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# Fixed point results for cyclic mappings in modular metric space

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**Abstract**

In this paper, we prove the existence and uniqueness of fixed points for cyclic mappings in modular metric space. We obtain the results for interpolative Kannan type cyclic contractions when the sum of the interpolative exponents is less than and greater than one.

*Keywords:* Modular metric space, Fixed point
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## 1. Introduction

In 2011, Karapınar et al. introduced the concept of Kannan-type cyclic contraction and established results in [3].

Moreover, Edraoui et al. [2] extended the results of Karapınar et al. [3] to interpolative Kannan type cyclic contractions , and they established related results therein.

In this paper, we prove the existence and uniqueness of fixed points for cyclic mappings in Modular metric space. We obtain the results for interpolative Kannan type cyclic contractions with including examples.

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## 2. Primarily

Let $m : (0, \infty) \times X \times X \to [0, \infty)$ be a function provided to be the $X$ is a non-empty set. If so, for clarity, we will prefer the notions of $m_b(x, y)$ rather than $m(b, x, y)$ for all $b > 0$ and $x, y \in X$.

**Definition 2.1** ([1]). Let $m : (0, \infty) \times X \times X \to [0, \infty]$ be a function where $X$ is a non-empty set. Thereby, m is entitled to modular metric provided that for all $x, y, z \in X$, the circumstances

**(m1)** $m_b(x, y) = 0$ for all $b > 0$ if and only if $x = y$,

**(m2)** $m_b(x, y) = m_b(y, x)$ for all $b > 0$,

**(m3)** $m_{b+c}(x, y) \leq m_b(x, z) + m_c(z, y)$ for all $b, c > 0$.

are satisfied. Thereupon, $(X, m)$ is a modular metric space, which abbreviates as MMS.

**Example 2.2.** Let $(X, d)$ be a metric space. If we put, $m_b(x, y) = \frac{d(x,y)}{\varphi(b)}$, where $\varphi : (0, \infty) \to (0, \infty)$ is a nondecreasing function. Then $(X, m)$ is a MMS.

**Definition 2.3.** $m$ is called convex modular if for $b, c > 0$ and $x, y, z \in X$,

$$m_{b+c}(x, y) \leq \frac{b}{b+c} m_b(x, y) + \frac{c}{b+c} m_c(x, y). \tag{1}$$

On the other hand, the function $b \to m_b(x, y)$ is non-increasing on $(0, \infty)$ for any $x, y \in X$, where $m$ is a metric pseudo-modular on a set $X$. Undoubtedly, for $0 < c < b$, it is verified by

$$m_b(x, y) \leq m_{b-c}(x, x) + m_c(x, y) = m_c(x, y).$$

Let $m$ be a modular on $X$ and $s_0 \in X$ be a fix. Thereby, the following sets are mentioned as modular spaces (around $s_0$):

$$X_m = X_m(x_0) = \{x \in X : \lim_{b\to\infty} m_b(x, x_0) \to 0\}$$

and

$$X_m^* = X_m^*(x_0) = \{x \in X : \exists b(x) \ s.t. \ m_b(x, x_0) < \infty\}$$

we know that $X_m \subset X_m^*$.

A nontrivial metric $d_m$, which is generated by the modular $m$,

$$d_m(x, y) = \inf\{b > 0 : m_b(x, y) \leq b\} \quad \forall x, y \in X_m$$

is identified on $X_m$

Furthermore, if we consider a convex modular $m$ on $X$, then $X_m = X_m^*$ thereupon, these sets are endowed with the metric

$$d_m(x, y) = \inf\{b > 0 : m_b(x, y) \leq 1\} \quad \forall x, y \in X_m$$

withal proverbial as the Luxembourg distance.

**Definition 2.4.** Let $X_m^*$ be an MMS, $\{x_n\}_{n\in\mathbb{N}} \in X_m^*$ be a sequence and $M$ be a subset of $X_m^*$.

1. $\{x_n\}_{n\in\mathbb{N}}$ is an $m$-convergent sequence to $x_0 \in X_m^*$ if and only if $m_b(x_n, x_0) \to 0$, as $n \to \infty$ for all $b > 0$ and $x_0$ is called $m$-limit of $\{x_n\}_{n\in\mathbb{N}}$.
2. If $\lim_{n,m\to\infty} m_b(x_n, x_m) = 0$, for all $b > 0$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X_m^*$ is named as an $m$-Cauchy sequence.
3. If any $m$-Cauchy sequence in $X_m^*$ is $m$-convergent to the point of $X_m^*$, then $X_m^*$ is called $m$-complete space.
4. The set $M$ is $m$-closed provided that the $m$-limit of an $m$-convergent sequence of $M$ all the time belongs to $M$.
5. $f : X_m^* \to X_m^*$ is an $m$-continuous mapping if $m_b(x_n, x_0) \to 0$, provided to $m_b(fx_n, fx_0) \to 0$ as $n \to \infty$.

