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On Periodic Solutions for Coupled Systems with Caputo Tempered Fractional Derivative

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Abstract

The main goal of this paper is to study the existence and uniqueness of periodic solutions for coupled system with Caputo tempered fractional derivative. The proofs are based upon the coincidence degree theory of Mawhin. An example is constructed to authenticate and affirm the main findings.

Keywords: Coincidence degree theory, existence, uniqueness, Tempered fractional operators, coupled system.
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1. Introduction

Fractional calculus goes beyond classical differentiation and integration by containing non-integer orders, a concept that has captured both theoretical interest and practical relevance across diverse research fields. Its adaptability has elevated it to a pivotal role in the domain. Recent times have witnessed a notable surge in research dedicated to fractional calculus, investigating various outcomes within distinct scenarios

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and manifestations of fractional differential equations and inclusions. To delve deeper into the practical implementations of fractional calculus, readers are referred to works by Herrmann [9] and Samko *et al.* [23]. The works of Benchohra *et al.* [2, 3] have spotlighted the existence, uniqueness, and stability of diverse problem classes, each adhering to distinct conditions. They introduced an extension of the renowned Hilfer fractional derivative, seamlessly combining the Riemann-Liouville and Caputo fractional derivatives.

Tempered fractional calculus has emerged as an important class of fractional calculus operators in recent years. This class can generalize various forms of fractional calculus and possesses analytic kernels, making it an extension of fractional calculus that can describe the transition between normal and anomalous diffusion. The definitions of fractional integration with weak singular and exponential kernels were initially established by Buschman in [6], and further elaboration on this topic can be found in [1, 10, 11, 12, 15, 16, 18, 19, 22, 24]. Although the Caputo tempered fractional derivative has not been extensively explored in the literature, it holds the potential to significantly contribute to this field. By studying this derivative, we aim to better understand its properties and potential applications in this unique mathematical notion, thus advancing fractional calculus.

The introduction of the coincidence degree theory by Mawhin [8, 14] has been widely employed to analyze different classes of nonlinear differential equations. This approach becomes especially advantageous in cases where classical techniques like the fixed point principle cannot be used. In references [4, 5, 7, 20, 21], the application of the coincidence degree theory led to outcomes concerning fractional-order nonlinear differential equations that would have been impossible to achieve using alternate methods such as the fixed point principle.

In [20], the authors studied the existence and uniqueness results for a coupled system of nonlinear k-generalized ψ-Hilfer type implicit fractional differential equations and periodic conditions as follows:

$$\begin{cases} \left({}^{H}_{k}\mathfrak{D}^{\vartheta_1,\varkappa_1;\psi}_{\omega_1{}^+}\mathfrak{w}_1\right)(\delta) = \chi_1\left(\delta, \mathfrak{w}_1(\delta), \mathfrak{w}_2(\delta), \left({}^{H}_{k}\mathfrak{D}^{\vartheta_1,\varkappa_1;\psi}_{\omega_1{}^+}\mathfrak{w}_1\right)(\delta), \left({}^{H}_{k}\mathfrak{D}^{\vartheta_2,\varkappa_2;\psi}_{\omega_1{}^+}\mathfrak{w}_2\right)(\delta)\right), \\ \left({}^{H}_{k}\mathfrak{D}^{\vartheta_2,\varkappa_2;\psi}_{\omega_1{}^+}\mathfrak{w}_2\right)(\delta) = \chi_2\left(\delta, \mathfrak{w}_1(\delta), \mathfrak{w}_2(\delta), \left({}^{H}_{k}\mathfrak{D}^{\vartheta_1,\varkappa_1;\psi}_{\omega_1{}^+}\mathfrak{w}_1\right)(\delta), \left({}^{H}_{k}\mathfrak{D}^{\vartheta_2,\varkappa_2;\psi}_{\omega_1{}^+}\mathfrak{w}_2\right)(\delta)\right), \end{cases}$$

$$\mathfrak{I}^{k(1-\delta_1),k;\psi}_{\omega_1{}^+}\mathfrak{w}_1(\omega_1) = \mathfrak{I}^{k(1-\delta_1),k;\psi}_{\omega_1{}^+}\mathfrak{w}_1(\omega_2) \text{ and } \mathfrak{I}^{k(1-\delta_2),k;\psi}_{\omega_1{}^+}\mathfrak{w}_2(\omega_1) = \mathfrak{I}^{k(1-\delta_2),k;\psi}_{\omega_1{}^+}\mathfrak{w}_2(\omega_2),$$

where $\delta \in (\omega_1, \omega_2]$, ${}^{H}_{k}\mathfrak{D}^{\vartheta_i,\varkappa_i;\psi}_{\omega_1{}^+}$ and $\mathfrak{I}^{k(1-\delta_i),k;\psi}_{\omega_1{}^+}$ are the k-generalized ψ-Hilfer fractional derivative of order $0 < \vartheta_i \leq k$ and type $\varkappa_i \in [0,1]$ and the k-generalized ψ-fractional integral of order $k(1-\delta_i)$, $\delta_i = \frac{1}{k}(\vartheta_i + k\varkappa_i - \vartheta_i\varkappa_i)$, $i \in \{1,2\}$, respectively. Moreover $\chi_1, \chi_2 : \Theta \times \mathbb{R}^4 \to \mathbb{R}$ are continuous functions.

In [4], by using the coincidence degree theory of Mawhin, the authors studied the nonlinear pantograph fractional equations with Ψ-Hilfer fractional derivative:

$$\begin{cases} {}^{H}\mathfrak{D}^{\varrho,\beta;\Psi}_{\mathfrak{a}^+}\mathfrak{w}(\delta) = \aleph\left(\delta, \mathfrak{w}(\delta), \mathfrak{w}(\varepsilon\delta)\right), \ \delta \in (0,\varkappa], \\ \mathfrak{I}^{1-\nu,\Psi}_{0^+}\mathfrak{w}(0) = \mathfrak{I}^{1-\nu,\Psi}_{0^+}\mathfrak{w}(\varkappa), \end{cases}$$

where ${}^{H}\mathfrak{D}^{\varrho,\beta;\Psi}_{0^+}$ denote the Ψ-Hilfer fractional derivative of order $0 < \varrho \leq 1$, $0 < \varepsilon < 1$ and type $\beta \in [0,1]$. $\mathfrak{I}^{1-\nu,\Psi}_{0^+}$ is the Ψ-Riemann–Liouville fractional integral of order $1-\nu$, $(\nu = \varrho + \beta - \varrho\beta)$. Moreover, $\aleph : (0,\varkappa] \times \mathfrak{R}^2 \to \mathfrak{R}$ is a given continuous function.

In [11], the authors investigated the following class of Caputo tempered fractional differential equation with finite delay:

$$\begin{cases} \left({}_0^C\mathfrak{D}_\delta^{\kappa,\varepsilon}\mathfrak{w}\right)(\delta) = \aleph\left(\delta, \mathfrak{w}_\delta, \mathfrak{D}_0^\kappa \mathfrak{w}(\delta)\right);\ \delta \in \Theta := [0, \varpi], \\ \mathfrak{w}(\delta) = \wp(\delta), \quad \delta \in [-\kappa, 0], \\ \jmath_1 \mathfrak{w}(0) + \jmath_2 \mathfrak{w}(\varpi) = \jmath_3, \end{cases}$$

where $0 < \kappa < 1$, $\varepsilon \geq 0$, ${}_0^C\mathfrak{D}_\delta^{\kappa,\varepsilon}$ is the Caputo tempered fractional derivative, $\aleph : \Theta \times C([-\kappa, 0], \mathbb{R}) \times \mathbb{R}$ is a continuous function, $\wp \in C([-\kappa, \varpi], \mathbb{R})$, $0 < \varpi < +\infty$, $\jmath_1, \jmath_2, \jmath_3$ are real constants, and $\kappa > 0$ is the time delay. The results are based on the fixed point theorems of Banach, Schauder and Schaefer. Observe that this problem encompasses initial, terminal, and anti-periodic problems; however, the employed approach does not yield solutions for the periodic problem.

In this paper, we study the existence and uniqueness of periodic solutions for the coupled system with the Caputo tempered fractional derivative:

$$\begin{cases} \left({}_0^C\mathfrak{D}_\delta^{\varrho_1,\wp_1}\mathfrak{w}_1\right)(\delta) = \aleph_1\left(\delta, \mathfrak{w}_1(\delta), \mathfrak{w}_2(\delta)\right), \\ \\ \left({}_0^C\mathfrak{D}_\delta^{\varrho_2,\wp_2}\mathfrak{w}_2\right)(\delta) = \aleph_2\left(\delta, \mathfrak{w}_1(\delta), \mathfrak{w}_2(\delta)\right), \end{cases} \quad \delta \in \nabla := [0, \varkappa], \tag{1.1}$$

$$\mathfrak{w}_1(0) = \mathfrak{w}_1(\varkappa) = 0 \text{ and } \mathfrak{w}_2(0) = \mathfrak{w}_2(\varkappa) = 0, \tag{1.2}$$

where $0 < \varrho_i < 1,\ i \in \{1, 2\}$, $\wp_i \geq 0$, ${}_0^C\mathfrak{D}_\delta^{\varrho_i,\wp_i}, i \in \{1, 2\}$ is the Caputo tempered fractional derivative, and $\aleph_i : \nabla \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function.

It's important to highlight that although the Caputo tempered fractional derivative has received relatively little attention in existing literature, it possesses the potential to offer significant advancements in this particular domain. Our research intends to delve into the characteristics and possible practical applications of the Caputo tempered fractional derivative. This endeavor seeks to not only augment our comprehension of this distinctive mathematical notion but also to propel the progress of fractional calculus. Moreover, our study is novel in that it addresses a certain category of problems, specifically, coupled systems involving the Caputo tempered fractional derivative alongside periodic conditions that have yet to be explored within existing literature. This renders our contribution a natural extension in the development of this dynamic field.

The structure of this paper is as follows: Section 2 presents certain notations and preliminaries about the tempered fractional derivatives used throughout this manuscript. In Section 3, we present existence and uniqueness result for the system (1.1)-(1.2) that are based upon the coincidence degree theory of Mawhin. Ultimately, an illustrative example will be provided to demonstrate our outcomes.

