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Cascade search for zeros of functionals in quasi-metric and cone metric spaces. Survey of the results

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Abstract

In 2009–2013 the author introduced a concept of an (α, β)-search functional on a metric space. Zero existence theorem was proved for such functionals, and fixed points and coincidence theorems were obtained as consequences. These results generalized several known theorems.

Then this idea was expanded by the author to the more general class of spaces. Here we consider a (b_1, b_2)-quasimetric space and survey some topological properties of such spaces. It turns out that the zero existence problem for (α, β)-search functionals is solved in a (b_1, b_2)-quasimetric space rather similarly, though there are some features concerning the proof of the Cauchy property and convergence of correspondent sequences. Like as in usual metric space, we obtain in (b_1, b_2)-quasimetric space fixed point and coincidence theorems for multivalued mappings as consequences of the zero existence theorem for an (α, β)-search functional. These results also generalize some previous theorems of other authors.

As well, it is of an interest to consider analogous concepts and constructions on a cone-valued metric space, where the metric takes its values in a given cone in a normed space. We consider such cone metric space, investigate its properties and expand the idea of an (α, β)-search functional for such spaces. Moreover, we define a concept of a multivalued (A, B)-search conic function on a cone metric space using positive linear operators A, B, instead of number coefficients (α, β), for the characteristics of a search conic function. Zero existence theorem is proved for multivalued (A, B)-search conic functions. As consequences of this theorem, coincidence and fixed point theorems are presented for multivalued mappings defined on a cone metric space.

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1. Introduction

Zero search principle for so called (α, β)-search functionals on a metric space [1, 2, 3, 4] is a useful investigation method. It is important because it implies a series of fixed point and coincidence theorems both for single-valued and multivalued mappings of metric spaces and theorems on the existence of preimages of a closed subspace of a metric space, under a given mapping, generalizing some known results of other authors.

Let us recall the concept of a multivalued (α, β)-search functional and some other necessary definitions (see [3, 4]).

The convergence and Cauchy property of sequences in $X \times \mathbb{R}_+$ are considered with respect to the metric $D : (X \times \mathbb{R}_+)^2 \to \mathbb{R}_+$, where $D((x', c'), (x'', c'')) := d(x', x'') + |c' - c''|$.

Definition 1.1. Let (X, d) be a metric space, $0 \le \beta < \alpha$. We say a multivalued functional $\Phi : X \rightrightarrows \mathbb{R}_+$ is (α, β)-*search on* X, if for any element $(x, c) \in Gr(\Phi) := \{(x, c) \in X \times \mathbb{R}_+ | c \in \Phi(x)\}$ there is an element $(x', c') \in Gr(\Phi)$ such that $d(x, x') \le \frac{1}{\alpha} c$, and $c' \le \frac{\beta}{\alpha} c$.

Definition 1.2. Let (X, d) be a metric space, $\Phi : X \rightrightarrows \mathbb{R}_+$ be a multivalued (α, β)-search functional on X. We say the graph $Gr(\Phi)$ of the functional Φ is *search-complete*, if every search sequence $\{(x_n, c_n)\} \subseteq Gr(\Phi)$ (that is a sequence with $\rho(x_n, x_{n+1}) \le \frac{c_n}{\alpha}$, $c_{n+1} \le \frac{\beta}{\alpha} \cdot c_n, n \in \{0\} \cup \mathbb{N}$) converges to some element of the graph. We say the graph of an (α, β)-search functional is *search-closed*, if it contains limits of all search sequences.

The following theorem is a slightly amended version of the correspondent theorem from [4].

Theorem 1.3. *Let (X, d) be a metric space, $\Phi : X \rightrightarrows \mathbb{R}_+$ be a multivalued (α, β)-search functional on X with its graph being search-complete, $0 \le \beta < \alpha$. Then, for any pair $(x_0, c_0) \in Gr(\Phi)$, there exists a pair $(\xi, 0) \in Gr(\Phi)$ that is $0 \in \Phi(\xi)$, and $d(x_0, \xi) \le \dfrac{c_0}{\alpha - \beta}$. In addition, it is clear that if $c_0 \le R \cdot (\alpha - \beta)$, it is true that $\xi \in \overline{B_R(x_0)}$.*

2. (α, β)-search functionals on (b_1, b_2)-quasimetric space. Basic concepts and some results

Now, let us turn to the concept of a (b_1, b_2)-quasimetric space. In 1989 Voronezh (Russia) mathematician I.A. Bakhtin introduced a concept of a b-metric space and generalized Banach contraction principle for such spaces [5]. In the paper by 4 authors [6], and in several papers by A.V. Arutyunov and A.V. Greshnov (see [7, 8, 9] and the bibliographies in these papers), there were considered and investigated more general spaces, namely (b_1, b_2)-quasimetric spaces.

Let us give corresponding definitions and some examples.

Definition 2.1. [9] Let b_1, b_2, s be positive numbers, X be a set containing at least two points. A function $\rho : X^2 \to \mathbb{R}_+$ is called an (b_1, b_2)-*quasimetric* on X if the following conditions hold for any $x, y, z \in X$.

