d(x_{m-1}, x_m).$$

We deduce that

$$d(x_n, x_{n+1}) = d(x_m, x_{m+1}) \leq \lambda^{m-n} d(x_n, x_{n+1}).$$

which is absurd unless $d(x_n, x_{n+1}) = 0$, that is, x_n is a fixed point of f.

We will assume from now on that $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$, and we shall show that $\{x_n\}$ is a Cauchy sequence. By using (1), we get for all $n \in \mathbb{N}$ that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \lambda \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= \lambda d(x_{n-1}, x_n), \end{aligned}$$

thus, by induction we obtain

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1). \tag{2}$$

Now, by using (1), we have

$$d(x_n, x_m) \leq \lambda \max\{d(x_{n-1}, x_n), d(x_{m-1}, x_m)\}, \text{ for all } n \in \mathbb{N}. \tag{3}$$

Thus, from (2) and (3) we conclude that

$$d(x_n, x_m) \leq \max\{\lambda^n, \lambda^m\} d(x_0, x_1), \text{ for all } n, m \in \mathbb{N}. \tag{4}$$

So, $\lim_{n,m\to+\infty} d(x_n, x_m) = 0$, that is, $\{x_n\}$ is a Cauchy sequence and since X is complete, so there exists $x_* \in X$ such that

$$\lim_{n\to+\infty} d(x_n, x_*) = 0. \tag{5}$$

If for some n, $x_n = x_*$ or $x_{n+1} = fx_*$, we obtain

$$d(x_*, fx_*) = d(x_n, x_{n+1}) \text{ or } d(x_*, fx_*) = d(x_*, x_{n+1})$$

which tend to 0 as n tends to infinity, thus we deduce that $x_* = fx_*$.

Assume next that $x_n \neq x_*$ and $x_{n+1} \neq fx_*$ for all $n \in \mathbb{N}$, then by using (t_3) we have

$$\begin{aligned} d(x_*, fx_*) &\leq c\, d(x_*, x_n) d(x_n, fx_*) \\ &\leq c^2\, d(x_*, x_n) d(x_n, x_{n+1}) d(x_{n+1}, fx_*). \end{aligned}$$

In other hand, from (4) we have

$$\begin{aligned} d(x_{n+1}, fx_*) &\leq \lambda \max\{d(x_n, x_{n+1}),\, d(x_*, fx_*)\} \\ &\leq \lambda \max\{\lambda^n d(x_0, x_1),\, d(x_*, fx_*)\}, \end{aligned}$$

so it follows that

$$d(x_*, fx_*) \leq c^2 \lambda^{n+1} d(x_*, x_n) d(x_0, x_1) \max\{\lambda^n d(x_0, x_1),\, d(x_*, fx_*)\}. \tag{6}$$

If we assume that there exists a subsequence $\{n(k)\}$ such that

$$\max\{\lambda^{n(k)} d(x_0, x_1),\, d(x_*, fx_*)\} = d(x_*, fx_*),$$

that is,

$$d(x_*, fx_*) \leq \lambda^{n(k)} d(x_0, x_1),$$

then, if k tends to infinity, we get that $fx_* = x_*$.
Otherwise, if there exists an integer $N > 0$ such that for all $n > N$, we have

$$\max\{\lambda^n d(x_0, x_1),\, d(x_*, fx_*)\} = \lambda^n d(x_0, x_1),$$

then (6) becomes

$$d(x_*, fx_*) \leq c^2 \lambda^{2n+1} d(x_*, x_n) d(x_0, x_1)^2.$$

thus, if n tends to infinity, we get by (5) that $fx_* = x_*$. □

As an immediate consequence, we obtain a fixed point theorem of Kannan type [5] in triangular multiplicative metric spaces.

Corollary 2.2. *Let (X, d) be a complete triangular multiplicative metric space and let $f\colon X \to X$ be a given mapping. Assume there exists $\lambda \in \left[0, \frac{1}{2}\right)$ such that*

$$d(fx, fy) \leq \lambda\big(d(x, fx) + d(y, fy)\big) \text{ for all } x, y \in X.$$

Then f has a unique fixed point $x_* \in X$ and the sequence $\{f^n x_0\}$ converges to x_* for every $x_0 \in X$.

Next, we present a fixed point theorem in rectangular multiplicative metric space. The proof is similar to that of the previous theorem, but for the sake of completeness, we give a full proof here.

Theorem 2.3. *Let (X, d) be a complete rectangular multiplicative metric space and let $f\colon X \to X$ be a given mapping. Assume there exists $\lambda \in \left[0, \frac{1}{2}\right)$ such that*

$$d(fx, fy) \leq \lambda \max\{d(x, fx), d(y, fy)\} \text{ for all } x, y \in X. \tag{7}$$

Then f has a unique fixed point $x_* \in X$ and the sequence $\{f^n x_0\}$ converges to x_* for every $x_0 \in X$.

