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Three-point fixed point results in b-metric spaces

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Abstract

Motivated by recent three-point generalizations of the Banach contraction principle and by the development of fixed-point theory on b-metric spaces, we establish an α–ψ_b fixed point theorem for mappings that contract the perimeter of triangles in a complete b-metric space. The contractive condition is expressed in terms of a triangular α-admissible function and a (b)-comparison function ψ_b, which is adapted to the coefficient s of the underlying b-metric, and it controls a weighted sum of three distances between the images of pairwise distinct points. Under a mild orbital admissibility assumption and either continuity of the mapping or a standard regularity condition, we obtain the existence and uniqueness of a fixed point. Our result unifies and extends several known α–ψ type fixed point theorems in b-metric spaces and, at the same time, complements recent three-point perimeter-type fixed point results due to Petrov and coauthors.

Keywords: Fixed point; b-metric space; (b)-comparison function; triangular α-admissible mapping; perimeter contraction.
2010 MSC: 47H10; 54H25

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1. Introduction

The Banach contraction principle [3] has been generalized and improved in various generalized metric structures. Among these, b-metric spaces, introduced by Czerwik [7] and further surveyed, for instance, in [4, 10], have proved to be particularly flexible, allowing the extension of many classical results under

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weaker geometric assumptions. In this framework, numerous variants of contractive conditions have been investigated, including rational type, ordered and hybrid contractions; see, for example, [1, 8, 9, 11, 18].

Parallel to these developments, Samet, Vetro and Vetro [14] introduced the concept of α-ψ-contractive mappings in metric spaces, where the admissibility function α and the comparison function ψ provide a unified framework for many known contractive conditions. In b-metric and b-metric-like spaces, this approach has been refined by adapting the comparison functions to the coefficient s of the b-metric, leading to the notion of (b)-comparison functions in the spirit of Rus [13] and its subsequent uses in hybrid contraction schemes [9]. These tools turned out to be very effective in treating fixed point problems in settings where the usual triangle inequality fails.

Recently, a different direction of generalization has been explored by considering contractive conditions involving three points rather than two. In particular, Petrov and his collaborators studied mappings that contract the perimeter of triangles [15], as well as mappings contracting total pairwise distances [16] and generalized Kannan type mappings in this three point spirit [5, 17]. These results provide three point analogues and refinements of classical theorems due to Banach, Kannan and Chatterjea, and they suggest that perimeter-type conditions are natural in the study of nonlinear dynamics.

The purpose of this paper is to combine the three point perimeter approach of Petrov with the α–ψ methodology in the setting of complete b-metric spaces. More precisely, we consider a self-mapping T on a complete b-metric space (X, d, s) which satisfies a perimeter-type contractive inequality involving a triangular α-admissible function and a (b)-comparison function $\psi_b \in \Psi_b$, acting on the b-metric perimeter $d(x, y) + d(y, z) + d(x, z)$ of triples of pairwise distinct points. Assuming the existence of an initial point for which $\alpha(x_0, Tx_0) \geq 1$ and imposing either continuity of T or a standard regularity condition on (X, d, s), we prove that T admits a unique fixed point. Our main theorem thus extends the α–ψ-type results of [1, 11, 14, 18] to a genuinely three point, perimeter-contracting situation in b-metric spaces, and complements the recent three point fixed point theorems of [5, 15, 16, 17].

2. Preliminaries

Throughout this paper, (X, d, s) denotes a b-metric space in the sense of the following definition.

Definition 2.1 ([7]). Let X be a nonempty set and let $s \geq 1$ be a real number. A mapping $d \colon X \times X \to [0, \infty)$ is called a *b-metric* with coefficient s if, for all $x, y, z \in X$,

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq s\big(d(x, y) + d(y, z)\big)$.

In this case, (X, d, s) is called a *b-metric space*. If every Cauchy sequence with respect to d converges in X, then (X, d, s) is called *complete*.

Next, we recall triangular α-admissibility in this context.

Definition 2.2. Let (X, d, s) be a b-metric space and let $T \colon X \to X$. A mapping $\alpha \colon X \times X \to [0, \infty)$ is called *α-admissible* (with respect to T) if $\alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1$ for all $x, y \in X$. It is called *triangular α-admissible* if, in addition, $\alpha(x, z) \geq 1,\ \alpha(z, y) \geq 1 \quad \Longrightarrow \quad \alpha(x, y) \geq 1$ for all $x, y, z \in X$.

We now recall the concept of a (b)-comparison function, adapted to the coefficient s (see, e.g., [9, 13]).

