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Fixed point results for cyclic mappings in modular metric space

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Abstract

In this paper, we prove the existence and uniqueness of fixed points for cyclic mappings in modular metric space. We obtain the results for interpolative Kannan type cyclic contractions when the sum of the interpolative exponents is less than and greater than one.

Keywords: Modular metric space, Fixed point
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1. Introduction

In 2011, Karapınar et al. introduced the concept of Kannan-type cyclic contraction and established results in [3].

Moreover, Edraoui et al. [2] extended the results of Karapınar et al. [3] to interpolative Kannan type cyclic contractions , and they established related results therein.

In this paper, we prove the existence and uniqueness of fixed points for cyclic mappings in Modular metric space. We obtain the results for interpolative Kannan type cyclic contractions with including examples.

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2. Primarily

Let $m : (0,\infty) \times X \times X \to [0,\infty)$ be a function provided to be the X is a non-empty set. If so, for clarity, we will prefer the notions of $m_b(x,y)$ rather than $m(b,x,y)$ for all $b > 0$ and $x, y \in X$.

Definition 2.1 ([1]). Let $m : (0,\infty) \times X \times X \to [0,\infty]$ be a function where X is a non-empty set. Thereby, m is entitled to modular metric provided that for all $x, y, z \in X$, the circumstances

(m1) $m_b(x,y) = 0$ for all $b > 0$ if and only if $x = y$,

(m2) $m_b(x,y) = m_b(y,x)$ for all $b > 0$,

(m3) $m_{b+c}(x,y) \leq m_b(x,z) + m_c(z,y)$ for all $b, c > 0$.

are satisfied. Thereupon, (X,m) is a modular metric space, which abbreviates as MMS.

Example 2.2. Let (X,d) be a metric space. If we put, $m_b(x,y) = \frac{d(x,y)}{\varphi(b)}$, where $\varphi : (0,\infty) \to (0,\infty)$ is a nondecreasing function. Then (X,m) is a MMS.

Definition 2.3. m is called convex modular if for $b, c > 0$ and $x, y, z \in X$,

$$m_{b+c}(x,y) \leq \frac{b}{b+c} m_b(x,y) + \frac{c}{b+c} m_c(x,y). \tag{1}$$

On the other hand, the function $b \to m_b(x,y)$ is non-increasing on $(0,\infty)$ for any $x, y \in X$, where m is a metric pseudo-modular on a set X. Undoubtedly, for $0 < c < b$, it is verified by

$$m_b(x,y) \leq m_{b-c}(x,x) + m_c(x,y) = m_c(x,y).$$

Let m be a modular on X and $s_0 \in X$ be a fix. Thereby, the following sets are mentioned as modular spaces (around s_0):

$$X_m = X_m(x_0) = \{x \in X : \lim_{b\to\infty} m_b(x,x_0) \to 0\}$$

and

$$X_m^* = X_m^*(x_0) = \{x \in X : \exists b(x) \ s.t. \ m_b(x,x_0) < \infty\}$$

we know that $X_m \subset X_m^*$.

A nontrivial metric d_m, which is generated by the modular m,

$$d_m(x,y) = \inf\{b > 0 : m_b(x,y) \leq b\} \quad \forall x, y \in X_m$$

is identified on X_m

Furthermore, if we consider a convex modular m on X, then $X_m = X_m^*$ thereupon, these sets are endowed with the metric

$$d_m(x,y) = \inf\{b > 0 : m_b(x,y) \leq 1\} \quad \forall x, y \in X_m$$

withal proverbial as the Luxembourg distance.

Definition 2.4. Let X_m^* be an MMS, $\{x_n\}_{n\in\mathbb{N}} \in X_m^*$ be a sequence and M be a subset of X_m^*.

1. $\{x_n\}_{n\in\mathbb{N}}$ is an m-convergent sequence to $x_0 \in X_m^*$ if and only if $m_b(x_n,x_0) \to 0$, as $n \to \infty$ for all $b > 0$ and x_0 is called m-limit of $\{x_n\}_{n\in\mathbb{N}}$.
2. If $\lim_{n,m\to\infty} m_b(x_n,x_m) = 0$, for all $b > 0$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_m^* is named as an m-Cauchy sequence.
3. If any m-Cauchy sequence in X_m^* is m-convergent to the point of X_m^*, then X_m^* is called m-complete space.
4. The set M is m-closed provided that the m-limit of an m-convergent sequence of M all the time belongs to M.
5. $f : X_m^* \to X_m^*$ is an m-continuous mapping if $m_b(x_n,x_0) \to 0$, provided to $m_b(fx_n,fx_0) \to 0$ as $n \to \infty$.