2. Preliminaries

First, we give the definitions and the notations that we will use throughout this paper. We denote by $C(\nabla, \mathbb{R})$ the Banach space of all continuous functions from ∇ into $\mathbb{R}$ with the following norm

$$\|\aleph\|_\infty = \sup_{\delta \in \nabla}\{|\aleph(\delta)|\}.$$

As usual, $AC(\nabla)$ denotes the space of absolutely continuous functions from ∇ into $\mathbb{R}$. For any $\jmath \in \mathbb{N}^*$, we denote by $AC^\jmath(\nabla)$ the space defined by

$$AC^\jmath(\nabla) := \left\{\mathfrak{w} : \nabla \to \mathbb{R} : \frac{d^\jmath}{dt^\jmath}\mathfrak{w}(\delta) \in AC(\nabla)\right\}.$$

Consider the space $X_b^p(0,\varkappa)$, $(b\in\mathbb{R},\ 1\le p\le\infty)$ of those real-valued Lebesgue measurable functions $\mathfrak{w}$ on $[0,\varkappa]$ for which $\|\mathfrak{w}\|_{X_b^p}<\infty$, where the norm is defined by:

$$\|\mathfrak{w}\|_{X_b^p}=\left(\int_0^{\varkappa}|\delta^b\mathfrak{w}(\delta)|^p\frac{d\delta}{\delta}\right)^{\frac{1}{p}},\quad (1\le p<\infty, b\in\mathbb{R}).$$

Definition 2.1 (The Riemann-Liouville tempered fractional integral [12, 18, 24])**.** Suppose that the real function $\mathfrak{w}$ is piecewise continuous on $[0,\varkappa]$ and $\mathfrak{w}\in X_b^p(0,\varkappa)$, $\wp>0$. Then, the Riemann-Liouville tempered fractional integral of order ϱ is defined by

$$ {}_0\mathcal{I}_\delta^{\varrho,\wp}\mathfrak{w}(\delta)=e^{-\wp\delta}\ {}_0\mathcal{I}_\delta^{\varrho}\left(e^{\wp\delta}\mathfrak{w}(\delta)\right)=\frac{1}{\Gamma(\varrho)}\int_0^\delta\frac{e^{-\wp(\delta-s)}\mathfrak{w}(s)}{(\delta-s)^{1-\varrho}}ds, \tag{2.1}$$

where ${}_0\mathcal{I}_\delta^{\varrho}$ denotes the Riemann-Liouville fractional integral, defined by

$$ {}_0\mathcal{I}_\delta^{\varrho}\mathfrak{w}(\delta)=\frac{1}{\Gamma(\varrho)}\int_0^\delta\frac{\mathfrak{w}(s)}{(\delta-s)^{1-\varrho}}ds. \tag{2.2}$$

Obviously, the tempered fractional integral (2.1) reduces to the Riemann-Liouville fractional integral (2.2) if $\wp=0$.

Definition 2.2 (The Riemann-Liouville tempered fractional derivative [12, 18])**.** For $\jmath-1<\varrho<\jmath$; $\jmath\in\mathbb{N}^+,\wp\ge0$. The Riemann-Liouville tempered fractional derivative is defined by

$$ {}_0\mathfrak{D}_\delta^{\varrho,\wp}\mathfrak{w}(\delta)=e^{-\wp\delta}{}_0\mathfrak{D}_\delta^{\varrho}\left(e^{\wp\delta}\mathfrak{w}(\delta)\right)=\frac{e^{-\wp\delta}}{\Gamma(\jmath-\varrho)}\frac{d^\jmath}{d\delta^\jmath}\int_0^\delta\frac{e^{\wp s}\mathfrak{w}(s)}{(\delta-s)^{\varrho-\jmath+1}}ds,$$

where ${}_0\mathfrak{D}_\delta^{\varrho}\left(e^{\wp\delta}\mathfrak{w}(\delta)\right)$ denotes the Riemann-Liouville fractional derivative, given by

$$ {}_0\mathfrak{D}_\delta^{\varrho}\left(e^{\wp\delta}\mathfrak{w}(\delta)\right)=\frac{d^\jmath}{d\delta^\jmath}\left({}_0\mathcal{I}_\delta^{\jmath-\varrho}\left(e^{\wp\delta}\mathfrak{w}(\delta)\right)\right)=\frac{1}{\Gamma(\jmath-\varrho)}\frac{d^\jmath}{d\delta^\jmath}\int_0^\delta\frac{(e^{\wp s}\mathfrak{w}(s))}{(\delta-s)^{\varrho-\jmath+1}}ds.$$

Definition 2.3 (The Caputo tempered fractional derivative [12, 24])**.** For $\jmath-1<\varrho<\jmath$; $\jmath\in\mathbb{N}^+,\wp\ge0$. The Caputo tempered fractional derivative is defined as

$$ {}_0^C\mathfrak{D}^{\varrho,\wp}\mathfrak{w}(\delta)=e^{-\wp\delta}\ {}_0^C\mathfrak{D}^{\varrho}\left(e^{\wp\delta}\mathfrak{w}(\delta)\right)=\frac{e^{-\wp\delta}}{\Gamma(\jmath-\varrho)}\int_0^\delta\frac{1}{(\delta-s)^{\varrho-\jmath+1}}\frac{d^\jmath\left(e^{\wp s}\mathfrak{w}(s)\right)}{ds^\jmath}ds,$$

where ${}_0^C\mathfrak{D}_\delta^{\varrho,\wp}\left(e^{\wp\delta}\mathfrak{w}(\delta)\right)$ denotes the Caputo fractional derivative, given by

$$ {}_0^C\mathfrak{D}_\delta^{\varrho}\left(e^{\wp\delta}\mathfrak{w}(\delta)\right)=\frac{1}{\Gamma(\jmath-\varrho)}\int_0^\delta\frac{1}{(\delta-s)^{\varrho-\jmath+1}}\frac{d^\jmath\left(e^{\wp s}\mathfrak{w}(s)\right)}{ds^\jmath}ds.$$

Lemma 2.4 ([12])**.** *For a constant C,*

$$ {}_0\mathfrak{D}_\delta^{\varrho,\wp}C=Ce^{-\wp\delta}{}_0\mathfrak{D}_\delta^{\varrho}e^{\wp\delta},\quad {}_0^C\mathfrak{D}_\delta^{\varrho,\wp}C=Ce^{-\wp\delta}\ {}_0^C\mathfrak{D}_\delta^{\varrho}e^{\wp\delta}.$$

Obviously, ${}_0\mathfrak{D}_\delta^{\varrho,\wp}(C)\neq{}_0^C\mathfrak{D}_\delta^{\varrho,\wp}(C)$. And, ${}_0^C\mathfrak{D}_\delta^{\varrho,\wp}(C)$ is no longer equal to zero, being different from ${}_0^C\mathfrak{D}_\delta^{\varrho}(C)\equiv 0$.

Lemma 2.5 ([12, 24])**.** *Let $\mathfrak{w}(\delta)\in AC^\jmath[0,\varkappa]$ and $\jmath-1<\varrho<\jmath$. Then the Caputo tempered fractional derivative and the Riemann-Liouville tempered fractional integral have the composite properties*

$$ {}_0\mathcal{I}_\delta^{\varrho,\wp}\left[{}_0^C\mathfrak{D}_\delta^{\varrho,\wp}\mathfrak{w}(\delta)\right]=\mathfrak{w}(\delta)-\sum_{k=0}^{\jmath-1}e^{-\wp\delta}\frac{(\delta-0)^k}{k!}\left[\left.\frac{d^k\left(e^{\wp\delta}\mathfrak{w}(\delta)\right)}{d\delta^k}\right|_{\delta=0}\right],$$

and

$$ {}_0^C\mathfrak{D}_\delta^{\varrho,\wp}\left[{}_0\mathcal{I}_\delta^{\varrho,\wp}\mathfrak{w}(\delta)\right]=\mathfrak{w}(\delta),\ \textit{for}\ \varrho\in(0,1).$$

Theorem 2.6 ([13])**.** *Let $\mathfrak{w}, \upsilon \in AC^{\jmath}(\nabla, \mathfrak{R}), \jmath - 1 < \varrho \leq \jmath,\ (\jmath \in \mathbb{N}), \wp \in [0, +\infty)$ and $\Psi \in C^{\jmath}(\nabla, \mathfrak{R})$, be a non-decreasing function such that $\Psi' \neq 0$ on ∇. Then we have*

$$ {}_0^C\mathfrak{D}_{\Psi(\delta)}^{\varrho,\wp}\mathfrak{w}(\delta) = {}_0^C\mathfrak{D}_{\Psi(\delta)}^{\varrho,\wp}\upsilon(\delta) \Longleftrightarrow \mathfrak{w}(\delta) = \upsilon(\delta) + e^{-\wp\Psi(\delta)} \sum_{k=0}^{\jmath-1} c_k \left(\psi(\delta) - \psi(0)\right)^k,\ \delta \in \nabla, $$

where

$$ c_k = \frac{1}{k!}\left[\left(\frac{1}{\psi'(\delta)}\frac{d}{d\delta}\right)^k \left(e^{\wp\Psi(\delta)}[\mathfrak{w}(\delta) - \upsilon(\delta)]\right)\right]_{\delta=0}. $$

Remark 2.7. If we pose $\omega = \mathfrak{w} - \upsilon \in C^1(\nabla, \mathfrak{R}),\ \Psi(\delta) = \delta$ and $0 < \varrho \leq 1$. Then we have

$$ {}_0^C\mathfrak{D}_{\delta}^{\varrho,\wp}\omega(\delta) = 0 \Longleftrightarrow \omega(\delta) = e^{-\wp\delta}\omega(0),\ \delta \in \nabla. $$

Definition 2.8 ([8, 14])**.** We consider the normed spaces $\mathfrak{I}$ and $\widehat{\mathfrak{I}}$. A Fredholm operator of index zero is a linear operator $\mho : Dom(\mho) \subset \mathfrak{I} \to \widehat{\mathfrak{I}}$ such that

a) $\mathfrak{dim}\ker\mho = \mathfrak{codim}\mathfrak{Img}\mho < +\infty$.

b) $\mathfrak{Img}\mho$ is a closed subset of $\widehat{\mathfrak{I}}$.