$\rho(x, y) \ge 0, \rho(x, y) = 0 \iff x = y$;

$\rho(x, y) \le b_1 \rho(x, z) + b_2 \rho(z, y)$.

If in addition the condition $\rho(x, y) \le s\rho(y, x)$ holds for any $x, y \in X$, we say that the b_1, b_2-quasimetric ρ is *s-symmetric.*

In the case when $s = 1$, X is called *symmetric* (b_1, b_2)-*quasimetric space.*

It is easy to see that the coefficients b_1, b_2 in the previous definition are not less than 1.

A b-metric space is a particular case of a symmetric (b_1, b_2)-quasimetric space, when $b_1 = b_2 = b$. As it was already mentioned above, b-metric spaces were introduced by russian mathematician I.A. Bakhtin from Voronezh in 1989 [5].

It should be also noticed that if $s = b_1 = b_2 = 1$, then ρ is a metric, and (X, ρ) is an ordinary metric space.

Let us also notice that a (b_1, b_2)-quasimetric (and the corresponding space (X, ρ)) is called *weakly symmetric* if $\underline{lim}_{n\to\infty}\, \rho(x_n, \xi) = 0$ implies that $\underline{lim}_{n\to\infty}\, \rho(\xi, x_n) = 0$.

We remark that for s-symmetric (b_1, b_2)-quasimetric space it is true that

$$\underset{n\to\infty}{\underline{lim}}\, \rho(x_n, \xi) = 0 \Longleftrightarrow \underset{n\to\infty}{\underline{lim}}\, \rho(\xi, x_n) = 0.$$

So, every s-symmetric (b_1, b_2)-quasimetric space is weakly symmetric. But it is known that the converse fails (see [6], Example 2.2).

Quasimetric spaces are studied in topology, functional and metric analysis, and have some applications in optimization and approximation theory, and in convex analysis.

The authors of the paper [9] give the following nontrivial example of a (b_1, b_2)-quasimetric space, where $b_1 \neq b_2$. The space $L_p(E)$ is considered, where E is a measurable bounded set in $\mathbb{R}^n$, and $0 < p < 1$.

Such spaces are quasinormed, because it is true that $||f_1 + f_2||_p \leq 2^{-1/p'}(||f_1||_p + ||f_2||_p)$ for all $f_1, f_2 \in L_p(E)$, where $p' < 0$ is the number conjugate to p, that is $1/p + 1/p' = 1$.

Moreover, for every $\varepsilon > 0$, it is true that

$$||f_1 + f_2||_p \leq (1 + \varepsilon)||f_1||_p + C(\varepsilon, p)||f_2||_p, \forall f_1, f_2 \in L_p(E),$$

where $C(\varepsilon, p) = (1-(1+\varepsilon)^{p'})^{1/p'}$. Thus, $L_p(E)$ with $0 < p < 1$ is a symmetric (b_1, b_2)-quasimetric space, and the quasimetric $\rho_{L_p(E)}$ is given by the formula $\rho_{L_p(E)}(f_1, f_2) := ||f_1 - f_2||_p$, $b_1 = b_2 = 2^{1/p'}$. Moreover, for every $\varepsilon > 0$, the (b_1, b_2)-generalized triangle inequality holds in $L_p(E)$ provided that $b_1 = 1+\varepsilon, b_2 = C(\varepsilon, p)$.

It is noticed in [6] that similar properties hold for the spaces l_p with $0 < p < 1$.

As well, in analysis and geometry there are other examples of (b_1, b_2)-quasimetric spaces with $b_1 \neq b_2$.

Definition 2.2. In a (b_1, b_2)-quasimetric space (X, ρ), a sequence $\{x_k\}$ is called *convergent to* $a \in X$, if $\rho(x_k, a) \underset{k\to\infty}{\longrightarrow} 0$. We say a sequence $\{x_k\}$ in (X, ρ) is *Cauchy* if, for any $\varepsilon > 0$ there exists a number $N = N(\varepsilon) \in \mathbb{N}$, such that for all $n, m \in \mathbb{N}, m \geq n > N$, it is true that $\rho(x_n, x_m) < \varepsilon$.

Like as for an ordinary metric space, a (b_1, b_2)-quasimetric space (X, ρ) is *complete* if any Cauchy sequence has at least one limit in X.

It should be mentioned that in a weakly symmetric (b_1, b_2)-quasimetric space (X, ρ) any convergent sequence has a unique limit.

Now, let us consider a zero search theorem for a multivalued (α, β)-search functional in (b_1, b_2)-quasimetric space.

Let (X, ρ) be a (b_1, b_2)-quasimetric space, and $0 \leq \beta < \alpha$.

The following definition of an (α, β)-search multivalued functional $\varphi : X \rightrightarrows \mathbb{R}_+$ on a (b_1, b_2)-quasimetric space (X, ρ) is quite the same as on a usual metric space (see definition 1.1).

Definition 2.3. A multivalued functional $\varphi : X \rightrightarrows \mathbb{R}_+$ is called (α, β)-search if for any $x \in X$ and any $c \in \varphi(x)$ there is $x' \in X, c' \in \varphi(x')$ such that $\rho(x, x') \leq \frac{c}{\alpha}$, $c' \leq \frac{\beta}{\alpha} \cdot c$.