3. Main results

Definition 3.1. Let (X,m) be a MMS, A and B be nonempty subsets of X. A cyclic map $T : A\cup B \to A\cup B$ ($T(A) \subseteq B$ and $T(B) \subseteq A$) is said to be a $(k, \alpha+\beta < 1)$-interpolative Kannan type cyclic contraction, if there exists $k \in [0,1)$ and $\alpha, \beta \in (0,1)$ with $\alpha+\beta < 1$ such that

$$m_b(Tx,Ty) \leq km_b(x,Tx)^{\alpha} m_b(y,Ty)^{\beta} \quad x \in A, y \in B, \tag{2}$$

where $x, y \notin FixT$.

Theorem 3.2. *Let (X,m) be a complete MMS, A and B be nonempty closed subsets of X and let $T : A\cup B \to A\cup B$ ($T(A) \subseteq B$ be a $(k, \alpha+\beta < 1)$-interpolative Kannan type cyclic contraction. If for any $x \in A\cup B$, $m_b(x,Tx) \geq 1$. Then T has a unique fixed point in $A\cap B$.*

Proof. Let $x \in A$, by (2)

$$m_b(Tx,T^2x) = m_b(Tx,T(Tx)) \leq km_b(x,Tx)^{\alpha} m_b(Tx,T^2x)^{\beta},$$

which implies that

$$m_b(Tx,T^2x)^{1-\beta} \leq km_b(x,Tx)^{\alpha}.$$

Since $\alpha+\beta < 1$ and $m_b(x,Tx) \geq 1$, so

$$m_b(Tx,T^2x)^{1-\beta} \leq km_b(x,Tx)^{1-\beta}, \quad \forall x \in A\cup B$$

or

$$m_b(Tx,T^2x) \leq k^{\frac{1}{1-\beta}} m_b(x,Tx) \leq km_b(x,Tx).$$

Also

$$m_b(T^2x,T^3x) \leq k^{\frac{1}{1-\beta}} m_b(Tx,T^2x) \leq km_b(Tx,T^2x) \leq k^2 m_b(x,Tx).$$

Thus

$$m_b(T^nx,T^{n+1}x) \leq k^n m_b(x,Tx).$$

$$m_b(T^nx,T^{n+2}x) \leq m_{\frac{b}{2}}(T^nx,T^{n+1}x) + m_{\frac{b}{2}}(T^{n+1}x,T^{n+2}x) \leq k^n m_{\frac{b}{2}}(x,Tx) + k^{n+1} m_{\frac{b}{2}}(x,Tx).$$

$$m_b(T^nx,T^{n+2}x) \leq (k^n + k^{n+1}) m_{\frac{b}{2}}(x,Tx).$$

Moreover, for each $n, m \in \mathbb{N}$ with $n \geq m$, by using triangular MMS inequality, we get

$$m_b(T^nx,T^mx) \leq (k^n + k^{n+1} + \cdots + k^{m-1}) m_{\frac{b}{2^{n-m-1}}}(x,Tx).$$

$$m_b(T^nx,T^mx) \leq (k^n + k^{n+1} + \cdots + k^{m-1}) m_{\frac{b}{2^{n-m-1}}}(x,Tx).$$

$$m_b(T^nx,T^mx) \leq (k^n + k^{n+1} + \cdots) m_{\frac{b}{2^{n-m-1}}}(x,Tx).$$

$$m_b(T^nx,T^mx) \leq \frac{k^n}{1-k} m_{\frac{b}{2^{n-m-1}}}(x,Tx).$$

We note that $m_{\frac{b}{2^{n-m-1}}}(x,Tx) < \infty$ in X_m^*. Hence $\{T^nx\}$ is a Cauchy sequence in $A\cup B$. Since A, B are closed subsets of X, so every Cauchy sequence in $A\cup B$ converges to an element of $A\cup B$. So $T^nx \to z$. Moreover, $\{T^{2n}x\}$ is a sequence in A and $\{T^{2n+1}x\}$ is a sequence in B and both sequences having the same limit z. As A and B are closed we conclude that $z \in A \cap B$ that is $A \cap B$ is nonempty. Now, we show that $Tz = z$, since