By Definition 2.8, there exist continuous projectors $\widehat{\mho} : \widehat{\mathfrak{I}} \to \widehat{\mathfrak{I}}$ and $\overline{\mho} : \mathfrak{I} \to \mathfrak{I}$ satisfying

$$ \mathfrak{Img}\mho = \ker\widehat{\mho}, \qquad \ker\mho = \mathfrak{Img}\overline{\mho}, \qquad \widehat{\mathfrak{I}} = \mathfrak{Img}\widehat{\mho} \oplus \mathfrak{Img}\mho, \qquad \mathfrak{I} = \ker\overline{\mho} \oplus \ker\mho. $$

Thus, the restriction of $\mho$ to $Dom\mho \cap \ker\overline{\mho}$, denoted by $\mho_{\overline{\mho}}$, is an isomorphism onto its image.

Definition 2.9 ([8, 14])**.** Let $\mathfrak{Z} \subseteq \mathfrak{I}$ be a bounded subset and $\mho$ be a Fredholm operator of index zero with $Dom\mho \cap \mathfrak{Z} \neq \emptyset$. Then, the operator $\Bbbk : \overline{\mathfrak{Z}} \to \widehat{\mathfrak{I}}$ is called to be $\mho-$compact in $\overline{\mathfrak{Z}}$ if

a) the mapping $\widehat{\mho}\Bbbk : \overline{\mathfrak{Z}} \to \widehat{\mathfrak{I}}$ is continuous and $\widehat{\mho}\Bbbk\left(\overline{\mathfrak{Z}}\right) \subseteq \widehat{\mathfrak{I}}$ is bounded.

b) the mapping $\left(\mho_{\overline{\mho}}\right)^{-1}(id - \widehat{\mho})\Bbbk : \overline{\mathfrak{Z}} \to \mathfrak{I}$ is completely continuous.

Lemma 2.10. *[17] Let $\mathfrak{I}, \widehat{\mathfrak{I}}$ be a Banach spaces, $\mathfrak{Z} \subset \mathfrak{I}$ a bounded open set and symmetric with $0 \in \mathfrak{Z}$. Suppose that $\mho : Dom\mho \subset \mathfrak{I} \to \widehat{\mathfrak{I}}$ is a Fredholm operator of index zero with $Dom\mho \cap \overline{\mathfrak{Z}} \neq \emptyset$ and $\Bbbk : \mathfrak{I} \to \widehat{\mathfrak{I}}$ is a $\mho-$compact operator on $\overline{\mathfrak{Z}}$. Assume, moreover, that*

$$ \mho\mathfrak{w} - \Bbbk\mathfrak{w} \neq -\kappa(\mho\mathfrak{w} + \Bbbk(-\mathfrak{w})), $$

for any $\mathfrak{w} \in Dom\mho \cap \partial\mathfrak{Z}$ and any $\kappa \in (0, 1]$, where $\partial\mathfrak{Z}$ is the boundary of $\mathfrak{Z}$ with respect to $\mathfrak{I}$. If these conditions are verified, then there exist at least one solution of the equation $\mho\mathfrak{w} = \Bbbk\mathfrak{w}$ on $Dom\mho \cap \overline{\mathfrak{Z}}$.

3. Main Results

Let the spaces

$$ \mathfrak{I} = \Big\{\mathfrak{w} = (\mathfrak{w}_1, \mathfrak{w}_2) \in C(\nabla, \mathfrak{R}^2) : (\mathfrak{w}_1, \mathfrak{w}_2) = \left({}_0\mathcal{I}_{\delta}^{\varrho_1,\wp_1}\upsilon_1, {}_0\mathcal{I}_{\delta}^{\varrho_2,\wp_2}\upsilon_2\right) \text{ where } \upsilon = (\upsilon_1, \upsilon_2) \in C(\nabla, \mathfrak{R}^2)\Big\}, $$

and

$$ \widehat{\mathfrak{I}} = C(\nabla, \mathfrak{R}^2), $$

be endowed with the norms

$$\|\mathfrak{w}\|_{\Im} = \|\mathfrak{w}\|_{\widehat{\Im}} = \max_{1\leq j\leq 2}\left\{\sup_{\delta\in\nabla}|\mathfrak{w}_j(\delta)|\right\}.$$

We give now the definition of the operator $\mho : Dom\mho \subseteq \Im \to \widehat{\Im}$

$$\mho\mathfrak{w} = (\mho_1\mathfrak{w}_1, \mho_2\mathfrak{w}_2) := \left({}_0^C\mathfrak{D}_\delta^{\varrho_1,\wp_1}\mathfrak{w}_1, {}_0^C\mathfrak{D}_\delta^{\varrho_2,\wp_2}\mathfrak{w}_2\right), \tag{3.1}$$

where

$$Dom\mho = \{\mathfrak{w} \in \Im : \mho\mathfrak{w} \in \widehat{\Im} \text{ and } \mathfrak{w}_1(0) = \mathfrak{w}_1(\varkappa) = \mathfrak{w}_2(0) = \mathfrak{w}_2(\varkappa) = 0\}.$$

Lemma 3.1. *Using the definition of* $\mho$ *given in (3.1). Then*

$$\ker\mho = \{\mathfrak{w} \in \Im : \mathfrak{w}(\delta) = (\mathfrak{w}_1(\delta), \mathfrak{w}_2(\delta)) = (0,0),\ \delta \in \nabla\},$$

and

$$\mathfrak{Img}\mho = \left\{\upsilon = (\upsilon_1, \upsilon_2) \in \widehat{\Im} : \int_0^\varkappa (\varkappa - s)^{\varrho_i-1}e^{-\wp_i(\varkappa-s)}\upsilon_i(s)ds = 0, i \in \{1,2\}\right\}.$$

Proof. By Remark 2.7, we have for all $\mathfrak{w} \in Dom\mho \subset \Im$ the equation

$$\mho\mathfrak{w} = \left({}_0^C\mathfrak{D}_\delta^{\varrho_1,\wp_1}\mathfrak{w}_1, {}_0^C\mathfrak{D}_\delta^{\varrho_2,\wp_2}\mathfrak{w}_2\right) = (0,0)$$

has a solution of the form

$$\mathfrak{w}(\delta) = (\mathfrak{w}_1(\delta), \mathfrak{w}_2(\delta)) = \left(e^{-\wp_1\delta}\mathfrak{w}_1(0), e^{-\wp_2\delta}\mathfrak{w}_2(0)\right) = (0,0),\ \delta \in \nabla,$$

then

$$\ker\mho = \{\mathfrak{w} \in \Im : \mathfrak{w}(\delta) = (\mathfrak{w}_1(\delta), \mathfrak{w}_2(\delta)) = (0,0),\ \delta \in \nabla\}.$$

For $\upsilon = (\upsilon_1, \upsilon_2) \in \mathfrak{Img}\mho$, there exists $\mathfrak{w} = (\mathfrak{w}_1, \mathfrak{w}_2) \in Dom\mho$ such that $(\upsilon_1, \upsilon_2) = (\mho_1\mathfrak{w}_1, \mho_2\mathfrak{w}_2) \in \widehat{\Im}$. Using Lemma 2.5, we obtain for every $\delta \in \nabla$ and $i \in \{1,2\}$:

$$\begin{aligned}\mathfrak{w}_i(\delta) &= e^{-\wp_i\delta}\mathfrak{w}_i(0) + {}_0\mathcal{I}_\delta^{\varrho_i,\wp_i}\upsilon_i(\delta)\\ &= e^{-\wp_i\delta}\mathfrak{w}_i(0) + \frac{1}{\Gamma(\varrho_i)}\int_0^\delta (\delta - s)^{\varrho_i-1}e^{-\wp_i(\delta-s)}\upsilon_i(s)ds.\end{aligned}$$

Since $\mathfrak{w} \in Dom\mho$, then we have $\mathfrak{w}_i(0) = 0$. Thus

$$\int_0^\varkappa (\varkappa - s)^{\varrho_i-1}e^{-\wp_i(\varkappa-s)}\upsilon_i(s)ds = 0.$$

Furthermore, if $\upsilon = (\upsilon_1, \upsilon_2) \in \widehat{\Im}$, and satisfies

$$\int_0^\varkappa (\varkappa - s)^{\varrho_i-1}e^{-\wp_i(\varkappa-s)}\upsilon_i(s)ds = 0,\ i \in \{1,2\},$$

then for any $\mathfrak{w}(\delta) = (\mathfrak{w}_1(\delta), \mathfrak{w}_2(\delta)) = \left({}_0\mathcal{I}_\delta^{\varrho_1,\wp_1}\upsilon_1(\delta), {}_0\mathcal{I}_\delta^{\varrho_2,\wp_2}\upsilon_2(\delta)\right)$, using Lemma 2.5, we get

$$(\upsilon_1(\delta), \upsilon_2(\delta)) = \left({}_0^C\mathfrak{D}_\delta^{\varrho_1,\wp_1}\mathfrak{w}_1(\delta), {}_0^C\mathfrak{D}_\delta^{\varrho_2,\wp_2}\mathfrak{w}_2(\delta)\right) = (\mho_1\mathfrak{w}_1(\delta), \mho_2\mathfrak{w}_2(\delta)).$$

Therefore

$$\mathfrak{w}_i(\varkappa) = \mathfrak{w}_i(0) = 0,\ i \in \{1,2\},$$

which implies that $\mathfrak{w} \in Dom\mho$. So $\upsilon \in \mathfrak{Img}\mho$.
So

$$\mathfrak{Img}\mho = \left\{\upsilon = (\upsilon_1, \upsilon_2) \in \widehat{\Im} : \int_0^\varkappa (\varkappa - s)^{\varrho_i-1}e^{-\wp_i(\varkappa-s)}\upsilon_i(s)ds = 0, i \in \{1,2\}\right\}.$$

□

Lemma 3.2. *Let $\mho$ be defined by (3.1). Then $\mho$ is a Fredholm operator of index zero, and the linear continuous projector operators $\widehat{\mho} : \widehat{\Im} \to \widehat{\Im}$ and $\overline{\mho} : \Im \to \Im$ can be written as*

$$\widehat{\mho}(\upsilon) = \Big(\widehat{\mho}_1\upsilon_1, \widehat{\mho}_2\upsilon_2\Big),$$

such that

$$\widehat{\mho}_i\upsilon_i = \frac{1}{\varpi_i(\varkappa)}\int_0^{\varkappa}(\varkappa - s)^{\varrho_i - 1}e^{-\wp_i(\varkappa - s)}\upsilon_i(s)ds, i \in \{1, 2\},$$

where for $i \in \{1, 2\}$ we have

$$\varpi_i(\varkappa) = \int_0^{\varkappa}(\varkappa - s)^{\varrho_i - 1}e^{-\wp_i(\varkappa - s)}ds.$$