In addition, concepts of a search sequence and of a search-complete space, for a (b_1, b_2)-quasimetric space with a given (α, β)-search multivalued functional, are also the same as for an ordinary metric space (see above definition 1.2).

Theorem 2.4. *Let (X,ρ) be a (b_1,b_2)-quasimetric space, and $\varphi : X \rightrightarrows \mathbb{R}_+$ be a multivalued (α,β)-search functional. We suppose that X is a search complete (b_1,b_2)-quasimetric space and the graph* Graph(φ) *is* 0*-closed. Then, for any point $x_0 \in X$ and any $c_0 \in \varphi(x_0)$ one can find a point $\xi \in \mathrm{Nil}(\varphi)$, such that*

$$\underline{\lim}_{\gamma\to\xi} \rho(x_0,\gamma) \le \frac{(b_1)^2 Q(b_2\frac{\beta}{\alpha}, k_0-1) + b_1(b_2\frac{\beta}{\alpha})^{k_0-1}}{\alpha(1-b_2(\frac{\beta}{\alpha})^{k_0})} c_0, \tag{1}$$

where $Q(b_2\frac{\beta}{\alpha},k) := 1 + b_2\frac{\beta}{\alpha} + (b_2\frac{\beta}{\alpha})^2 + ... + (b_2\frac{\beta}{\alpha})^{k-1}$.

In addition, if the quasimetric ρ is lower semicontinuous in the second variable, the estimate (1) *is true for $\rho(x_0,\xi) = \underline{\lim}_{\gamma\to\xi} \rho(x_0,\gamma)$.*

Proof. The reasonings are quite similar to the proof of the zero search theorem for a single-valued (α,β)-search functional in a (b_1,b_2)-quasimetric space [10]. Nevertheless, for the completeness, we give here the complete proof.

We shall take into account the specifics of a (b_1,b_2)-quasimetric ρ which tell it from a usual metric. As in the case of an usual metric, one can construct a sequence of points $\{x_m\}_{m=0,1,2,...}$ in X, meeting the following conditions:

(a) $\rho(x_m,x_{m+1}) \le \frac{c_m}{\alpha}$;

(b) $c_{m+1} \le \frac{\beta}{\alpha} c_m$.

If a point x_m has already been chosen, and $c_m = 0$, we put $x_j = x_m$ for all $j > m$. If $c_m > 0$, then by the conditions of the theorem and the definition of an (α,β)-search functional φ (relative to the (b_1,b_2)-quasimetric ρ) there exist a point x_{m+1} and a value c_{n+1}, satisfying the conditions (a) and (b).

Now, we need to prove the Cauchy property of the constructed sequence $\{x_m\}_{m=0,1,2,...}$, with respect to the quasimetric ρ. Below we give the standart arguments. Similar reasonings are contained in [9] (see also the papers [7, 8] where analogous sequences are considered).

At first we mention that conditions (a) and (b) imply the following inequalities

$$\rho(x_m,x_{m+1}) \le \frac{c_m}{\alpha} \le \frac{1}{\alpha}(\frac{\beta}{\alpha})^m c_0, \quad m \ge 1. \tag{2}$$

It follows that

$$\rho(x_n,x_m) \le b_1\rho(x_n,x_{n+1}) + b_2\rho(x_{n+1},x_m) \le b_1\rho(x_n,x_{n+1}) + b_2(b_1\rho(x_{n+1},x_{n+2}) + b_2\rho(x_{n+2},x_m)) =$$
$$= b_1\rho(x_n,x_{n+1}) + b_2b_1\rho(x_{n+1},x_{n+2}) + (b_2)^2\rho(x_{n+2},x_m)) \le ... \le b_1\rho(x_n,x_{n+1}) +$$
$$+ b_2b_1\rho(x_{n+1},x_{n+2}) + b_1(b_2)^2\rho(x_{n+2},x_{n+3}) + ... + (b_2)^{m-n-2}b_1\rho(x_{m-2},x_{m-1}) + (b_2)^{m-1}\rho(x_{m-1},x_m).$$

Using the inequalities (2), we have:

$$\alpha\rho(x_n,x_m) \le b_1c_0(\frac{\beta}{\alpha})^n[1 + b_2\frac{\beta}{\alpha} + (b_2\frac{\beta}{\alpha})^2 + ...(b_2\frac{\beta}{\alpha})^{m-n-2} + \frac{1}{b_1}(b_2\frac{\beta}{\alpha})^{m-n-1}].$$

As above, we denote $Q(b_2\frac{\beta}{\alpha},k) := 1 + b_2\frac{\beta}{\alpha} + (b_2\frac{\beta}{\alpha})^2 + ... + b_2\frac{\beta}{\alpha})^{k-1}$. Then

$$\alpha\rho(x_n,x_m) \le b_1c_0(\frac{\beta}{\alpha})^n[Q(b_2\frac{\beta}{\alpha}, m-n-1) + \frac{1}{b_1}(b_2\frac{\beta}{\alpha})^{m-n-1}]. \tag{3}$$

For any $k \in \mathbb{N}$, we denote $\Phi(k) := Q(b_2\frac{\beta}{\alpha}, k-1) + \frac{1}{b_1}(b_2\frac{\beta}{\alpha})^{k-1}$, and put $\Phi(0) = 0$.