## 3. Main results

**Definition 3.1.** Let $(X,m)$ be a MMS, $A$ and $B$ be nonempty subsets of $X$. A cyclic map $T: A\cup B \to A\cup B$ ($T(A)\subseteq B$ and $T(B)\subseteq A$) is said to be a $(k,\alpha+\beta<1)$-interpolative Kannan type cyclic contraction, if there exists $k\in[0,1)$ and $\alpha,\beta\in(0,1)$ with $\alpha+\beta<1$ such that

$$m_b(Tx,Ty)\le k m_b(x,Tx)^{\alpha} m_b(y,Ty)^{\beta} \quad x\in A, y\in B, \tag{2}$$

where $x,y\notin FixT$.

**Theorem 3.2.** *Let $(X,m)$ be a complete MMS, $A$ and $B$ be nonempty closed subsets of $X$ and let $T: A\cup B\to A\cup B$ ($T(A)\subseteq B$ be a $(k,\alpha+\beta<1)$-interpolative Kannan type cyclic contraction. If for any $x\in A\cup B$, $m_b(x,Tx)\ge 1$. Then $T$ has a unique fixed point in $A\cap B$.*

*Proof.* Let $x\in A$, by (2)

$$m_b(Tx,T^2x)=m_b(Tx,T(Tx))\le k m_b(x,Tx)^{\alpha} m_b(Tx,T^2x)^{\beta},$$

which implies that

$$m_b(Tx,T^2x)^{1-\beta}\le k m_b(x,Tx)^{\alpha}.$$

Since $\alpha+\beta<1$ and $m_b(x,Tx)\ge 1$, so

$$m_b(Tx,T^2x)^{1-\beta}\le k m_b(x,Tx)^{1-\beta}, \quad \forall x\in A\cup B$$

or

$$m_b(Tx,T^2x)\le k^{\frac{1}{1-\beta}} m_b(x,Tx)\le k m_b(x,Tx).$$

Also

$$m_b(T^2x,T^3x)\le k^{\frac{1}{1-\beta}} m_b(Tx,T^2x)\le k m_b(Tx,T^2x)\le k^2 m_b(x,Tx).$$

Thus

$$m_b(T^nx,T^{n+1}x)\le k^n m_b(x,Tx).$$

$$m_b(T^nx,T^{n+2}x)\le m_{\frac{b}{2}}(T^nx,T^{n+1}x)+m_{\frac{b}{2}}(T^{n+1}x,T^{n+2}x)\le k^n m_{\frac{b}{2}}(x,Tx)+k^{n+1}m_{\frac{b}{2}}(x,Tx).$$

$$m_b(T^nx,T^{n+2}x)\le (k^n+k^{n+1})m_{\frac{b}{2}}(x,Tx).$$

Moreover, for each $n,m\in\mathbb{N}$ with $n\ge m$, by using triangular MMS inequality, we get

$$m_b(T^nx,T^mx)\le (k^n+k^{n+1}+\cdots+k^{m-1})m_{\frac{b}{2^{n-m-1}}}(x,Tx).$$

$$m_b(T^nx,T^mx)\le (k^n+k^{n+1}+\cdots+k^{m-1})m_{\frac{b}{2^{n-m-1}}}(x,Tx).$$

$$m_b(T^nx,T^mx)\le (k^n+k^{n+1}+\cdots)m_{\frac{b}{2^{n-m-1}}}(x,Tx).$$

$$m_b(T^nx,T^mx)\le \frac{k^n}{1-k} m_{\frac{b}{2^{n-m-1}}}(x,Tx).$$

We note that $m_{\frac{b}{2^{n-m-1}}}(x,Tx)<\infty$ in $X_m^*$. Hence $\{T^nx\}$ is a Cauchy sequence in $A\cup B$. Since $A,B$ are closed subsets of $X$, so every Cauchy sequence in $A\cup B$ converges to an element of $A\cup B$. So $T^nx\to z$. Moreover, $\{T^{2n}x\}$ is a sequence in A and $\{T^{2n+1}x\}$ is a sequence in $B$ and both sequences having the same limit $z$. As $A$ and $B$ are closed we conclude that $z \in A \cap B$ that is $A \cap B$ is nonempty. Now, we show that $Tz = z$, since

$$\begin{aligned} 0 \leq d(z, Tz) &= \lim_{n\to\infty} m_b(Tz, T^{2n}x) \\ &= \lim_{n\to\infty} m_b(Tz, T(T^{2n-1}x)) \\ &\leq \lim_{n\to\infty} km_b(z, Tz)^{\alpha} m_b(T^{2n-1}x, T^{2n}x)^{\beta} \\ &\leq km_b(z, Tz)^{\alpha} \lim_{n\to\infty} m_b(T^{2n-1}x, T^{2n}x)^{\beta} = 0. \end{aligned}$$