And

$$\overline{\mho}(\mathfrak{w}) = \Big(\overline{\mho}_1\mathfrak{w}_1, \overline{\mho}_2\mathfrak{w}_2\Big) = (0, 0).$$

Furthermore, the operator $\mho_{\overline{\mho}}^{-1} : \mathfrak{Img}\mho \to \Im \cap \ker\overline{\mho}$ can be written by

$$\begin{aligned} \mho_{\overline{\mho}}^{-1}(\upsilon)(\delta) &= \Big(\mho_{\overline{\mho}1}{}^{-1}\upsilon_1(\delta), \mho_{\overline{\mho}2}{}^{-1}\upsilon_2(\delta)\Big) \\ &= \Big({}_0\mathcal{I}_{\delta}^{\varrho_1,\wp_1}\upsilon_1(\delta), {}_0\mathcal{I}_{\delta}^{\varrho_2,\wp_2}\upsilon_2(\delta)\Big),\ \delta \in \nabla. \end{aligned}$$

Proof. Obviously, for each $\upsilon \in \widehat{\Im}$, $\widehat{\mho}^2\upsilon = \widehat{\mho}\upsilon$ and $\upsilon = \widehat{\mho}(\upsilon) + (\upsilon - \widehat{\mho}(\upsilon))$, where $(\upsilon - \widehat{\mho}(\upsilon)) \in \ker\widehat{\mho} = \mathfrak{Img}\mho$. Using the fact that $\mathfrak{Img}\mho = \ker\widehat{\mho}$ and $\widehat{\mho}^2 = \widehat{\mho}$ then $\mathfrak{Img}\mho \cap \mathfrak{Img}\widehat{\mho} = 0$. So,

$$\widehat{\Im} = \mathfrak{Img}\mho \oplus \mathfrak{Img}\widehat{\mho}.$$

By the same way we get that $\mathfrak{Img}\overline{\mho} = \ker\mho$ and $\overline{\mho}^2 = \overline{\mho}$. It follows for each $\mathfrak{w} \in \Im$, that $\mathfrak{w} = \big(\mathfrak{w} - \overline{\mho}(\mathfrak{w})\big) + \overline{\mho}(\mathfrak{w})$ then $\Im = \ker\overline{\mho} + \ker\mho$. Clearly we have $\ker\overline{\mho} \cap \ker\mho = 0$. So

$$\Im = \ker\overline{\mho} \oplus \ker\mho.$$

Using Rank−nullity theorem, we get :

$$\begin{aligned} \mathfrak{codimImg}\mathfrak{L} &= \dim\widehat{\Im} - \dim\mathfrak{Img}\mho \\ &= \Big[\dim\ker\widehat{\mho} + \dim\mathfrak{Img}\widehat{\mho}\Big] - \dim\mathfrak{Img}\mho, \end{aligned}$$

and since $\mathfrak{Img}\mho = \ker\widehat{\mho}$, then

$$\mathfrak{codimImg}\mho = \dim\mathfrak{Img}\widehat{\mho}. \tag{3.2}$$

Using also Rank−nullity theorem, we obtain

$$\dim\ker\mho = \dim\Im - \dim\mathfrak{Img}\mho = \mathfrak{codimImg}\mho,$$

which implies that

$$\dim\ker\mho = \mathfrak{codimImg}\mho. \tag{3.3}$$

By (3.2) and (3.3) we have:

$$\dim\ker\mho = \mathfrak{codimImg}\mho = \dim\mathfrak{Img}\widehat{\mho},$$

and since $\dim\mathfrak{Img}\widehat{\mho} < \infty$, then

$$\dim\ker\mho = \mathfrak{codimImg}\mho < \infty.$$

And since $\mathfrak{Img}\mho$ is a closed subset of $\widehat{\Im}$, then $\mho$ is a Fredholm operator of index zero.

Now, we will show that the inverse of $\mho|_{Dom\mho\cap\ker\overline{\mho}}$ is $\mho_{\overline{\mho}}^{-1}$. Effectively, for $\upsilon \in \mathfrak{Img}\mho$, by Lemma 2.5, we have

$$\mho\mho_{\overline{\mho}}^{-1}(\upsilon) = \left({}_0^C\mathfrak{D}_\delta^{\varrho_1,\wp_1}\left({}_0\mathcal{I}_\delta^{\varrho_1,\wp_1}\upsilon_1\right), {}_0^C\mathfrak{D}_\delta^{\varrho_2,\wp_2}\left({}_0\mathcal{I}_\delta^{\varrho_2,\wp_2}\upsilon_1\right)\right) = (\upsilon_1,\upsilon_1) = \upsilon. \tag{3.4}$$

Furthermore, for $\mathfrak{w} \in Dom\mho \cap \ker\overline{\mho}$ we get

$$\begin{aligned}\mho_{\overline{\mho}}^{-1}\mho\mathfrak{w}(\delta) &= \left({}_0\mathcal{I}_\delta^{\varrho_1,\wp_1}{}_0^C\mathfrak{D}_\delta^{\varrho_1,\wp_1}\mathfrak{w}_1(\delta), {}_0\mathcal{I}_\delta^{\varrho_2,\wp_2}{}_0^C\mathfrak{D}_\delta^{\varrho_2,\wp_2}\mathfrak{w}_2(\delta)\right)\\ &= \left(\mathfrak{w}_1(\delta) - e^{-\wp_1\delta}\mathfrak{w}_1(0), \mathfrak{w}_2(\delta) - e^{-\wp_2\delta}\mathfrak{w}_2(0)\right),\ \delta\in\nabla.\end{aligned}$$

Using the fact that $\mathfrak{w} \in Dom\mho \cap \ker\overline{\mho}$, then

$$(\mathfrak{w}_1(0), \mathfrak{w}_2(0)) = (0,0).$$

Thus,

$$\mho_{\overline{\mho}}^{-1}\mho(\mathfrak{w}) = \mathfrak{w}. \tag{3.5}$$

Using (3.4) and (3.5) together, we get $\mho_{\overline{\mho}}^{-1} = \left(\mho|_{Dom\mho\cap\ker\overline{\mho}}\right)^{-1}$. □

Let the following hypothesis:

(*H1*) Assume that $\aleph_1(\delta,0,0) \neq 0$, $\aleph_2(\delta,0,0)\neq 0$ for $\delta\in\nabla$, and for each $j\in\{1,2\}$, there exist positive constants γ_j,β_j with
$$|\aleph_j(\delta,\mathfrak{w},\upsilon) - \aleph_j(\delta,\bar{\mathfrak{w}},\bar{\upsilon})| \leqslant \gamma_j|\mathfrak{w}-\bar{\mathfrak{w}}| + \beta_j|\bar{\upsilon}-\upsilon|,$$
for every $\delta\in\nabla$ and $\mathfrak{w},\bar{\mathfrak{w}},\upsilon,\bar{\upsilon}\in\mathfrak{R}$.

Define $\mathbb{k}:\Im\to\widehat{\Im}$ by

$$\mathbb{k}\mathfrak{w}(\delta) = (\mathbb{k}_1\mathfrak{w}(\delta), \mathbb{k}_2\mathfrak{w}(\delta)) := \Big(\aleph_1(\delta,\mathfrak{w}_1(\delta),\mathfrak{w}_2(\delta)), \aleph_2(\delta,\mathfrak{w}_1(\delta),\mathfrak{w}_2(\delta))\Big),\ \delta\in\nabla.$$

Then the problem (1.1)-(1.2) is equivalent to the problem $\mho\mathfrak{w} = \mathbb{k}\mathfrak{w}$.

Lemma 3.3. *Suppose that (H1) is satisfied then, for any bounded open set $\mathfrak{Z}\subset\Im$, the operator $\mathbb{k}$ is $\mho-$compact.*

Proof. Let $\mathfrak{K}>0$ and the bounded open set $\mathfrak{Z} = \{\mathfrak{w}\in\Im : \|\mathfrak{w}\|_\Im < \mathfrak{K}\}$.

Step 1: $\widehat{\mho}\mathbb{k}$ *is continuous.*
Let $(\upsilon_\jmath)_{\jmath\in\mathbb{N}}$ be a sequence such that $\upsilon_\jmath \longrightarrow \upsilon$ in $\widehat{\Im}$, thus for $j\in\{1,2\}$, and $\delta\in\nabla$, we have

$$\begin{aligned}&|\widehat{\mho}_j\mathbb{k}_j(\upsilon_\jmath)(\delta) - \widehat{\mho}_j\mathbb{k}_j(\upsilon)(\delta)|\\ \leqslant\ & \frac{1}{\varpi_j(\varkappa)}\int_0^\varkappa(\varkappa-s)^{\varrho_j-1}e^{-\wp_j(\varkappa-s)}\left|\mathbb{k}_j(\upsilon_\jmath)(s) - \mathbb{k}_j(\upsilon)(s)\right|ds.\end{aligned}$$

By (*H1*), we have

$$\begin{aligned}&|\widehat{\mho}_1\mathbb{k}_1(\upsilon_\jmath)(\delta) - \widehat{\mho}_1\mathbb{k}_1(\upsilon)(\delta)|\\ \leqslant\ & \frac{\gamma_1}{\varpi_1(\varkappa)}\int_0^\varkappa(\varkappa-s)^{\varrho_1-1}e^{-\wp_2(\varkappa-s)}\left|\upsilon_{\jmath 1}(s) - \upsilon_1(s)\right|ds\\ & + \frac{\beta_1}{\varpi_1(\varkappa)}\int_0^\varkappa(\varkappa-s)^{\varrho_1-1}e^{-\wp_2(\varkappa-s)}\left|\upsilon_{\jmath 2}(s) - \upsilon_2(s)\right|ds\\ \leqslant\ & (\gamma_1+\beta_1)\|\upsilon_\jmath - \upsilon\|_{\widehat{\Im}}\\ \leqslant\ & (\gamma^*+\beta^*)\|\upsilon_\jmath - \upsilon\|_{\widehat{\Im}},\end{aligned}$$

and

$$\begin{aligned}
&|\widehat{\mho}_2\mathbb{k}_2(\upsilon_\jmath)(\delta)-\widehat{\mho}_2\mathbb{k}_2(\upsilon)(\delta)| \\
\leqslant\ & \frac{\gamma_2}{\varpi_2(\varkappa)}\int_0^{\varkappa}(\varkappa-s)^{\varrho_2-1}e^{-\wp_j(\varkappa-s)}\,|\upsilon_{\jmath 1}(s)-\upsilon_1(s)|\,ds \\
& +\frac{\beta_2}{\varpi_2(\varkappa)}\int_0^{\varkappa}(\varkappa-s)^{\varrho_2-1}e^{-\wp_j(\varkappa-s)}\,|\upsilon_{\jmath 2}(s)-\upsilon_2(s)|\,ds \\
\leqslant\ & (\gamma_2+\beta_2)\|\upsilon_\jmath-\upsilon\|_{\widehat{\Im}} \\
\leqslant\ & (\gamma^*+\beta^*)\|\upsilon_\jmath-\upsilon\|_{\widehat{\Im}},
\end{aligned}$$

where $\gamma^*=\max\limits_{1\leq j\leq 2}\gamma_j$ and $\beta^*=\max\limits_{1\leq j\leq 2}\beta_j$. Thus for each $j\in\{1,2\}$, we get

$$\sup_{\delta\in\nabla}|\widehat{\mho}_j\mathbb{k}_j(\upsilon_\jmath)(\delta)-\widehat{\mho}_j\mathbb{k}_j(\upsilon)(\delta)|\leqslant(\gamma^*+\beta^*)\|\upsilon_\jmath-\upsilon\|_{\widehat{\Im}},$$

and hence

$$\|\widehat{\mho}\mathbb{k}(\upsilon_\jmath)-\widehat{\mho}\mathbb{k}(\upsilon)\|_{\widehat{\Im}}\longrightarrow 0 \text{ as } \jmath\longrightarrow+\infty.$$

We deduce that $\widehat{\mho}\mathbb{k}$ is continuous.