Then we have from (3) that

$$\alpha\rho(x_n,x_m) \le b_1c_0(\frac{\beta}{\alpha})^n\Phi(m-n). \tag{4}$$

It was mentioned above that as $\frac{\beta}{\alpha} < 1$ and hence $(\frac{\beta}{\alpha})^k \to 0$, $k\to\infty$, it follows that there is the least natural number k_0, such that for any $k \ge k_0$ it is true that $b_2(\frac{\beta}{\alpha})^k < 1$. Using this number k_0 one can divide the part of the constructed sequence between points x_n and x_m into "peaces" in which of them the difference of numbers is nongreater than k_0. After that we obtain the following estimates, for any $n, m \in \mathbb{N}, m > n > N$. Let $m - n = qk_0 + r$, where r be the residue of the division $m - n$ by k_0, $0 \le r < k_0$. Then

$$\rho(x_n, x_m) \le b_1\rho(x_n, x_{n+k_0}) + b_2\rho(x_{n+k_0}, x_m) \le b_1\rho(x_n, x_{n+k_0}) + b_2b_1\rho(x_{n+k_0}, x_{n+2k_0}) + (b_2)^2\rho(x_{n+2k_0}, x_m) \le ...$$

$$... \le b_1\rho(x_n, x_{n+k_0}) + b_2b_1\rho(x_{n+k_0}, x_{n+2k_0}) + (b_2)^2 b_1\rho(x_{n+2k_0}, x_{n+3k_0}) + ...$$

$$... + (b_2)^{q-1} b_1\rho(x_{n+(q-1)k_0}, x_{n+qk_0}) + (b_2)^q\rho(x_{n+qk_0}, x_m). \tag{5}$$

We apply the estimates of the form (4) to every summand of the inequalities (5). One can notice that

$$\alpha\rho(x_{n+jk_0}, x_{n+(j+1)k_0}) \le b_1c_0(\frac{\beta}{\alpha})^{n+jk_0}\Phi(k_0), j = 0, 1, ..., q-1.$$

In addition,

$$\alpha\rho(x_{n+qk_0}, x_m) \le b_1c_0(\frac{\beta}{\alpha})^{n+qk_0}\Phi(r).$$

So, it follows from (5) that

$$\alpha\rho(x_n, x_m) \le b_1b_1c_0(\frac{\beta}{\alpha})^n\Phi(k_0) + b_2b_1b_1c_0(\frac{\beta}{\alpha})^{n+k_0}\Phi(k_0) + (b_2)^2b_1b_1c_0(\frac{\beta}{\alpha})^{n+2k_0}\Phi(k_0) + ...$$

$$... + b_1(b_2)^{q-1}b_1c_0(\frac{\beta}{\alpha})^{n+(q-1)k_0}\Phi(k_0) + b_2^q b_1c_0(\frac{\beta}{\alpha})^{n+qk_0}\Phi(r).$$

Further we have (using the inequality $b_2(\frac{\beta}{\alpha})^{k_0} < 1$):

$$\alpha\rho(x_n, x_m) \le (b_1)^2c_0\Phi(k_0)(\frac{\beta}{\alpha})^n[1 + b_2(\frac{\beta}{\alpha})^{k_0} + (b_2)^2(\frac{\beta}{\alpha})^{2k_0} + ...$$

$$... + (b_2)^{q-1}(\frac{\beta}{\alpha})^{(q-1)k_0}] + b_2^q b_1c_0(\frac{\beta}{\alpha})^{n+qk_0}\Phi(r) =$$

$$= (b_1)^2c_0(\frac{\beta}{\alpha})^n\Phi(k_0)Q(b_2(\frac{\beta}{\alpha})^{k_0}, q) + b_2^q b_1c_0(\frac{\beta}{\alpha})^{n+qk_0}\Phi(r) \le$$

$$\le (b_1)^2c_0(\frac{\beta}{\alpha})^n\frac{\Phi(k_0)}{1 - b_2(\frac{\beta}{\alpha})^{k_0}} + b_2^q b_1c_0(\frac{\beta}{\alpha})^{n+qk_0}\Phi(r) =$$

$$= (b_1)^2c_0(\frac{\beta}{\alpha})^n[\frac{\Phi(k_0)}{1 - b_2(\frac{\beta}{\alpha})^{k_0}} + b_2^q(\frac{\beta}{\alpha})^{qk_0}(b_1)^{-1}\Phi(r)] \le (b_1)^2c_0(\frac{\beta}{\alpha})^n[\frac{\Phi(k_0)}{1 - b_2(\frac{\beta}{\alpha})^{k_0}} + (b_1)^{-1}\Phi(r)].$$

So, finally we obtain

$$\alpha\rho(x_n, x_m) \le (b_1)^2c_0(\frac{\beta}{\alpha})^n[\frac{\Phi(k_0)}{1 - b_2(\frac{\beta}{\alpha})^{k_0}} + (b_1)^{-1}\Phi(r)]. \tag{6}$$

As $(\frac{\beta}{\alpha})^n \to 0$ when $n \to \infty$, and the rest expression in the right part of (6) is bounded, we have $\rho(x_n, x_m) \to 0$ when $n \to \infty$. Therefore, the sequence $\{x_m\}$ is Cauchy, hence it converges in the search-complete space (X, ρ) to some element $\xi \in X$. Since $c_m \to 0$, we have $\varphi(\xi) = 0$ by the conditions of the theorem, that is $\xi \in \mathrm{Nil}(\varphi)$.

