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Three-point fixed point results in b-metric spaces

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Abstract

In this paper, we explore fixed-point theorems in b-metric spaces, focusing on mappings that contract the perimeters of triangles and their generalizations. We establish new fixed-point results for such mappings under specific conditions, demonstrating their existence and uniqueness in b-metric spaces. Additionally, we introduce three-point Kannan-type generalized mappings, extending classical fixed-point theorems to a broader context. These results not only generalize existing theorems but also open new directions for further research in fixed-point theory within the b-metric framework.

Keywords: Fixed point, b-metric, mappings contracting perimeters of triangles
2010 MSC: 47H10, 54E50, 47S20

1. Introduction

The concept of b-metric spaces was introduced by S. Czerwik (see [5]) in 1993 to generalize the traditional notion of metric spaces. The concept of b-metric spaces has since garnered significant attention in the field of mathematical analysis, particularly in the study of fixed-point theorems and their applications (see [4, 6, 7, 11, 12, 13] and the references cited herein). An interesting exposition of the early developments of the concept can be found in [2].

Very recently, Petrov introduced in [16] a new type of mappings called mappings contracting perimeters of triangles, which are a three-point analogue of Banach contractions [1]:

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Definition 1.1 (Petrov [16])**.** Let (X, d) be a metric space with $|X| \geq 3$. We shall say that $T \colon X \to X$ is a mapping contracting perimeters of triangles on X if there exists $\alpha \in [0, 1)$ such that the inequality

$$d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx) \leq \alpha[d(x, y) + d(y, z) + d(z, x)],$$

holds for all three pairwise distinct points $x, y, z \in X$.

Petrov proved in [16] a fixed point theorem for these kind of mappings:

Theorem 1.2 (Petrov [16])**.** *Let (X, d), $|X| \geq 3$ be a complete metric space and let $T \colon X \to X$ be a mapping contracting perimeters of triangles on X. Then, T has a fixed point if and only if T does not possess periodic points of prime period 2. The number of fixed points is at most 2.*

The newly introduced mappings were further studied and extended in [3, 8, 9, 14, 15, 17, 18, 19].

2. Preliminaries

Let us recall some basic notions related to b-metric spaces. We make the notation $\mathbb{R}^+ = [0, \infty)$.

Let $d : X \times X \to \mathbb{R}^+$ and $s \geq 1$. We say that d is a b-metric on X, if for all $x, y \in X$, we have

(d_1) $d(x, y) = 0$ if and only if $x = y$;

(d_2) $d(x, y) = d(y, x)$;

(d_3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

Definition 2.1. Let (X, d) be a b-metric. A sequence (x_n) in X is said to be

a. *b-convergent* to a point $x \in X$ if for every $\epsilon > 0$, there exists a positive integer N such that for all $n \geq N$,
$$d(x_n, x) < \epsilon.$$

b. *b-Cauchy sequence* if for every $\epsilon > 0$, there exists a positive integer N such that for all $m, n \geq N$,
$$d(x_n, x_m) < \epsilon.$$

Definition 2.2. Let (X, d) be a b-metric space. The space (X, d) is said to be *b-complete* if every b-Cauchy sequence in X is b-convergent in X.

In [13], the following lemma related to b-Cauchy sequences was proved:

Lemma 2.3 ([13])**.** *Every sequence $(x_n)_{n \in \mathbb{N}}$ of elements from a b-metric space (X, d) of constant s, having the property that there exists $\gamma \in [0, 1)$ such that*

$$d(x_{n+1}, x_n) \leq \gamma d(x_n, x_{n-1}),$$

for every $n \in \mathbb{N}$, is b-Cauchy.

3. Mappings contracting perimeters of triangles in b-metric spaces

Definition 3.1. Let (X,d) be a b-metric space with $|X| \geq 3$. We shall say that $T\colon X \to X$ is a mapping contracting perimeters of triangles in b-metric spaces if there exists $\lambda \in [0, \frac{1}{s})$ such that the inequality

$$d(Tx,Ty) + d(Ty,Tz) + d(Tz,Tx) \leq \lambda[d(x,y) + d(y,z) + d(z,x)], \tag{1}$$

holds for all three pairwise distinct points $x, y, z \in X$.