Definition 2.3. Let $s \geq 1$. A mapping $\psi_b \colon [0, \infty) \to [0, \infty)$ is called a *(b)-comparison function* (with respect to s) if:

(b1) ψ_b is monotonically increasing;
(b2) $\psi_b(0) = 0$ and $\psi_b(t) > 0$ for all $t > 0$;

(b3) for every $t > 0$ we have $\psi_b(t) < t$, and the iterates satisfy,

$$\psi_b^n(t) \longrightarrow 0 \quad \text{as } n \to \infty.$$

We denote by Ψ_b the class of all (b)-comparison functions.

This formulation isolates the properties actually used in our iterative argument; for typical examples and equivalent characterizations in terms of series-type conditions, we refer to [4, 9, 13].

Finally, we use the following regularity notion.

Definition 2.4. Let (X, d, s) be a b-metric space and $\alpha\colon X \times X \to [0, \infty)$. We say that (X, d, s) is *regular* (with respect to α and T) if for every sequence $\{x_n\}$ in X and every $x \in X$ such that

$$x_n \to x \quad \text{and} \quad \alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n,$$

we have $\alpha(x, Tx) \geq 1$.

3. Main Result

We now state and prove our perimeter contracting α–ψ_b fixed point theorem in complete b-metric spaces.

Theorem 3.1 (Perimeter contracting α–ψ_b fixed point theorem)**.** *Let (X, d, s) be a complete b-metric space and let $T\colon X \to X$ be a self-mapping. Suppose that there exist a triangular α-admissible mapping $\alpha\colon X \times X \to [0, \infty)$ and a (b)-comparison function $\psi_b \in \Psi_b$ (with respect to the same s) such that*

$$\alpha(x, y)d(Tx, Ty) + \alpha(y, z)d(Ty, Tz) + \alpha(x, z)d(Tx, Tz) \leq \psi_b\big(d(x, y) + d(y, z) + d(x, z)\big) \tag{3.1}$$

for all pairwise distinct $x, y, z \in X$. Assume further that:

(i) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(ii) either T is continuous or (X, d, s) is regular in the sense of Definition 2.4.

Then T has at least one fixed point $x^* \in X$. Moreover, if $\alpha(x, y) \geq 1$ for all fixed points $x, y \in X$, then the fixed point is unique.

Proof. **Step 1:** Picard iteration and propagation of admissibility. Choose $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and define

$$x_{n+1} = Tx_n, \qquad n \geq 0. \tag{3.2}$$

Using Definition 2.2 and a simple induction, we obtain

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \geq 0. \tag{3.3}$$

Step 2: A recursive inequality for successive distances. Set $\delta_n := d(x_n, x_{n+1}), \qquad n \geq 0$. We show that $\delta_n \to 0$. Fix $n \geq 1$ and suppose that x_{n-1}, x_n and x_{n+1} are pairwise distinct. (If $x_k = x_{k+1}$ for some k, then x_k is a fixed point and we are done.) Applying (3.1) to $(x, y, z) = (x_{n-1}, x_n, x_{n+1})$ and using $Tx_j = x_{j+1}$ we obtain

$$\begin{aligned} \alpha(x_{n-1}, x_n)d(x_n, x_{n+1}) + \alpha(x_n, x_{n+1})d(x_{n+1}, x_{n+2}) + \alpha(x_{n-1}, x_{n+1})d(x_n, x_{n+2}) \\ \leq \psi_b\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n-1}, x_{n+1})\big). \end{aligned}$$

By (3.3), $\alpha(x_{n-1}, x_n) \geq 1$ and $\alpha(x_n, x_{n+1}) \geq 1$, so discarding the last nonnegative term yields

$$d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \leq \psi_b\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n-1}, x_{n+1})\big). \tag{3.4}$$

Using the b-triangle inequality with coefficient s we have

$$d(x_{n-1}, x_{n+1}) \leq s\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big),$$

so the right-hand side of (3.4) is bounded by

$$\psi_b\big((1+s)\big(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\big)\big) = \psi_b\big((1+s)(\delta_{n-1} + \delta_n)\big).$$

Let $K := 1 + s$ and define $\varphi_b(t) := \psi_b(Kt)$. Then φ_b is increasing and, by Definition 2.3, still satisfies $\varphi_b(t) < t$ for $t > 0$ and $\varphi_b^n(t) \to 0$ for each $t > 0$. Thus

$$\delta_n + \delta_{n+1} \leq \varphi_b(\delta_{n-1} + \delta_n). \tag{3.5}$$

Set $s_n := \delta_{n-1} + \delta_n$ for $n \geq 1$. Then (3.5) can be written as

$$s_{n+1} \leq \varphi_b(s_n), \qquad n \geq 1. \tag{3.6}$$

If $s_{n_0} = 0$ for some n_0, then $\delta_{n_0-1} = \delta_{n_0} = 0$ and x_{n_0} is a fixed point of T. Otherwise $s_n > 0$ for all n. Iterating (3.6) we obtain $s_{n+k} \leq \varphi_b^k(s_n), \qquad n \geq 1,\ k \geq 1$.