Further, under $n = 0$ the inequality (6) implies the following inequality:

$$\alpha\rho(x_0, x_m) \le (b_1)^2c_0[\frac{\Phi(k_0)}{1 - b_2(\frac{\beta}{\alpha})^{k_0}} + (b_1)^{-1}\Phi(r)]. \tag{7}$$

If m is divisible by k_0, $r = 0$, hence $\Phi(r) = 0$, and the inequality (7) takes the form

$$\alpha\rho(x_0, x_m) \le (b_1)^2c_0\frac{\Phi(k_0)}{1 - b_2(\frac{\beta}{\alpha})^{k_0}}. \tag{8}$$

We consider the subsequence $\{x_{jk_0}\}$ of the above constructed sequence $\{x_m\}$, not supposing that the metric ρ is semicontinuous in one of its variables. It's clear that this subsequence converges to ξ, so we obtain from (8):

$$\varliminf_{\gamma\to\xi} \rho(x_0,\gamma) \le \lim_{j\to\infty} \rho(x_0, x_{jk_0}) \le \alpha^{-1}(b_1)^2 c_0 \frac{\Phi(k_0)}{1-b_2(\frac{\beta}{\alpha})^{k_0}} = \alpha^{-1}(b_1)^2 c_0 \frac{Q(b_2\frac{\beta}{\alpha}, k_0-1) + \frac{1}{b_1}(b_2\frac{\beta}{\alpha})^{k_0-1}}{1-b_2(\frac{\beta}{\alpha})^{k_0}} =$$

$$= \frac{\alpha^{k_0-1}(b_1)^2 Q(b_2\frac{\beta}{\alpha}, k_0-1) + b_1(b_2\beta)^{k_0-1}}{\alpha^{k_0} - b_2\beta^{k_0}} c_0.$$

□

Now, basing on the proved zero existence theorem, we can suggest the following coincidence theorem.

Let $(X,\rho), (Y,\nu)$ be (b_1,b_2)-quasimetric spaces. In the product $Y^n = Y \times ... \times Y$ we consider a metric $\hat{\nu}$, where $\hat{\nu}(y,z) := \sum_{k=1}^{n} \nu(y_k, z_k)$ for any $y=(y_1,...,y_n), z=(z_1,...,z_n) \in Y^n$. We denote by $C(Y)$ the totality of closed subsets of Y, and Δ_n stands for the main diagonal in Y^n that is

$$\Delta_n = \{y=(y_1,...,y_n) \in Y^n | y_1 = ... = y_n\}.$$

Theorem 2.5. *(see also [10]) Let in the described situation $F_1,...,F_n : X \rightrightarrows C(Y)$, $F = F_1\times ...\times F_n : X \rightrightarrows C(Y^n)$, and the graph* Graph$(F)$ *of the mapping F be Δ_n-closed that is all limit points $(x,y) \in X\times Y^n$ of the graph* Graph(F)*, with $y\in\Delta_n$, be contained in* Graph(F)*. In addition, we suppose that at least one of the graphs* Graph(F_i)*, $i = 1,...,n$, is complete.*

Let numbers α,β,γ, $\gamma>0, 0<\beta<\alpha$, be such that for any $x\in X$, $y\in F(x)$ there exist points $x'\in X$, $y'\in F(x')$, such that $\rho(x,x') \le \frac{\hat{\nu}(y,\Delta_n)}{\alpha}$, $\hat{\nu}(y,y') \le \gamma\cdot\hat{\nu}(y,\Delta_n)$, and $\hat{\nu}(y',\Delta_n) \le \frac{\beta}{\alpha}\cdot\hat{\nu}(y,\Delta_n)$.

Then, for any point $(x_0,y_0) = (x_0,(y_{10},...,y_{n0})) \in$ Graph(F), that is $y_{i0}\in F_i(x_0), i=1,...,n$, there exists a convergent sequence $\{(x_k,y_k)\} = \{(x_k,(y_{1k},...,y_{nk}))\}_{k=0,1,...}$, where $x_k \underset{m\to\infty}{\longrightarrow} \xi$, $y_{ik} \underset{m\to\infty}{\longrightarrow} \eta \in Y$, $i=1,...,n$, and the point $(\xi,(\eta,...,\eta)) \in$ Graph(F), that is $\xi\in$ Coin$(F_1,...,F_n)$, $\eta\in F_1(\xi)\cap...\cap F_n(\xi)$. In addition, the following estimates are valid:

$$\varliminf_{w\to\xi} \rho(x_0,w) \le \frac{(b_1)^2 Q(b_2\frac{\beta}{\alpha}, k_0-1) + b_1(b_2\frac{\beta}{\alpha})^{k_0-1}}{\alpha(1-b_2(\frac{\beta}{\alpha})^{k_0})} \hat{\nu}(y_0,\Delta_n).$$

$$\varliminf_{v\to\eta} \rho(y_0,v) \le \gamma \frac{(b_1)^2 Q(b_2\frac{\beta}{\alpha}, k_0-1) + b_1(b_2\frac{\beta}{\alpha})^{k_0-1}}{(1-b_2(\frac{\beta}{\alpha})^{k_0})} \hat{\nu}(y_0,\Delta_n).$$