Step 2: $\widehat{\mho}\mathbb{k}(\overline{\mathfrak{Z}})$ *is bounded*
For $\delta\in\nabla$ and $\upsilon\in\overline{\mathfrak{Z}}$, we have

$$\begin{aligned}
|\widehat{\mho}_1\mathbb{k}_1(\upsilon)(\delta)| &\leqslant \frac{1}{\varpi_1(\varkappa)}\int_0^{\varkappa}(\varkappa-s)^{\varrho_1-1}e^{-\wp_1(\varkappa-s)}|\mathbb{k}_1(\upsilon)(s)|ds \\
&\leqslant \frac{1}{\varpi_1(\varkappa)}\int_0^{\varkappa}(\varkappa-s)^{\varrho_1-1}e^{-\wp_1(\varkappa-s)}|\aleph_1(s,\upsilon_1(s),\upsilon_2(s))-\aleph_1(s,0,0)|ds \\
&\quad+\frac{1}{\varpi_1(\varkappa)}\int_0^{\varkappa}(\varkappa-s)^{\varrho_1-1}e^{-\wp_1(\varkappa-s)}|\aleph_1(s,0,0)|ds \\
&\leqslant \aleph_1^*+\frac{\gamma_1}{\varpi_1(\varkappa)}\int_0^{\varkappa}(\varkappa-s)^{\varrho_1-1}e^{-\wp_1(\varkappa-s)}|\upsilon_1(s)|ds \\
&\quad+\frac{\beta_1}{\varpi_1(\varkappa)}\int_0^{\varkappa}(\varkappa-s)^{\varrho_1-1}e^{-\wp_1(\varkappa-s)}|\upsilon_2(s)|ds \\
&\leqslant \aleph_1^*+(\gamma_1+\beta_1)\mathfrak{K} \\
&\leqslant \aleph^{**}+(\gamma^*+\beta^*)\mathfrak{K},
\end{aligned}$$

and

$$\begin{aligned}
|\widehat{\mho}_2\mathbb{k}_2(\upsilon)(\delta)| &\leqslant \frac{1}{\varpi_2(\varkappa)}\int_0^{\varkappa}(\varkappa-s)^{\varrho_2-1}e^{-\wp_2(\varkappa-s)}|\mathbb{k}_2(\upsilon)(s)|ds \\
&\leqslant \frac{1}{\varpi_2(\varkappa)}\int_0^{\varkappa}(\varkappa-s)^{\varrho_2-1}e^{-\wp_2(\varkappa-s)}|\aleph_2(s,\upsilon_1(s),\upsilon_2(s))-\aleph_2(s,0,0)|ds \\
&\quad+\frac{1}{\varpi_2(\varkappa)}\int_0^{\varkappa}(\varkappa-s)^{\varrho_2-1}e^{-\wp_2(\varkappa-s)}|\aleph_2(s,0,0)|ds \\
&\leqslant \aleph_2^*+\frac{\gamma_2}{\varpi_2(\varkappa)}\int_0^{\varkappa}(\varkappa-s)^{\varrho_2-1}e^{-\wp_2(\varkappa-s)}|\upsilon_1(s)|ds \\
&\quad+\frac{\beta_2}{\varpi_2(\varkappa)}\int_0^{\varkappa}(\varkappa-s)^{\varrho_2-1}e^{-\wp_2(\varkappa-s)}|\upsilon_2(s)|ds \\
&\leqslant \aleph_2^*+(\gamma_2+\beta_2)\mathfrak{K} \\
&\leqslant \aleph^{**}+(\gamma^*+\beta^*)\mathfrak{K},
\end{aligned}$$

where $\aleph_j^* = \sup_{\delta\in\nabla} |\aleph_j(\delta,0,0)|, j \in \{1,2\}$ and $\aleph^{**} = \max_{1\le j\le 2} \aleph_j^*$.
Thus for each $j \in \{1,2\}$

$$\sup_{\delta\in\nabla} |\widehat{\mho}_j \Bbbk_j(\upsilon)(\delta)| \leqslant \aleph^{**} + (\gamma^* + \beta^*)\mathfrak{K},$$

which implies that

$$\|\widehat{\mho}\Bbbk(\upsilon)\|_{\widehat{\Im}} \leqslant \aleph^{**} + (\gamma^* + \beta^*)\mathfrak{K}.$$

So, $\widehat{\mho}\Bbbk(\overline{\mathfrak{Z}})$ is a bounded set in $\widehat{\Im}$.

Step 3: $\mho_{\overline{\mho}}^{-1}(id - \widehat{\mho})\Bbbk : \overline{\mathfrak{Z}} \to \Im$ *is completely continuous.*
By the Arzelà-Ascoli theorem, we show that $\mho_{\overline{\mho}}^{-1}(id - \widehat{\mho})\Bbbk(\overline{\mathfrak{Z}}) \subset \Im$ is equicontinuous and bounded. Firstly, for any $\mathfrak{w} \in \overline{\mathfrak{Z}}$ and $\delta \in \nabla$, we get

$$\mho_{\overline{\mho}}^{-1}(\Bbbk\mathfrak{w}(\delta) - \widehat{\mho}\Bbbk\mathfrak{w}(\delta)) = \left(\mho_{\overline{\mho}1}{}^{-1}\Big(\Bbbk_1\mathfrak{w}(\delta) - \widehat{\mho}_1\Bbbk_1\mathfrak{w}(\delta)\Big), \mho_{\overline{\mho}2}{}^{-1}\Big(\Bbbk_2\mathfrak{w}(\delta) - \widehat{\mho}_2\Bbbk_2\mathfrak{w}(\delta)\Big)\right),$$

where for each $j \in \{1,2\}$

$$\begin{aligned}
&\mho_{\overline{\mho}j}{}^{-1}\Big(\Bbbk_j\mathfrak{w}(\delta) - \widehat{\mho}_j\Bbbk_j\mathfrak{w}(\delta)\Big)\\
&= {}_0\mathcal{I}_\delta^{\varrho_j,\wp_j}\left[\aleph_j(\delta,\mathfrak{w}_1(\delta),\mathfrak{w}_2(\delta)) - \frac{1}{\varpi_j(\varkappa)}\int_0^\varkappa (\varkappa - s)^{\varrho_j-1}e^{-\wp_j(\varkappa-s)}\aleph_j(s,\mathfrak{w}_1(s),\mathfrak{w}_2(s))ds\right]\\
&= \frac{1}{\Gamma(\varrho_j)}\int_0^\delta (\delta - s)^{\varrho_j-1}e^{-\wp_j(\delta-s)}\aleph_j(s,\mathfrak{w}_1(s),\mathfrak{w}_2(s))ds\\
&\quad - \frac{\varpi(\delta)}{\Gamma(\varrho_j)\varpi_j(\varkappa)}\int_0^\varkappa (\varkappa - s)^{\varrho_j-1}e^{-\wp_j(\varkappa-s)}\aleph_j(s,\mathfrak{w}_1(s),\mathfrak{w}_2(s))ds.
\end{aligned}$$

For all $\mathfrak{w} \in \overline{\mathfrak{Z}}$, $\delta \in \nabla$, and $j \in \{1,2\}$, we get

$$\begin{aligned}
&|\mho_{\overline{\mho}j}{}^{-1}(id - \widehat{\mho}_j)\Bbbk_j\mathfrak{w}(\delta)|\\
\leqslant\;& \frac{1}{\Gamma(\varrho_j)}\int_0^\delta (\delta - s)^{\varrho_j-1}e^{-\wp_j(\delta-s)}\Big|\aleph_j(s,\mathfrak{w}_1(s),\mathfrak{w}_2(s)) - \aleph_j(s,0,0)\Big|ds\\
&+\frac{1}{\Gamma(\varrho_j)}\int_0^\delta (\delta - s)^{\varrho_j-1}e^{-\wp_j(\delta-s)}|\aleph_j(s,0,0)|ds\\
&+\frac{\varpi_j(\delta)}{\Gamma(\varrho_j)\varpi_j(\varkappa)}\int_0^\varkappa (\varkappa - s)^{\varrho_j-1}e^{-\wp_j(\varkappa-s)}\Big|\aleph_j(s,\mathfrak{w}_1(s),\mathfrak{w}_2(s)) - \aleph_j(s,0,0)\Big|ds\\
&+\frac{\varpi_j(\delta)}{\Gamma(\varrho_j)\varpi_j(\varkappa)}\int_0^\varkappa (\varkappa - s)^{\varrho_j-1}e^{-\wp_j(\varkappa-s)}|\aleph_j(s,0,0)|ds,\\
\leqslant\;& \frac{2\aleph_j^*\varpi_j(\delta)}{\Gamma(\varrho_j)} + \frac{\gamma_j}{\Gamma(\varrho_j)}\int_0^\delta (\delta - s)^{\varrho_j-1}e^{-\wp_j(\delta-s)}|\mathfrak{w}_1(s)|ds\\
&+\frac{\beta_j}{\Gamma(\varrho_j)}\int_0^\delta (\delta - s)^{\varrho_j-1}e^{-\wp_j(\delta-s)}|\mathfrak{w}_2(s)|ds\\
&+\frac{\gamma_j\varpi_j(\delta)}{\Gamma(\varrho_j)\varpi_j(\varkappa)}\int_0^\varkappa (\varkappa - s)^{\varrho_j-1}e^{-\wp_j(\varkappa-s)}|\mathfrak{w}_1(s)|ds\\
&+\frac{\beta_j\varpi_j(\delta)}{\Gamma(\varrho_j)\varpi_j(\varkappa)}\int_0^\varkappa (\varkappa - s)^{\varrho_j-1}e^{-\wp_j(\varkappa-s)}|\mathfrak{w}_2(s)|ds\\
\leqslant\;& \frac{2\varkappa^{\varrho_j}}{\Gamma(\varrho_j+1)}\Big[\aleph_j^* + (\gamma_j + \beta_j)\mathfrak{K}\Big]
\end{aligned}$$