Theorem 3.2. *Let (X,d) be a complete b-metric space with $|X| \geq 3$ and let the mapping $T\colon X \to X$ satisfy the following two conditions:*

(i) $T(Tx) \neq x$ for all $x \in X$ such that $Tx \neq x$;

(ii) T is a mapping contracting perimeters of triangles in b-metric spaces.

Then, T has a fixed point. The number of fixed points is at most two.

Proof. Let $x_0 \in X$, arbitrarily chosen, but fixed and the Picard iteration

$$x_{n+1} = Tx_n, \quad \forall n \geq 0.$$

We shall show that T has at least one fixed point. Suppose that x_n is not a fixed point of the mapping T for every $n = 0, 1, \ldots$ Then, we have $x_n = Tx_{n-1} \neq x_{n-1}$ and $x_{n+1} = T(Tx_{n-1}) \neq x_{n-1}$ for every $n = 1, 2, \ldots$ Hence, by condition (i), x_{n-1}, x_n and x_{n+1} are pairwise distinct. Then taking in (1) $x = x_{n-1}$, $y = x_n$, $z = x_{n+1}$ we obtain

$$\begin{aligned} d(Tx_{n-1},Tx_n) + d(Tx_n,Tx_{n+1}) + d(Tx_{n+1},Tx_{n-1}) \leq \\ \leq \lambda[d(x_{n-1},x_n) + d(x_n,x_{n+1}) + d(x_{n+1},x_{n-1})]. \end{aligned} \tag{2}$$

For every $n = 0, 1, \ldots,$ let

$$p_n = d(x_n,x_{n+1}) + d(x_n,x_{n+2}) + d(x_{n+1},x_{n+2}).$$

Then, (2) becomes $p_n \leq \lambda p_{n-1}$ for every $n = 1, 2, \ldots$. Hence, by induction, we get

$$p_n \leq \lambda^n p_0. \tag{3}$$

Now, let $p \in \mathbb{N}, p \geq 1$. By (d_3) we have

$$\begin{aligned} d(x_n,x_{n+p}) &\leq s[d(x_n,x_{n+1}) + d(x_{n+1},x_{n+p})] \leq \\ &\leq sd(x_n,x_{n+1}) + s^2[d(x_{n+1},x_{n+2}) + d(x_{n+2},x_{n+p})] \leq \\ &\ldots\ldots\ldots \\ &\leq sd(x_n,x_{n+1}) + s^2 d(x_{n+1},x_{n+2}) + \cdots + s^p d(x_{n+p-1},x_{n+p}) \leq \\ &\overset{(3)}{\leq} s\lambda^n p_0 + s^2\lambda^{n+1}p_0 + \cdots + s^p\lambda^{n+p-1}p_0 = \\ &= p_0 s\lambda^n(1 + s\lambda + \cdots + s^{p-1}\lambda^{p-1}) \leq \\ &\overset{s\lambda<1}{\leq} p_0 s\lambda^n \frac{1}{1-s\lambda}. \end{aligned}$$

Now, since $s\lambda < 1$, passing to limit as $n \to \infty$, we obtain that $\lim_{n\to\infty} d(x_n,x_{n+p}) = 0$, which implies that $\{x_n\}$ is a **b**-Cauchy sequence. By b-completeness of (X,d), we obtain that $\{x_n\}$ is b-convergent and we get that $\{x_n\}$ has a limit $x^* \in X$. Let us prove that $Tx^* = x^*$.

Since x_n, x_{n+1} and x_{n+2} are pairwise distinct for every $n = 0, 1, \ldots$, there exists a subsequence $\{x_{n(k)}\}_{k\geq 0}$ such that $x_{n(k)}$, $x_{n(k)+1}$ and x^* are pairwise distinct for every $k = 0, 1, \ldots$. Indeed, if x^* does not belong to the sequence $\{x_{n(k)}\}$, then since $x_{n(k)} \neq Tx_{n(k)} = x_{n(k)+1}$, the points $x_{n(k)}$, $x_{n(k)+1}$ and x^* are pairwise distinct. If x^* belongs to the sequence $\{x_{n(k)}\}$, then since $x_{n(k)} = Tx_{n(k)-1} \neq x_{n(k)-1}$ and $x_{n(k)+1} = T(Tx_{n(k)}) \neq x_{n(k)}$ for every $k = 0, 1, \ldots$, the points $x_{n(k)-1}$, $x_{n(k)}$, $x_{n(k)+1}$ are pairwise distinct. Suppose k is the smallest possible index for which $x^* = x_{n(k)-1}$ then $x_{n(k)}$, $x_{n(k)+1}$ and x^* are pairwise distinct for every $k = 0, 1, \ldots$.