It is not difficult to see that this theorem is a consequence of theorem 2.4, for the functional $\varphi : X \rightrightarrows Y^n$, where $\varphi(x) := \{c = \hat{\nu}(y,\Delta_n) | y\in F(x)\}$. So, we do not give the proof here.

Recently, A.V. Arutyunov and A.V. Greshnov proved a coincidence theorem for two multivalued mappings between (b_1,b_2)-quasimetric spaces $(X,\rho),(Y,\nu)$ [7, 8, 9].

Before its formulation, we need to give necessary definitions.

We consider two multivalued mappings $F,G : X \rightrightarrows Y$, where $F(x), G(x)$ are non-empty closed subsets in Y, for any $x\in X$. Suppose numbers α,β be given, with $0\le\beta<\alpha$.

Definition 2.6. A point $x\in X$ is called *a coincidence point* of multi-valued mappings F,G if $x\in Coin(F,G)$, that is $F(x)\cap G(x) \ne \emptyset$.

Definition 2.7. A multivalued mapping $F : X \rightrightarrows Y$ is said to be α-covering if, for any $x\in X$ and any $r>0$, it is true that the neighbourhood $U_{\alpha r}(F(x))$ of the set $F(x)$ of radius αr is covered by the image $F(B_r(x))$ under the mapping F of the ball $B_r(x)$ centered in x of radius r. In other words,

$$U_{\alpha r}(F(x)) := \bigcup_{y\in F(x)} B_{\alpha r}(y) \subseteq F(B_r(x)).$$

Definition 2.8. A multi-valued mapping G is said to be β-Lipschitz if for any $x_1, x_2 \in X$ the following inequality holds.

$$H(G(x_1, G(x_2)) \leq \beta\rho(x_1, x_2)$$

Here $H(A, B)$ stands for the Hausdorff (generalized) $(\hat{b}_1, \hat{b}_2)$-quasimetric between closed subsets $A, B \subset Y$ relative to the given (b_1, b_2)-quasimetric ν on Y. One should notice that in general the constants $\hat{b}_1, \hat{b}_2$ might be different from b_1, b_2 (see the detailed definition of the Hausdorff (generalized) $(\hat{b}_1, \hat{b}_2)$-quasimetric $H(A, B)$ in [7, 8].

In what follows we consider (b_1, b_2)-quasimetric spaces $(X, \rho), (Y, \nu)$. We put $m_0 = min\{j \in \mathbb{N} | b_2\beta^j < \alpha^j\}$, and under the assumption that $q_0^2\beta < \alpha$, we put $n_0 = min\{j \in \mathbb{N} | b_1(q_0^2\beta)^j < \alpha^j\}$.

Theorem 2.9. *([7, theorem 1], [8, theorem 5.7], [9, theorem 3.2]) Consider two multivalued mappings $F, G : X \rightrightarrows Y$. Let F be an α-covering multivalued mapping with closed graph $Graph(F)$, and G be a β-Lipschitz multivalued mapping, $0 \leq \beta < \alpha$. Suppose also that at least one of the graphs $Graph(F)$, $Graph(G)$ is a complete subspace in Y.*

Then, for an arbitrary point $x_0 \in X$ and arbitrary positive number $\varepsilon > 0$, the mappings F, G have a coincidence point $\xi \in X$ such that

$$\underline{\lim}_{\eta \to \xi} \rho(x_0, \eta) \leq \frac{b_1^2 \alpha^{m_0 - 1} S(b_2 \frac{\beta}{\alpha}, m_0 - 1) + b_1(b_2\beta)^{m_0 - 1}}{\alpha^{m_0} - b_2\beta^{m_0}} \nu(F(x_0), G(x_0)) + \varepsilon. \tag{9}$$

If the space (X, ρ) is weakly symmetric, and there is the limit uniqueness, then from (9) one can obtain the following inequality for ξ.

$$\rho(x_0, \xi) \leq b_1 \frac{b_1^2 \alpha^{m_0 - 1} S(b_2 \frac{\beta}{\alpha}, m_0 - 1) + b_1(b_2\beta)^{m_0 - 1}}{\alpha^{m_0} - b_2\beta^{m_0}} \nu(F(x_0), G(x_0)) + \varepsilon. \tag{10}$$

If the space (X, ρ) is q_0-symmetric, with $q_0^2\beta < \alpha$, then besides (9) and (10), ξ also satisfies the following inequalities (11) and (12).