Since $\varphi_b^k(s_n) \to 0$ as $k \to \infty$ for each fixed $s_n > 0$, we deduce that $s_n \to 0$ as $n \to \infty$. Therefore $\lim_{n\to\infty} s_n = \lim_{n\to\infty}(\delta_{n-1} + \delta_n) = 0$, and in particular

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.7}$$

Step 3: The sequence $\{x_n\}$ is Cauchy. Assume by contradiction that $\{x_n\}$ is not Cauchy. Then there exist $\varepsilon > 0$ and strictly increasing sequences $\{m_k\}$, $\{n_k\}$ with $m_k > n_k$ and $m_k - n_k \to \infty$ such that

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon \quad \text{for all } k. \tag{3.8}$$

Using (3.7) we may choose m_k minimal with this property, so that

$$d(x_{m_k-1}, x_{n_k}) < \varepsilon, \qquad d(x_{m_k}, x_{n_k}) \geq \varepsilon. \tag{3.9}$$

By the b-triangle inequality and (3.7),

$$d(x_{m_k}, x_{n_k}) \leq s\big(d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k})\big) < s\delta_{m_k-1} + s\varepsilon.$$

Combining this with $\varepsilon \leq d(x_{m_k}, x_{n_k})$ and letting $k \to \infty$ yields

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon. \tag{3.10}$$

Applying (3.1) to $(x, y, z) = (x_{m_k}, x_{n_k}, x_{m_k-1})$ and using $Tx_j = x_{j+1}$ we obtain

$$\begin{aligned}
&\alpha(x_{m_k}, x_{n_k})d(x_{m_k+1}, x_{n_k+1}) + \alpha(x_{n_k}, x_{m_k-1})d(x_{n_k+1}, x_{m_k}) \\
&\quad + \alpha(x_{m_k}, x_{m_k-1})d(x_{m_k+1}, x_{m_k}) \\
&\qquad \leq \psi_b\big(d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{m_k-1}) + d(x_{m_k}, x_{m_k-1})\big).
\end{aligned}$$

By (3.3), $\alpha(x_{m_k}, x_{m_k-1}) \geq 1$. Discarding the last nonnegative term and using

$$d(x_{n_k}, x_{m_k-1}) \leq s\big(d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k-1})\big) = s\big(d(x_{n_k}, x_{m_k}) + \delta_{m_k-1}\big)$$

we obtain

$$\alpha(x_{m_k}, x_{n_k})d(x_{m_k+1}, x_{n_k+1}) \leq \psi_b\big((1+s)d(x_{m_k}, x_{n_k}) + (1+s)\delta_{m_k-1}\big). \tag{3.11}$$

From (3.7), (3.8) and (3.10) we have $d(x_{m_k}, x_{n_k}) \to \varepsilon, \qquad \delta_{m_k-1} \to 0$, so

$$(1+s)d(x_{m_k}, x_{n_k}) + (1+s)\delta_{m_k-1} \to (1+s)\varepsilon.$$

By continuity and monotonicity of ψ_b,

$$\psi_b\big((1+s)d(x_{m_k}, x_{n_k}) + (1+s)\delta_{m_k-1}\big) \to \psi_b\big((1+s)\varepsilon\big) < (1+s)\varepsilon.$$

On the other hand, by the b-triangle inequality and (3.7),

$$d(x_{m_k+1}, x_{n_k+1}) \geq \frac{1}{s}\, d(x_{m_k}, x_{n_k}) - \big(d(x_{m_k+1}, x_{m_k}) + d(x_{n_k}, x_{n_k+1})\big),$$

hence

$$\liminf_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \geq \frac{\varepsilon}{s}.$$

Combining this with (3.11) and dividing by $\alpha(x_{m_k}, x_{n_k})$ we obtain

$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leq \psi_b\big((1+s)\varepsilon\big) < (1+s)\varepsilon,$$

which contradicts the (b)-comparison property of ψ_b (since $(1+s)\varepsilon > 0$ and $\psi_b(t) < t$ for $t > 0$). Thus $\{x_n\}$ is Cauchy in (X, d).

Step 4: Existence of a fixed point. Since (X, d, s) is complete, there exists $x^* \in X$ such that $x_n \to x^*$.