$$\leqslant \frac{2\varkappa^*}{\Gamma^*}\Big[\aleph^{**}+(\gamma^*+\beta^*)\mathfrak{K}\Big],$$

where $\varkappa^* = \max\limits_{1\leq j\leq 2}\varkappa^{\varrho_j}$ and $\Gamma^* = \max\limits_{1\leq j\leq 2}\Gamma(\varrho_j+1)$.
Which implies that, for each $j \in \{1,2\}$, we get

$$\sup_{\delta\in\nabla}\Big|\mho_{\overline{\mho}j}{}^{-1}(id-\widehat{\mho}_j)\mathbb{k}_j\mathfrak{w}(\delta)\Big| \leqslant \frac{2\varkappa^*}{\Gamma^*}\Big[\aleph^{**}+(\gamma^*+\beta^*)\mathfrak{K}\Big].$$

Therefore

$$\|\mho_{\overline{\mho}}^{-1}(id-\widehat{\mho})\mathbb{k}\mathfrak{w}\|_{\Im} \leqslant \frac{2\varkappa^*}{\Gamma^*}\Big[\aleph^{**}+(\gamma^*+\beta^*)\mathfrak{K}\Big].$$

This means that $\mho_{\overline{\mho}}^{-1}(id-\widehat{\mho})\mathbb{k}(\overline{\mathfrak{Z}})$ is uniformly bounded in $\Im$.
It remains to show that $\mho_{\overline{\mho}}^{-1}(id-\widehat{\mho})\mathbb{k}(\overline{\mathfrak{Z}})$ is equicontinuous.
For $0<\delta_1<\delta_2\leqslant\varkappa, \mathfrak{w}\in\overline{\mathfrak{Z}}$, and $j\in\{1,2\}$, we have

$$\begin{aligned}
&|\mho_{\overline{\mho}j}{}^{-1}(id-\widehat{\mho}_j)\mathbb{k}_j\mathfrak{w}(\delta_2)-\mho_{\overline{\mho}j}{}^{-1}(id-\widehat{\mho}_j)\mathbb{k}_j\mathfrak{w}(\delta_1)|\\
\leqslant\ & \frac{1}{\Gamma(\varrho_j)}\int_0^{\delta_1}\Big|(\delta_2-s)^{\varrho_j-1}e^{-\wp_j(\delta_2-s)}-(\delta_1-s)^{\varrho_j-1}e^{-\wp_j(\delta_1-s)}\Big|\Big|\aleph_j(s,\mathfrak{w}_1(s),\mathfrak{w}_2(s))\Big|ds\\
&+\frac{1}{\Gamma(\varrho_j)}\int_{\delta_1}^{\delta_2}(\delta_2-s)^{\varrho_j-1}e^{-\wp_j(\delta_2-s)}\Big|\aleph_j(s,\mathfrak{w}_1(s),\mathfrak{w}_2(s))\Big|ds\\
&+\frac{|\varpi_j(\delta_2)-\varpi_j(\delta_1)|}{\Gamma(\varrho_j)\varpi_j(\varkappa)}\int_0^{\varkappa}(\varkappa-s)^{\varrho_j-1}e^{-\wp_j(\varkappa-s)}\Big|\aleph_j(s,\mathfrak{w}_1(s),\mathfrak{w}_2(s))\Big|ds\\
\leqslant\ & \frac{1}{\Gamma(\varrho_j)}\int_0^{\delta_1}\Big|(\delta_2-s)^{\varrho_j-1}e^{-\wp_j(\delta_2-s)}-(\delta_1-s)^{\varrho_j-1}e^{-\wp_j(\delta_1-s)}\Big|\\
&\times\Big|\aleph_j(s,\mathfrak{w}_1(s),\mathfrak{w}_2(s))-\aleph_j(s,0,0)\Big|ds\\
&+\frac{1}{\Gamma(\varrho_j)}\int_0^{\delta_1}\Big|(\delta_2-s)^{\varrho_j-1}e^{-\wp_j(\delta_2-s)}-(\delta_1-s)^{\varrho_j-1}e^{-\wp_j(\delta_1-s)}\Big||\aleph_j(s,0,0)|ds\\
&+\frac{1}{\Gamma(\varrho_j)}\int_{\delta_1}^{\delta_2}(\delta_2-s)^{\varrho_j-1}e^{-\wp_j(\delta_2-s)}\Big|\aleph_j(s,\mathfrak{w}_1(s),\mathfrak{w}_2(s))-\aleph_j(s,0,0)\Big|ds\\
&+\frac{1}{\Gamma(\varrho_j)}\int_{\delta_1}^{\delta_2}(\delta_2-s)^{\varrho_j-1}e^{-\wp_j(\delta_2-s)}\Big|\aleph_j(s,0,0)\Big|ds\\
&+\frac{|\varpi_j(\delta_2)-\varpi_j(\delta_1)|}{\Gamma(\varrho_j)\varpi_j(\varkappa)}\int_0^{\varkappa}(\varkappa-s)^{\varrho_j-1}e^{-\wp_j(\varkappa-s)}\Big|\aleph_j(s,\mathfrak{w}_1(s),\mathfrak{w}_2(s))-\aleph_j(s,0,0)\Big|ds\\
&+\frac{|\varpi_j(\delta_2)-\varpi_j(\delta_1)|}{\Gamma(\varrho_j)\varpi_j(\varkappa)}\int_0^{\varkappa}(\varkappa-s)^{\varrho_j-1}e^{-\wp_j(\varkappa-s)}\Big|\aleph_j(s,0,0)\Big|ds\\
\leqslant\ & \frac{(\gamma_j+\beta_j)\mathfrak{K}+\aleph_j^*}{\Gamma(\varrho_j)}\int_0^{\delta_1}\Big|(\delta_2-s)^{\varrho_j-1}e^{-\wp_j(\delta_2-s)}-(\delta_1-s)^{\varrho_j-1}e^{-\wp_j(\delta_1-s)}\Big|ds\\
&+\frac{2\gamma\mathfrak{K}+\aleph_j^*}{\Gamma(\varrho_j)}\int_{\delta_1}^{\delta_2}(\delta_2-s)^{\varrho_j-1}e^{-\wp_j(\delta_2-s)}ds+\frac{(\gamma_j+\beta_j)\mathfrak{K}+\aleph_j^*}{\Gamma(\varrho_j)}|\varpi_j(\delta_2)-\varpi_j(\delta_1)|.
\end{aligned}$$

The operator $\mho_{\overline{\mho}}^{-1}(id-\widehat{\mho})\mathbb{k}(\overline{\mathfrak{Z}})$ is equicontinuous in $\Im$ because the right-hand side of the above inequality tends to zero as $\delta_1\to\delta_2$ and the limit is independent of $\mathfrak{w}$. The Arzelà-Ascoli theorem implies that $\mho_{\overline{\mho}}^{-1}(id-\widehat{\mho})\mathbb{k}(\overline{\mathfrak{Z}})$ is relatively compact in $\Im$. As a consequence of steps 1 to 3, we get that $\mathbb{k}$ is $\mho-$compact in $\overline{\mathfrak{Z}}$. □

Lemma 3.4. *Assume (H1). If the condition*

$$\frac{(\gamma^* + \beta^*)\varkappa^*}{\Gamma^*} < \frac{1}{2}, \tag{3.6}$$

is satisfied, then there exists $\mathfrak{W} > 0$, which is independent of κ, such that,

$$\mho(\mathfrak{w}) - \Bbbk(\mathfrak{w}) = -\kappa[\mho(\mathfrak{w}) + \Bbbk(-\mathfrak{w})] \Longrightarrow \|\mathfrak{w}\|_{\Im} \leqslant \mathfrak{W},\ \kappa \in (0, 1].$$

Proof. Let $\mathfrak{w} \in \Im$ satisfies

$$\mho(\mathfrak{w}) - \Bbbk(\mathfrak{w}) = -\kappa\mho(\mathfrak{w}) - \kappa\Bbbk(-\mathfrak{w}),$$

then

$$\mho(\mathfrak{w}) = \frac{1}{1+\kappa}\Bbbk(\mathfrak{w}) - \frac{\kappa}{1+\kappa}\Bbbk(-\mathfrak{w}).$$

So, we obtain for any $j \in \{1, 2\}$ and $\delta \in \nabla$:

$$\mho_j\mathfrak{w}_j(\delta) = {}_0^C\mathfrak{D}_\delta^{\varrho_j,\wp_j}\mathfrak{w}_j(\delta) = \frac{1}{1+\kappa}\aleph_j(\delta, \mathfrak{w}_1(\delta), \mathfrak{w}_2(\delta)) - \frac{\kappa}{1+\kappa}\aleph_j(\delta, -\mathfrak{w}_1(\delta), -\mathfrak{w}_2(\delta)).$$

By Lemma 2.5 we get

$$\begin{aligned}\mathfrak{w}_j(\delta) &= e^{-\wp_j\delta}\mathfrak{w}_j(0) \\ &\quad + \frac{1}{\kappa+1}\left[{}_0\mathcal{I}_\delta^{\varrho_j,\wp_j}\left(\aleph_j(s, \mathfrak{w}_1(s), \mathfrak{w}_2(s))\right)(\delta) - \kappa{}_0\mathcal{I}_\delta^{\varrho_j,\wp_j}\left(\aleph_j(s, -\mathfrak{w}_1(s), -\mathfrak{w}_2(s))\right)(\delta)\right].\end{aligned}$$