$$\bar{\rho}(x_0, \xi) \leq q_0 b_2^2 \frac{b_2 \alpha^{n_0 - 1} S(b_1 q_0^2 \frac{\beta}{\alpha}, n_0 - 1) + (b_1 q_0^2 \beta)^{n_0 - 1}}{\alpha^{n_0} - b_1(q_0^2\beta)^{n_0}} \nu(F(x_0), G(x_0)) + \varepsilon, \tag{11}$$

$$\underline{\lim}_{\eta \to \xi} \bar{\rho}(x_0, \eta) \leq q_0 b_2 \frac{b_2 \alpha^{n_0 - 1} S(b_1 q_0^2 \frac{\beta}{\alpha}, n_0 - 1) + (b_1 q_0^2 \beta)^{n_0 - 1}}{\alpha^{n_0} - b_1(q_0^2\beta)^{n_0}} \nu(F(x_0), G(x_0)) + \varepsilon. \tag{12}$$

In the case when $X = Y$ is complete, $F = Id_X$, $0 \leq \beta < 1$, and G is a β-Lipschitz multivalued mapping, the previous theorem turns to the fixed point theorem for a multivalued contraction mapping G. We do not give here this formulation.

Below we demonstrate the connection between theorems 2.4, 2.5 and 2.9.

Before this, we need the following auxiliary statement.

Lemma 1. *In an s-symmetric (b_1, b_2)-quasimetric space Y, for any $y = (y_1, y_2) \in Y^2$, the following estimate holds.*

$$\hat{\nu}(y, \Delta_2) \leq \frac{1+s}{2} \nu(y_1, y_2) \leq \frac{1+s}{2} \bar{b} \hat{\nu}(y, \Delta_2),$$

where $\bar{b} = max\{b_1, sb_2)$.

Proof. For any $z \in Y$, $\nu(y_1, y_2) \leq b_1\nu(y_1, z) + b_2\nu(z, y_2) \leq b_1\nu(y_1, z) + b_2 s\nu(y_2, z) \leq \bar{b}(\nu(y_1, z) + \nu(y_2, z)) = \bar{b}\hat{\nu}(y, (z, z))$, where $\bar{b} = max\{b_1, sb_2\}$. As $z \in Y$ is an arbitrary element, it follows that $\nu(y_1, y_2) \leq \bar{b} \cdot \hat{\nu}(y, \Delta_2)$. On the other hand, $\hat{\nu}(y, \Delta_2) \leq \frac{1}{2}[\hat{\nu}((y_1, y_2), (y_1, y_1)) + \hat{\nu}((y_1, y_2), (y_2, y_2))] \leq = \frac{1+s}{2}\nu(y_1, y_2)$. So, we have $\hat{\nu}(y, \Delta_2) \leq \frac{1+s}{2}\nu(y_1, y_2) \leq \frac{1+s}{2}\bar{b}\hat{\nu}(y, \Delta_2)$. □

Theorem 2.10. *The main statement of theorem 2.9 with inequality (9) follows from theorem 2.4 and from the version of theorem 2.5, in the case when $n = 2$ and the functional $\varphi : X \rightrightarrows Y^n$, $\varphi(x) := \{c \in \mathbb{R}_+ | \exists y = (y_1, y_2), y_1 \in F_1(x), y_2 \in F_2(x), \hat{\nu}(y, \Delta_n) = c\}$ is replaced with the functional $\psi : X \rightrightarrows \mathbb{R}_+$, $\psi(x) := \{c \in R_+ | \exists y = (y_1, y_2), y_1 \in F(x), y_2 \in G(x), \nu(y_1, y_2) = c\}$.*

Proof. Let $(X, \rho), (Y, \nu)$ be (b_1, b_2)-quasimetric s-symmetric spaces, $\bar{b} = max\{b_1, sb_2)$, and $F_1, F_2 : X \rightrightarrows Y$ be given multivalued mappings satisfying the conditions of theorem 2.9. It means that the mapping F_1 is $\tilde{\alpha}$-covering and closed (that is its graph is closed), the mapping F_2 is $\tilde{\beta}$-Lipshitsz, $0 < \tilde{\beta} < \tilde{\alpha}$, and in addition, at least one of the graphs $\mathrm{Graph(F_1)}, \mathrm{Graph(F_2)}$ is complete. We show that then all conditions of theorem 2.5, for $n = 2$, are fulfilled for the multivalued functional $\psi : X \rightrightarrows \mathbb{R}_+$ where $\psi(x) = \{c \in \mathbb{R}_+ | \exists y \in F(x) = (y_1, y_2) \in (F_1, F_2)(x), \nu(y_1, y_2) = c\}$, instead of the functional φ.

At first, we show that the graph $\mathrm{Graph}(F_1 \times F_2)$ of the mapping $F = (F_1, F_2)$ is closed. Indeed, as the mapping F_2 is $\tilde{\beta}$-Lipshitsz, it is sequentially upper semicontinuous. By this reason, it is true for any convergent sequence $\{(x_k, y_{2k})\}_{k=0,1,\dots} \subseteq \mathrm{Graph}(F_2)$ that $\lim\limits_{k\to\infty} \nu(y_{2k}, F_2(\xi)) = 0$, if $\xi = \lim\limits_{k\to\infty} x_k$. Consequently, $\lim\limits_{k\to\infty} (x_k, y_{2k}) \in \mathrm{Graph}(F_2)$. So, $\mathrm{Graph}(F_2)$ is closed, hence $\mathrm{Graph}(F) = \mathrm{Graph}(F_1 \times F_2)$ is also closed.