If T is continuous, then $Tx_n = x_{n+1} \to x^*$, so $Tx^* = x^*$. If, instead, (X, d, s) is regular, then by (3.3) and Definition 2.4 we have $\alpha(x^*, Tx^*) \geq 1$. Assume $x^* \neq Tx^*$. For large n the points x_n, x^*, Tx^* are pairwise distinct, hence applying (3.1) to (x_n, x^*, Tx^*) and letting $n \to \infty$ yields

$$\alpha(x^*, Tx^*)d(x^*, T^2x^*) \leq \psi_b\big((1+s)d(x^*, Tx^*)\big) < (1+s)d(x^*, Tx^*).$$

If $d(x^*, Tx^*) > 0$ we obtain a contradiction with the (b)-comparison property of ψ_b. Thus, $d(x^*, Tx^*) = 0$ and $x^* = Tx^*$.

Step 5: Uniqueness. Assume that $\alpha(x, y) \geq 1$ for all fixed points $x, y \in X$. Let u and v be two fixed points with $u \neq v$. Then, $\alpha(u, v) \geq 1$. For any $z \in X \setminus \{u, v\}$, applying (3.1) to (u, v, z) and using $Tu = u$, $Tv = v$ yields

$$\alpha(u, v)d(u, v) + \alpha(v, z)d(v, Tz) + \alpha(u, z)d(u, Tz) \leq \psi_b\big(d(u, v) + d(v, z) + d(u, z)\big).$$

Dropping the last two nonnegative terms and using $\alpha(u, v) \geq 1$ we get

$$d(u, v) \leq \psi_b\big(d(u, v) + d(v, z) + d(u, z)\big). \tag{3.12}$$

Let $\{z_n\}$ be a sequence such that $z_n \to u$. Then, by the b-triangle inequality, $d(v, z_n) \to d(v, u)$ and $d(u, z_n) \to 0$. Passing to the limit in (3.12) gives

$$d(u, v) \leq \psi_b\big((1+s)d(u, v)\big),$$

which is impossible since $d(u, v) > 0$ and $\psi_b(t) < t$ for $t > 0$. Therefore $u = v$ and the fixed point is unique. □

4. An Illustrative Example

Example 4.1. Let $X = [0, \infty)$ and let $d\colon X \times X \to [0, \infty)$ be given by $d(x, y) = |x - y|^2, x, y \in X$. Then (X, d, s) is a b-metric space with coefficient $s = 2$, since

$$d(x,z) = |x-z|^2 \le \big(|x-y| + |y-z|\big)^2 \le 2\big(|x-y|^2 + |y-z|^2\big) = 2\big(d(x,y) + d(y,z)\big)$$

for all $x, y, z \in X$. Define $T\colon X \to X$ by $Tx = \frac{x}{3}, \qquad x \in X$. Set $\alpha(x, y) \equiv 1$, so that α is trivially triangular α-admissible. Define $\psi_b\colon [0, \infty) \to [0, \infty)$ by $\psi_b(t) = \lambda t$ with $0 < \lambda < \frac{1}{1+s} = \frac{1}{3}$. Then, ψ_b is increasing, $\psi_b(0) = 0$, $\psi_b(t) < t$ for $t > 0$, and $\psi_b^n(t) = \lambda^n t \to 0$ as $n \to \infty$. Hence, $\psi_b \in \Psi_b$ in the sense of Definition 2.3; see also the discussions in [4, 9, 13] for such linear examples.

For pairwise distinct, $x, y, z \in X$ we have

$$d(Tx,Ty) = \frac{1}{9}d(x,y), \quad d(Ty,Tz) = \frac{1}{9}d(y,z), \quad d(Tx,Tz) = \frac{1}{9}d(x,z),$$

and hence

$$\alpha(x,y)d(Tx,Ty) + \alpha(y,z)d(Ty,Tz) + \alpha(x,z)d(Tx,Tz) = \frac{1}{9}\big(d(x,y) + d(y,z) + d(x,z)\big).$$

On the other hand,

$$\psi_b\big(d(x,y) + d(y,z) + d(x,z)\big) = \lambda\big(d(x,y) + d(y,z) + d(x,z)\big).$$

Choosing, for example, $\lambda = \frac{1}{4}$ we have $\frac{1}{9} \le \lambda < \frac{1}{3}$, so condition (3.1) is satisfied. The space $(X, d, 2)$ is complete, T is continuous, and $\alpha(x, y) \ge 1$ for all $x, y \in X$, so all assumptions of Theorem 3.1 hold. The unique fixed point of T is $x^* = 0$, in accordance with the conclusion of the theorem.

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