Now, taking in (1) $x = x_{n(k)}$, $y = x_{n(k)+1}$ and $z = x^*$, we obtain

$$\begin{aligned} d(Tx_{n(k)}, Tx_{n(k)+1}) + d(Tx_{n(k)+1}, Tx^*) + d(Tx^*, Tx_{n(k)}) \leq \\ \leq \lambda[d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x^*) + d(x^*, x_{n(k)})], \end{aligned}$$

so ,

$$\begin{aligned} d(x_{n(k)+1}, x_{n(k)+2}) + d(x_{n(k)+2}, Tx^*) + d(Tx^*, x_{n(k)+1}) \leq \\ \leq \lambda[d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x^*) + (x^*, x_{n(k)})]. \end{aligned}$$

Taking the limit as $k \to \infty$ we get

$$2d(x^*, Tx^*) \leq 0,$$

by where $d(x^*, Tx^*) = 0$, so x^* is a fixed point of T.

Now, suppose that there exist three distinct fixed point of T, $x^*, y^*, z^* \in X$, then, by (1) we obtain

$$\begin{aligned} d(Tx^*, Ty^*) + d(Ty^*, Tz^*) + d(Tz^*, Tx^*) \leq \\ \leq \lambda[d(x^*, y^*) + d(y^*, z^*) + d(z^*, x^*)], \end{aligned}$$

so

$$(1 - \lambda)[d(x^*, y^*) + d(y^*, z^*) + d(z^*, x^*)] \leq 0,$$

which implies $1 - \lambda < 0$, which is a contradiction, so T has at most two fixed points. □

Example 3.3. Let $X = \{0, 1, 2\}$ and $d(0, 0) = d(1, 1) = d(2, 2) = 0$, $d(1, 0) = d(0, 1) = \frac{1}{2}$, $d(2, 0) = d(0, 2) = 5$ and $d(2, 1) = d(1, 2) = 6$. Then, d is a b-metric, but it is not a metric as

$$d(2, 1) = 6 > 5.5 = d(2, 0) + d(0, 1).$$

Let $T\colon X \to X$ such that $T0 = T2 = 0$ and $T1 = 1$. Then T satisfies (1) for $\lambda > \frac{2}{23}$ as

$$d(T0, T1) + d(T1, T2) + d(T2, T0) = 2d(0, 1) = 1 \leq 11.5 = d(0, 1) + d(1, 2) + d(2, 0).$$

Let us also note that $T(Tx) \neq x$ for every $x \neq Tx$, and T has two fixed points.

4. Three point Kannan generalized mappings in b-metric spaces

Definition 4.1. Let (X, d) be a b-metric space with $|X| \geq 3$. We shall say that $T\colon X \to X$ is a three point Kannan generalized mapping in b-metric spaces if there exists $\lambda \in [0, \frac{1}{2})$ such that the inequality

$$d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx) \leq \lambda[d(x, Tx) + d(y, Ty) + d(z, Tz)], \tag{4}$$

holds for all three pairwise distinct points $x, y, z \in X$.

Theorem 4.2. *Let (X, d) be a complete b-metric space with $|X| \geq 3$ and let the mapping $T\colon X \to X$ satisfy the following two conditions:*

(i) $T(Tx) \neq x$ for all $x \in X$ such that $Tx \neq x$;

(ii) T is a three point Kannan generalized mapping in b-metric spaces.

Then, T has a fixed point. The number of fixed points is at most two.