Thus for every $j \in \{1, 2\}$ and $\delta \in \nabla$ we obtain

$$\begin{aligned}&|\mathfrak{w}_j(\delta)| \\ &\leqslant |\mathfrak{w}_j(0)| + \frac{1}{(\kappa+1)\Gamma(\varrho_j)}\int_0^\delta (\delta - s)^{\varrho_j-1}e^{-\wp_j(\delta-s)}\left|\aleph_j(s, \mathfrak{w}_1(s), \mathfrak{w}_2(s))\right| ds \\ &\quad + \frac{\kappa}{(\kappa+1)\Gamma(\varrho_j)}\int_0^\delta (\delta - s)^{\varrho_j-1}e^{-\wp_j(\delta-s)}\left|\aleph_j\left(s, -\mathfrak{w}_1(s), -\mathfrak{w}_2(s)\right)\right| ds \\ &\leqslant |\mathfrak{w}_j(0)| \\ &\quad + \frac{1}{(\kappa+1)\Gamma(\varrho_j)}\int_0^\delta (\delta - s)^{\varrho_j-1}e^{-\wp_j(\delta-s)}\left|\aleph_j(s, \mathfrak{w}_1(s), \mathfrak{w}_2(s)) - \aleph_j(s, 0, 0)\right| ds \\ &\quad + \frac{1}{(\kappa+1)\Gamma(\varrho_j)}\int_0^\delta (\delta - s)^{\varrho_j-1}e^{-\wp_j(\delta-s)}\left|\aleph_j(s, 0, 0)\right| ds \\ &\quad + \frac{\kappa}{(\kappa+1)\Gamma(\varrho_j)}\int_0^\delta (\delta - s)^{\varrho_j-1}e^{-\wp_j(\delta-s)}\left|\aleph_j\left(s, -\mathfrak{w}_1(s), -\mathfrak{w}_2(s)\right) - \aleph_j(s, 0, 0)\right| ds \\ &\quad + \frac{\kappa}{(\kappa+1)\Gamma(\varrho_j)}\int_0^\delta (\delta - s)^{\varrho_j-1}e^{-\wp_j(\delta-s)}\left|\aleph_j(s, 0, 0)\right| ds \\ &\leqslant |\mathfrak{w}_j(0)| + \frac{2\aleph_j^*\varkappa^{\varrho_j}}{\Gamma(\varrho_j+1)} + \frac{2(\gamma_j + \beta_j)\varkappa^{\varrho_j}}{\Gamma(\varrho_j+1)}\|\mathfrak{w}\|_{\Im} \\ &\leqslant \frac{2\aleph^{**}\varkappa^*}{\Gamma^*} + \frac{2(\gamma^* + \beta^*)\varkappa^*}{\Gamma^*}\|\mathfrak{w}\|_{\Im},\end{aligned}$$

thus for every $j \in \{1, 2\}$, we have

$$\sup_{\delta\in\nabla}|\mathfrak{w}_j(\delta)| \leqslant \frac{2\aleph^{**}\varkappa^*}{\Gamma^*} + \frac{2(\gamma^* + \beta^*)\varkappa^*}{\Gamma^*}\|\mathfrak{w}\|_{\Im}.$$

We deduce that

$$\|\mathfrak{w}\|_{\Im} \leqslant \frac{\dfrac{2f^{**}\varkappa^*}{\Gamma^*}}{\left[1-\dfrac{2(\gamma^*+\beta^*)\varkappa^*}{\Gamma^*}\right]} := \mathfrak{W}.$$

□

Lemma 3.5. *If conditions (H1) and (3.6) are verified, then there exist a bounded open set $\mathfrak{Z} \subset \Im$ with*

$$\mho(\mathfrak{w}) - \Bbbk(\mathfrak{w}) \neq -\kappa[\mho(\mathfrak{w}) + \Bbbk(-\mathfrak{w})], \tag{3.7}$$

for any $\mathfrak{w} \in \partial\mathfrak{Z}$ and any $\kappa \in (0,1]$.

Proof. Using Lemma 3.4, then there exists a positive constant $\mathfrak{W}$ which is independent of κ such that, if $\mathfrak{w}$ verify

$$\mho(\mathfrak{w}) - \Bbbk(\mathfrak{w}) = -\kappa[\mho(\mathfrak{w}) + \Bbbk(-\mathfrak{w})],\ \kappa \in (0,1],$$

thus $\|\mathfrak{w}\|_{\Im} \leqslant \mathfrak{W}$. So, if

$$\mathfrak{Z} = \{\mathfrak{w} \in \Im; \|\mathfrak{w}\|_{\Im} < \vartheta\}, \tag{3.8}$$

such that $\vartheta > \mathfrak{W}$, we deduce that

$$\mho(\mathfrak{w}) - \Bbbk(\mathfrak{w}) \neq -\kappa[\mho(\mathfrak{w}) - \Bbbk(-\mathfrak{w})],$$

for all $\mathfrak{w} \in \partial\mathfrak{Z} = \{\mathfrak{w} \in \Im; \|\mathfrak{w}\|_{\Im} = \vartheta\}$ and $\kappa \in (0,1]$. □

Theorem 3.6. *Assume (H1) and (3.6), then the problem* (1.1)-(1.2) *has a unique solution in* $Dom\mho \cap \overline{\mathfrak{Z}}$.

Proof. It is clear that the set $\mathfrak{Z}$ defined in (3.8) is symmetric, $0 \in \mathfrak{Z}$ and $\Im \cap \overline{\mathfrak{Z}} = \overline{\mathfrak{Z}} \neq \emptyset$. In addition, By Lemma 3.5, assume (H1) and (3.6), then

$$\mho(\mathfrak{w}) - \Bbbk(\mathfrak{w}) \neq -\kappa[\mho(\mathfrak{w}) - \Bbbk(-\mathfrak{w})],$$

for each $\mathfrak{w} \in \Im \cap \partial\mathfrak{Z} = \partial\mathfrak{Z}$ and each $\kappa \in (0,1]$. By Lemma 2.10, problem (1.1)-(1.2) has at least one solution in $Dom\mho \cap \overline{\mathfrak{Z}}$.

Now, we prove the uniqueness result. Suppose that the problem (1.1)-(1.2) has two different solutions $\mathfrak{w}, \overline{\mathfrak{w}} \in Dom\mho \cap \overline{\mathfrak{Z}}$. Then, we have for each $\delta \in \nabla$ and $j \in \{1,2\}$

$${}_0^C\mathfrak{D}_\delta^{\varrho_j,\wp_j}\mathfrak{w}_j(\delta) = \aleph_j\left(\delta, \mathfrak{w}_1(\delta), \mathfrak{w}_2(\delta)\right),$$

$${}_0^C\mathfrak{D}_\delta^{\varrho_j,\wp_j}\overline{\mathfrak{w}}_j(\delta) = \aleph_j\left(\delta, \overline{\mathfrak{w}}_1(\delta), \overline{\mathfrak{w}}_2(\delta)\right),$$

and

$$\mathfrak{w}_j(0) = \mathfrak{w}_j(\varkappa) = 0,\ \overline{\mathfrak{w}}_j(0) = \overline{\mathfrak{w}}_j(\varkappa) = 0.$$

Let $\mathfrak{U}(\delta) = \mathfrak{w}(\delta) - \overline{\mathfrak{w}}(\delta)$, for all $\delta \in \nabla$, which means that

$$\mathfrak{U}(\delta) = (\mathfrak{U}_1(\delta), \mathfrak{U}_2(\delta)) = \left(\mathfrak{w}_1(\delta) - \overline{\mathfrak{w}}_1(\delta), \mathfrak{w}_2(\delta) - \overline{\mathfrak{w}}_2(\delta)\right), \text{for all } \delta \in \nabla.$$

Then

$$\begin{aligned}
\mho\mathfrak{U}(\delta) &= \Big(\mho_1\mathfrak{U}_1(\delta), \mho_2\mathfrak{U}_2(\delta)\Big) \\
&= \Big({}_0^C\mathfrak{D}_\delta^{\varrho_1,\wp_1}\mathfrak{U}_1(\delta), {}_0^C\mathfrak{D}_\delta^{\varrho_2,\wp_2}\mathfrak{U}_2(\delta)\Big) \\
&= \Big({}_0^C\mathfrak{D}_\delta^{\varrho_1,\wp_1}\mathfrak{w}_1(\delta) - {}_0^C\mathfrak{D}_\delta^{\varrho_1,\wp_1}\overline{\mathfrak{w}}_1(\delta), {}_0^C\mathfrak{D}_\delta^{\varrho_2,\wp_2}\mathfrak{w}_2(\delta) - {}_0^C\mathfrak{D}_\delta^{\varrho_2,\wp_2}\overline{\mathfrak{w}}_2(\delta)\Big)
\end{aligned}$$

$$= \Big(\aleph_1(\delta, \mathfrak{w}_1(\delta), \mathfrak{w}_2(\delta)) - \aleph_1(\delta, \overline{\mathfrak{w}}_1(\delta), \overline{\mathfrak{w}}_2(\delta)), \\ \aleph_2(\delta, \mathfrak{w}_1(\delta), \mathfrak{w}_2(\delta)) - \aleph_2(\delta, \overline{\mathfrak{w}}_1(\delta), \overline{\mathfrak{w}}_2(\delta))\Big). \tag{3.9}$$

On the other hand, by Lemma 2.5, we have

$${}_0\mathcal{I}_\delta^{\varrho_j,\wp_j}\, {}_0^C\mathfrak{D}_\delta^{\varrho_j,\wp_j}\mathfrak{U}_j(\delta) = \mathfrak{U}_j(\delta) - e^{-\wp_j\delta}\mathfrak{U}_j(0) = \mathfrak{U}_j(\delta),\ j \in \{1,2\}.$$