Proof. Let $x_0 \in X$. Define $x_n = f^n x_0$ for all $n \in \mathbb{N}$. Clearly by (7) the fixed point is unique whenever it exists. Assume now that there exists some $n, m \in \mathbb{N}$ such that $n < m$ and $x_n = x_m$. Thus, we have

$$\begin{aligned} d(x_n, x_{n+1}) = d(x_m, x_{m+1}) &\leq \lambda \max\{d(x_m, x_{m+1}), d(x_{m-1}, x_m)\} \\ &= \lambda\, d(x_{m-1}, x_m), \end{aligned}$$

We deduce that

$$d(x_n, x_{n+1}) = d(x_m, x_{m+1}) \leq \lambda^{m-n} d(x_n, x_{n+1}).$$

which is absurd unless $d(x_{n+1}, x_n) = 0$, that is, x_n is a fixed point of f.

We will assume from now on that $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$, and we shall show that $\{x_n\}$ is a Cauchy sequence. Observe first that from (7), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \lambda \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &= \lambda\, d(x_{n-1}, x_n), \end{aligned}$$

thus, we obtain

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1) \text{ for all } n \in \mathbb{N}, \tag{8}$$

Now, by using condition (7), we have

$$d(x_n, x_m) \leq \lambda \max\{d(x_{n-1}, x_n), d(x_{m-1}, x_m)\} \text{ for all } n, m \in \mathbb{N}. \tag{9}$$

Combining (8) and (9), we get

$$d(x_n, x_m) \leq \max\{\lambda^n, \lambda^m\} d(x_0, x_1) \text{ for all } n, m \in \mathbb{N}.$$

So, $\lim_{n,m\to+\infty} d(x_n, x_m) = 0$, that is, $\{x_n\}$ is a Cauchy sequence and since X is complete there exists $x_* \in X$ such that

$$\lim_{n\to+\infty} d(x_n, x_*) = 0. \tag{10}$$

If for some n, $x_n = x_*$ or $x_{n+1} = fx_*$, we obtain

$$d(x_*, fx_*) = d(x_n, x_{n+1}) \text{ or } d(x_*, fx_*) = d(x_*, x_{n+1})$$

which tend to 0 as n tends to infinity, thus we deduce that $x_* = fx_*$.

Assume next that $x_n \neq x_*$ and $x_{n+1} \neq fx_*$ for all $n \in \mathbb{N}$, then by using (r_3) we have

$$d(x_*, fx_*) \leq c\, d(x_*, x_n) d(x_n, x_{n+1}) d(x_{n+1}, fx_*).$$

Now, from (8) we have

$$\begin{aligned} d(x_{n+1}, fx_*) &\leq \lambda \max\{d(x_n, x_{n+1}),\, d(x_*, fx_*)\} \\ &= \lambda \max\{\lambda^n d(x_0, x_1),\, d(x_*, fx_*)\}. \end{aligned}$$

Thus, it follows that

$$d(x_*, fx_*) \leq c\, \lambda^{n+1} d(x_0, x_1) d(x_*, x_n) \max\{\lambda^n d(x_0, x_1),\, d(x_*, fx_*)\}. \tag{11}$$

If we assume that there exists a subsequence $\{n(k)\}$ such that

$$\max\{\lambda^{n(k)} d(x_0, x_1),\, d(x_*, fx_*)\} = d(x_*, fx_*),$$

that is,

$$d(x_*, fx_*) \leq \lambda^{n(k)} d(x_0, x_1),$$

thus, if k tends to infinity, we get that $fx_* = x_*$.
Otherwise, if there exists an integer $N > 0$ such that for all $n > N$, we have

$$\max\{\lambda^n d(x_0, x_1),\, d(x_*, fx_*)\} = \lambda^n d(x_0, x_1),$$

then (11) becomes

$$d(x_*, fx_*) \leq c\, \lambda^{2n+1} d(x_*, x_n) d(x_0, x_1)^2.$$

thus, if n tends to infinity, we get by (10) that $fx_* = x_*$. □

As an immediate consequence, we obtain a fixed point theorem of Kannan type in rectangular multiplicative metric spaces..

Corollary 2.4. *Let (X, d) be a complete rectangular multiplicative metric space and let $f\colon X \to X$ be a given mapping. Assume there exists $\lambda \in \left[0, \frac{1}{2}\right)$ such that*

$$d(fx, fy) \leq \lambda\big(d(x, fx) + d(y, fy)\big) \textit{ for all } x, y \in X.$$

Then f has a unique fixed point $x_* \in X$ and the sequence $\{f^n x_0\}$ converges to x_* for every $x_0 \in X$.

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