Proof. Let $x_0 \in X$, arbitrarily chosen, but fixed and the Picard iteration

$$x_{n+1} = Tx_n, \quad \forall n \geq 0.$$

We shall show that T has at least one fixed point. As in the proof of theorem 3.2, we suppose that x_n is not a fixed point of the mapping T for every $n = 0, 1, \dots$ Then, we have $x_n = Tx_{n-1} \neq x_{n-1}$ and $x_{n+1} = T(Tx_{n-1}) \neq x_{n-1}$ for every $n = 1, 2, \dots$ and by condition (i), x_{n-1}, x_n and x_{n+1} are pairwise distinct. Then taking in (4) $x = x_{n-1}$, $y = x_n$, $z = x_{n+1}$ we obtain

$$\begin{aligned} d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n-1}) \leq \\ \leq \lambda[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})], \end{aligned} \tag{5}$$

so

$$\begin{aligned} d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_n) \leq \\ \leq \lambda[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})], \end{aligned}$$

by which

$$(1-\lambda)[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + d(x_{n+2}, x_n) \leq \lambda d(x_{n-1}, x_n).$$

Since

$$(1-\lambda)d(x_n, x_{n+1}) \leq (1-\lambda)[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + d(x_{n+2}, x_n),$$

we obtain

$$(1-\lambda)d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n). \tag{6}$$

Since $\lambda < \frac{1}{2}$, we have $\gamma = \dfrac{\lambda}{1-\lambda} \in [0, 1)$. Then, (6) becomes

$$d(x_n, x_{n+1}) \leq \gamma d(x_{n-1}, x_n),$$

so by Lemma 2.3, we obtain that $\{x_n\}$ is a b-Cauchy sequence, and by completeness of X, we obtain that there exists $x^* \in X$ such that $\lim\limits_{n\to\infty} x_n = x^*$.

Now let us prove that $Tx^* = x^*$. As in the proof of theorem 3.2 there exists a subsequence $\{x_{n(k)}\}_{k\geq 0}$ such that $x_{n(k)}$, $x_{n(k)+1}$ and x^* are pairwise distinct for every $k = 0, 1, \dots$.

Now, taking in (4) $x = x_{n(k)}$, $y = x_{n(k)+1}$ and $z = x^*$, we obtain

$$\begin{aligned} d(Tx_{n(k)}, Tx_{n(k)+1}) + d(Tx_{n(k)+1}, Tx^*) + d(Tx^*, Tx_{n(k)}) \leq \\ \leq \lambda[d(x_{n(k)}, Tx_{n(k)}) + d(x_{n(k)+1}, Tx_{n(k)+1}) + d(x^*, Tx^*)]. \end{aligned}$$

Hence,

$$\begin{aligned} d(x_{n(k)+1}, x_{n(k)+2}) + d(x_{n(k)+2}, Tx^*) + d(Tx^*, x_{n(k)+1}) \leq \\ \leq \lambda[d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)+2}) + d(x^*, Tx^*)]. \end{aligned}$$

Taking the limit as $k \to \infty$ we get

$$2d(x^*, Tx^*) \leq \lambda d(x^*, Tx^*),$$

and since $\lambda \in [0, \frac{1}{2})$ we obtain $d(x^*, Tx^*) = 0$, so x^* is a fixed point of T.

Now, suppose that there exist three distinct fixed point of T, $x^*, y^*, z^* \in X$, then, by (4) we obtain

$$\begin{aligned} d(Tx^*, Ty^*) + d(Ty^*, Tz^*) + d(Tz^*, Tx^*) \leq \\ \leq \lambda[d(x^*, Tx^*) + d(y^*, Ty^*) + d(z^*, Tz^*)], \end{aligned}$$

so

$$d(x^*, y^*) + d(y^*, z^*) + d(z^*, x^*) \leq 0$$

which implies $d(x^*, y^*) = d(y^*, z^*) = d(z^*, x^*) = 0$, so T has at most two fixed points. □

Example 4.3. Let $X = \{0, 1, 2\}$ and $d(0,0) = d(1,1) = d(2,2) = 0$, $d(1,0) = d(0,1) = 10$, $d(2,0) = d(0,2) = 2$ and $d(2,1) = d(1,2) = 5$. Then, d is a b-metric, but it is not a metric as

$$d(0,1) = 10 > 7 = d(0,2) + d(2,1).$$

Let $T\colon X \to X$ such that $T0 = T1 = 0$ and $T2 = 2$. Then T satisfies (4) for $\lambda > \frac{2}{5}$ as

$$d(T0,T1) + d(T1,T2) + d(T2,T0) = 2d(0,2) = 4 \leq \lambda 10 = d(0,T0) + d(1,T1) + d(2,T2).$$

Let us also note that $T(Tx) \neq x$ for every $x \neq Tx$, and T has two fixed points.

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