Now we show that this fixed point is unique. Assume that there exist another fixed point $w \in A \cap B$ of $T$, such that $z \neq w$ and $Tw = w$. So, there exist $x, y \in A$ such that

$$\begin{aligned} m_b(z, w) &= m_b(Tz, Tw) = \lim_{n,m\to\infty} m_b(T^n x, T^m y) \\ &\leq \lim_{n,m\to\infty} km_b(T^{n-1}x, T^n x)^{\alpha} m_b(T^{m-1}y, T^n y)^{\beta} \\ &= km_b(z, z)^{\alpha} m_b(w, w)^{\beta} = 0, \end{aligned}$$

□

**Definition 3.3.** Let $(X, m)$ be a MMS, $A$ and $B$ be nonempty subsets of $X$. A cyclic map $T : A\cup B \to A\cup B$ ($T(A) \subseteq B$ and $T(B) \subseteq A$) is said to be a $(k, \alpha + \beta > 1)$-interpolative Kannan type cyclic contraction, if there exists $k \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta > 1$ such that

$$m_b(Tx, Ty) \leq km_b(x, Tx)^{\alpha} m_b(y, Ty)^{\beta} \quad x \in A, y \in B, \tag{3}$$

where $x, y \notin FixT$.

**Theorem 3.4.** *Let $(X, m)$ be a complete MMS, $A$ and $B$ be nonempty closed subsets of $X$ and let $T : A \cup B \to A \cup B$ ($T(A) \subseteq B$ and $TB \subseteq A$) be a $(k, \alpha + \beta > 1)$-interpolative Kannan type cyclic contraction. If for any $x \in A \cup B$, $m_b(x, Tx) \leq 1$. Then $T$ has a unique fixed point in $A \cap B$.*

*Proof.* It is similar to proof of Theorem 3.2. □

**Example 3.5.** Let $b > 0$ be an arbitrary and fixed, $X := \{-\frac{b}{4}, 0, \frac{b}{4}\}$ and $m_b(x, y) = \frac{|x-y|}{b}$, for all $x, y \in X$. Hence $(X, m)$ is a complete MMS. Consider $A = \{\frac{b}{4}, 0\}$, $B = \{-\frac{b}{4}, 0\}$ and $T : A \cup B \to A \cup B$ defined as $Tx = -x$. Then obviously $TA \subseteq B$ and $TB \subseteq A$ and $T$ is a cyclic mapping. Now for $\alpha = \frac{1}{3}$, $\beta = \frac{1}{3}$ and $k = \frac{1}{\sqrt[3]{2}}$, $T$ is a $(\frac{1}{\sqrt[3]{2}}, \alpha + \beta < 1)$-interpolative Kannan type cyclic contraction. Thus by Theorem 3.2, $0 \in A \cap B$ is a fixed point of $T$.

To show (2), i.e.:

$$\frac{|x - y|}{b} \leq k \left(\frac{|2x|}{b}\right)^{\frac{1}{3}} \left(\frac{|2y|}{b}\right)^{\frac{1}{3}} \quad x, y \notin FixT = \{0\}$$

we note that, if $x = y$, then it is clear. And if $x = -y = \frac{b}{4}$, then

$$|2x| \leq kb^{\frac{1}{3}} |2x|^{\frac{1}{3}} |2x|^{\frac{1}{3}} \iff \left|\frac{b}{2}\right|^{\frac{1}{3}} \leq \frac{1}{\sqrt[3]{2}} b^{\frac{1}{3}}.$$

## References


[1] Abdurrahman Büyükkaya, Mahpeyker Öztürk, *Multivalued Sehgal-Proinov type contraction mappings involving rational terms in modular metric spaces*, Filomat **38**(10) (2024), 3563–3576.

[2] M. Edraoui, S. Semami, *Fixed points results for various types of interpolative cyclic contraction.* Applied General Topology, **24**(2) (2023), 247-252.

[3] E. Karapınar, I.M. Erhan, *Best proximity point on different type contractions.* Appl. Math. Inf. Sci, **3**(3) (2011), 342-353.