By (3.9) and (H1), for all $\delta \in \nabla$ and $j \in \{1,2\}$ we have

$$\begin{aligned} |\mathfrak{U}_j(\delta)| &= \left|{}_0\mathcal{I}_\delta^{\varrho_j,\wp_j}\, {}_0^C\mathfrak{D}_\delta^{\varrho_j,\wp_j}\mathfrak{U}_j(\delta)\right| \\ &\le {}_0\mathcal{I}_\delta^{\varrho_j,\wp_j}\left[\Big|\aleph_j(s, \mathfrak{w}_1(s), \mathfrak{w}_2(s)) - \aleph_j(s, \overline{\mathfrak{w}}_1(s), \overline{\mathfrak{w}}_2(s))\Big|\right](\delta) \\ &\le \frac{1}{\Gamma(\varrho_j)}\int_0^\delta (\delta-s)^{\varrho_j-1}e^{-\wp_j(\delta-s)}\Big|\aleph_j(s, \mathfrak{w}_1(s), \mathfrak{w}_2(s)) \\ &\quad - \aleph_j(s, \overline{\mathfrak{w}}_1(s), \overline{\mathfrak{w}}_2(s))\Big| \\ &\le \frac{\gamma_j}{\Gamma(\varrho_j)}\int_0^\delta (\delta-s)^{\varrho_j-1}e^{-\wp_j(\delta-s)}|\mathfrak{U}_1(s)|ds \\ &\quad + \frac{\beta_j}{\Gamma(\varrho_j)}\int_0^\delta (\delta-s)^{\varrho_j-1}e^{-\wp_j(\delta-s)}|\mathfrak{U}_2(s)|ds \\ &\le \frac{(\gamma^*+\beta^*)\varkappa^*}{\Gamma^*}\|\mathfrak{U}\|_{\mathfrak{J}}. \end{aligned}$$

Therefore,

$$\|\mathfrak{U}\|_{\mathfrak{J}} \leqslant \frac{(\gamma^*+\beta^*)\varkappa^*}{\Gamma^*}\|\mathfrak{U}\|_{\mathfrak{J}}.$$

Hence, by (3.6), we conclude that

$$\|\mathfrak{U}\|_{\mathfrak{J}} = 0.$$

As a result, for any $\delta \in \nabla$, we get

$$\mathfrak{U}(\delta) = 0 \Longrightarrow \mathfrak{w}_1(\delta) = \mathfrak{w}_2(\delta).$$

□

4. An Example

Consider the following system:

$$\begin{cases} {}_0^C\mathfrak{D}_\delta^{\frac{1}{3};2}\mathfrak{w}_1(\delta) = \aleph_1(\delta, \mathfrak{w}_1(\delta), \mathfrak{w}_2(\delta)), \\ {}_0^C\mathfrak{D}_\delta^{\frac{1}{2};\frac{2}{3}}\mathfrak{w}_2(\delta) = \aleph_2(\delta, \mathfrak{w}_1(\delta), \mathfrak{w}_2(\delta)), \\ \mathfrak{w}_1(0) = \mathfrak{w}_1(1) = 0 \text{ and } \mathfrak{w}_2(0) = \mathfrak{w}_2(1) = 0, \end{cases} \quad \delta \in \nabla := [0,1],$$

where

$$\begin{aligned} \aleph_1(\delta, \mathfrak{w}_1(\delta), \mathfrak{w}_2(\delta)) &= \sqrt{\delta+2} + \frac{1}{23\sqrt{\pi}}\left(\sin\mathfrak{w}_1(\delta) + \frac{3}{2}\mathfrak{w}_1(\delta)\right) + \frac{e^{-11-\delta}}{37(1+\mathfrak{w}_2(\delta))}, \\ \aleph_2(\delta, \mathfrak{w}_1(\delta), \mathfrak{w}_2(\delta)) &= \frac{\ln(2+\delta)}{5} + \frac{e^{\frac{-\delta}{9}}}{7\left(1+\frac{\delta}{3}\right)}\mathfrak{w}_2(\delta) + \frac{\delta}{55\sqrt{\pi}}\cos\mathfrak{w}_1(\delta). \end{aligned}$$

Here $\varrho_1 = \frac{1}{3}, \varrho_2 = \frac{1}{2}$, $\wp_1 = 2, \wp_2 = \frac{2}{3}$ and $\varkappa = 1$.
It is easy to see that $\aleph_1, \aleph_2 \in C\big([0,1] \times \mathfrak{R}^2, \mathfrak{R}\big)$. Let $\mathfrak{w}, \bar{\mathfrak{w}}, \upsilon, \overline{\upsilon} \in \mathfrak{R}$ and $\delta \in \nabla$, then

$$\begin{cases} \left|\aleph_1\left(\delta, \mathfrak{w}, \upsilon\right) - \aleph_1\left(\delta, \bar{\mathfrak{w}}, \overline{\upsilon}\right)\right| \leqslant \frac{5}{46\sqrt{\pi}} |\mathfrak{w} - \bar{\mathfrak{w}}| + \frac{1}{37e^{11}} \left|\upsilon - \overline{\upsilon}\right|, \\ \\ \left|\aleph_2\left(\delta, \mathfrak{w}, \upsilon\right) - \aleph_2\left(\delta, \bar{\mathfrak{w}}, \overline{\upsilon}\right)\right| \leqslant \frac{1}{55\sqrt{\pi}} |\mathfrak{w} - \bar{\mathfrak{w}}| + \frac{1}{7} \left|\upsilon - \overline{\upsilon}\right|. \end{cases}$$

Hence, the assumption (H1) is satisfied with $\gamma_1 = \dfrac{5}{46\sqrt{\pi}}, \gamma_2 = \dfrac{1}{55\sqrt{\pi}}, \beta_1 = \dfrac{1}{37e^{11}}$ and $\beta_2 = \dfrac{1}{7}$ which implies that $\gamma^* = \dfrac{5}{46\sqrt{\pi}}$ and $\beta^* = \dfrac{1}{7}$ and $\varkappa^* = 1, \Gamma^* = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)$
By simple calculations, we see that

$$\frac{(\gamma^* + \beta^*)\varkappa^*}{\Gamma^*} \approx 0,229 < \frac{1}{2}.$$

With the use of Theorem 3.6 our problem has a unique solution.

Declarations

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References

[1] R. Almeida, M. L. Morgado, Analysis and numerical approximation of tempered fractional calculus of variations problems, *J. Comput. Appl. Math.* **361** (2019), 1-12.

[2] M. Benchohra, E. Karapınar, J. E. Lazreg and A. Salim, *Advanced Topics in Fractional Differential Equations: A Fixed Point Approach*, Springer, Cham, 2023.

[3] M. Benchohra, E. Karapınar, J. E. Lazreg and A. Salim, *Fractional Differential Equations: New Advancements for Generalized Fractional Derivatives*, Springer, Cham, 2023.

[4] D. Benzenati, S. Bouriah, A. Salim and M. Benchohra, On periodic solutions for some nonlinear fractional pantograph problems with Ψ-Hilfer derivative, *Lobachevskii J. Math.* **44** (2023), 1264-1279. https://doi.org/10.1134/S1995080223040054

[5] S. Bouriah, D. Foukrach, M. Benchohra, and J. Graef; Existence and uniqueness of periodic solutions for some nonlinear fractional pantograph differential equations with $\psi-$Caputo derivative. *Arab. J. Math.* **10** (3) (2021), 575-587.

[6] R. G. Buschman, Decomposition of an integral operator by use of Mikusinski calculus, *SIAM J. Math. Anal.* **3** (1972), 83-85.

[7] M. Chohri, S. Bouriah, A. Salim and M. Benchohra, On nonlinear periodic problems with Caputo's exponential fractional derivative, *ATNAA.* **7** (2023), 103-120. https://doi.org/10.31197/atnaa.1130743

[8] R.E. Gaines, J. Mawhin, *coincidence degree and nonlinear differential equations*, Lecture Notes in Math., Springer-Verlag, Berlin, 1977.

[9] R. Herrmann, *Fractional Calculus: An Introduction for Physicists.* Singapore: World Scientific Publishing Company, 2011.

[10] S. Krim, A. Salim and M. Benchohra, Nonlinear contractions and Caputo tempered implicit fractional differential equations in b-metric spaces with infinite delay. *Filomat.* **37** (22) (2023), 7491-7503. https://doi.org/10.2298/FIL2322491K

[11] S. Krim, A. Salim and M. Benchohra, On implicit Caputo tempered fractional boundary value problems with delay. *Lett. Nonlinear Anal. Appl.* **1** (1) (2023), 12-29. https://doi.org/10.5281/zenodo.7682064

[12] C. Li, W. Deng and L. Zhao, Well-posedness and numerical algorithm for the tempered fractional differential equations, *Discr. Contin. Dyn. Syst. Ser. B.* **24** (2019), 1989-2015.

[13] A. Mali, K. Kucche, A. Fernandez, H. Fahad, On tempered fractional calculus with respect to functions and the associated fractional differential equations. *Math. Methods Appl. Sci.* (2022), 1-24.

[14] J. Mawhin, *NSFCBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, 1979.

[15] M. Medved and E. Brestovanska, Differential Equations with Tempered ψ-Caputo Fractional Derivative, *Math. Model. Anal.* **26** (2021), 631-650.

[16] N. A. Obeidat, D. E. Bentil, New theories and applications of tempered fractional differential equations, *Nonlinear Dyn.* **105** (2021), 1689-1702.

[17] D. O'Regan, Y.J. Chao and Y.Q. Chen, *Topological Degree Theory and Application,* Taylor and Francis Group, Boca Raton, London, New York, 2006.

[18] F. Sabzikar, M. M. Meerschaert and J. Chen, Tempered fractional calculus, *J. Comput. Phys.* **293** (2015), 14-28.

[19] A. Salim and M. Benchohra, A study on tempered (k,ψ)-Hilfer fractional operator, *Lett. Nonlinear Anal. Appl.* **1** (3) (2023), 101-121. https://doi.org/10.5281/zenodo.8361961

[20] A. Salim, M. Benchohra and J. E. Lazreg, On implicit k-generalized ψ-Hilfer fractional differential coupled systems with periodic conditions, *Qual. Theory Dyn. Syst.* **22** (2023), 46 pages. https://doi.org/10.1007/s12346-023-00776-1

[21] A. Salim, S. Bouriah, M. Benchohra, J. E. Lazreg and E. Karapınar, A study on k-generalized ψ-Hilfer fractional differential equations with periodic integral conditions. *Math. Methods Appl. Sci.* (2023), 1-18. https://doi.org/10.1002/mma.9056

[22] A. Salim, S. Krim, J. E. Lazreg and M. Benchohra, On Caputo tempered implicit fractional differential equations in b-metric spaces. *Analysis.* **43** (2) (2023), 129-139. https://doi.org/10.1515/anly-2022-1114

[23] S. G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[24] B. Shiri, G. Wu and D. Baleanu, Collocation methods for terminal value problems of tempered fractional differential equations, *Appl. Numer. Math.* **156** (2020), 385-395.