$$\begin{aligned} 0 \le d(z,Tz) &= \lim_{n\to\infty} m_b(Tz, T^{2n}x) \\ &= \lim_{n\to\infty} m_b(Tz, T(T^{2n-1}x)) \\ &\le \lim_{n\to\infty} km_b(z,Tz)^{\alpha} m_b(T^{2n-1}x, T^{2n}x)^{\beta} \\ &\le km_b(z,Tz)^{\alpha} \lim_{n\to\infty} m_b(T^{2n-1}x, T^{2n}x)^{\beta} = 0. \end{aligned}$$

Now we show that this fixed point is unique. Assume that there exist another fixed point $w \in A \cap B$ of T, such that $z \ne w$ and $Tw = w$. So, there exist $x, y \in A$ such that

$$\begin{aligned} m_b(z,w) &= m_b(Tz,Tw) = \lim_{n,m\to\infty} m_b(T^n x, T^m y) \\ &\le \lim_{n,m\to\infty} km_b(T^{n-1}x, T^n x)^{\alpha} m_b(T^{m-1}y, T^n y)^{\beta} \\ &= km_b(z,z)^{\alpha} m_b(w,w)^{\beta} = 0, \end{aligned}$$

□

Definition 3.3. Let (X, m) be a MMS, A and B be nonempty subsets of X. A cyclic map $T : A \cup B \to A \cup B$ ($T(A) \subseteq B$ and $T(B) \subseteq A$) is said to be a $(k, \alpha + \beta > 1)$-interpolative Kannan type cyclic contraction, if there exists $k \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta > 1$ such that

$$m_b(Tx,Ty) \le km_b(x,Tx)^{\alpha} m_b(y,Ty)^{\beta} \quad x \in A, y \in B, \tag{3}$$

where $x, y \notin FixT$.

Theorem 3.4. *Let (X, m) be a complete MMS, A and B be nonempty closed subsets of X and let $T : A \cup B \to A \cup B$ ($T(A) \subseteq B$ and $TB \subseteq A$) be a $(k, \alpha + \beta > 1)$-interpolative Kannan type cyclic contraction. If for any $x \in A \cup B$, $m_b(x, Tx) \le 1$. Then T has a unique fixed point in $A \cap B$.*

Proof. It is similar to proof of Theorem 3.2. □

Example 3.5. Let $b > 0$ be an arbitrary and fixed, $X := \{-\frac{b}{4}, 0, \frac{b}{4}\}$ and $m_b(x, y) = \frac{|x-y|}{b}$, for all $x, y \in X$. Hence (X, m) is a complete MMS. Consider $A = \{\frac{b}{4}, 0\}$, $B = \{-\frac{b}{4}, 0\}$ and $T : A \cup B \to A \cup B$ defined as $Tx = -x$. Then obviously $TA \subseteq B$ and $TB \subseteq A$ and T is a cyclic mapping. Now for $\alpha = \frac{1}{3}$, $\beta = \frac{1}{3}$ and $k = \frac{1}{\sqrt[3]{2}}$, T is a $(\frac{1}{\sqrt[3]{2}}, \alpha + \beta < 1)$-interpolative Kannan type cyclic contraction. Thus by Theorem 3.2, $0 \in A \cap B$ is a fixed point of T.

To show (2), i.e.:

$$\frac{|x-y|}{b} \le k \left(\frac{|2x|}{b}\right)^{\frac{1}{3}} \left(\frac{|2y|}{b}\right)^{\frac{1}{3}} \qquad x, y \notin FixT = \{0\}$$

we note that, if $x = y$, then it is clear. And if $x = -y = \frac{b}{4}$, then

$$|2x| \le kb^{\frac{1}{3}} |2x|^{\frac{1}{3}} |2x|^{\frac{1}{3}} \iff \left|\frac{b}{2}\right|^{\frac{1}{3}} \le \frac{1}{\sqrt[3]{2}} b^{\frac{1}{3}}.$$

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