Now, let $x \in X$ be an arbitrary point and $y = (y_1, y_2) \in F(x) = (F_1, F_2)(x)$. As the mapping F_1 is $\tilde{\alpha}$-covering, there exists a point $x' \in X$, such that $\rho(x, x') \leq \frac{\nu(y_1, y_2)}{\tilde{\alpha}}$, and $y_2 \in F_1(x') \cap F_2(x)$.

Then, as F_2 is $\tilde{\beta}$-Lipshitsz mapping, we have

$$\nu(y_2, F_2(x')) \leq H(F_2(x), F_2(x')) \leq \tilde{\beta} \cdot \rho(x, x') < (\tilde{\beta} + \tilde{\alpha} \cdot \delta) \cdot \rho(x, x'), \tag{13}$$

where δ is a positive number, such that $\tilde{\beta} + \tilde{\alpha} \cdot \delta < \tilde{\alpha}$. That is $0 < \delta < 1 - \frac{\tilde{\beta}}{\tilde{\alpha}}$. Here above

$$H(F_2(x), F_2(x')) = \max\{ \sup_{y \in F_2(x')} \{\nu(y, F_2(x)\}, \sup_{z \in F_2(x)} \{\nu(z, F_2(x')\}\}$$

is the Hausdorff quasimetric defined by the given quasimetric ν.

It follows from (13), that there exists a point $y_3 \in F_2(x')$, such that $\nu(y_2, y_3) \leq (\tilde{\beta} + \tilde{\alpha}\delta) \cdot \rho(x, x')$. When denoting $y' = (y_2, y_3) \in (F_1 \times F_2)(x')$, we have

$$\hat{\nu}(y, y') = \nu(y_1, y_2) + \nu(y_2, y_3) \leq \nu(y_1, y_2) + (\tilde{\beta} + \tilde{\alpha}\delta) \cdot \rho(x, x') \leq$$

$$\leq \nu(y_1, y_2) + (\tilde{\beta} + \tilde{\alpha}\delta) \frac{\nu(y_1, y_2)}{\tilde{\alpha}} = (1 + \frac{\tilde{\beta} + \tilde{\alpha}\delta}{\tilde{\alpha}}) \nu(y_1, y_2) = \gamma \cdot \nu(y_1, y_2), \gamma = (1 + \frac{\tilde{\beta} + \tilde{\alpha}\delta}{\tilde{\alpha}}).$$

When denoting $c' = \nu(y_2, y_3)$, we also have

$$c' \leq (\tilde{\beta} + \tilde{\alpha}\delta) \cdot \rho(x, x') \leq (\tilde{\beta} + \tilde{\alpha}\delta) \frac{\nu(y_1, y_2)}{\tilde{\alpha}} = \frac{\tilde{\beta} + \tilde{\alpha}\delta}{\tilde{\alpha}} \nu(y_1, y_2) = \frac{\tilde{\beta} + \tilde{\alpha}\delta}{\tilde{\alpha}} \cdot c.$$

Now, consider the multivalued functional $\psi : X \rightrightarrows \mathbb{R}_+$, where $\psi(x) = \{c \in \mathbb{R}_+ | \exists y = (y_1, y_2) \in F(x), \nu(y_1, y_2) = c\}$. Let it be $\alpha = \tilde{\alpha}$, $\beta = \tilde{\beta} + \tilde{\alpha}\delta$, $\gamma = 1 + \frac{\beta}{\alpha}$.

We have obtained that, for any point $x \in X$ and any value $c = \nu(y_1, y_2) \in \psi(x)$, there exist a point $x' \in X$ and a value $c' \in \psi(x')$, such that $\rho(x, x') \leq \frac{c}{\alpha}$, $c' \leq \frac{\beta}{\alpha}$.

So, one can see that all conditions of theorem 2.4 are fulfilled, for (α, β)-search multivalued functional ψ.

The connection between multivalued functionals $\varphi, \psi : X \rightrightarrows \mathbb{R}_+$, $\varphi(x) = \{c \in \mathbb{R}_+ | \exists y = (y_1, y_2) \in F(x), \hat{\nu}(y, \Delta_2) = c\}$ is shown in lemma 1. In the case when X is a metric space, these functionals coincide (for $n = 2$). So, by virtue of theorem 2.4, for any point $x_0 \in X$

$$\varliminf_{w \to \xi} \rho(x_0, w) \leq \frac{(b_1)^2 Q(b_2 \frac{\beta}{\alpha}, k_0 - 1) + b_1 (b_2 \frac{\beta}{\alpha})^{k_0 - 1}}{\alpha(1 - b_2 (\frac{\beta}{\alpha})^{k_0})} c_0.$$

$$\varliminf_{v\to\eta}\rho(y_0,v)\le\gamma\frac{(b_1)^2Q(b_2\frac{\beta}{\alpha},k_0-1)+b_1(b_2\frac{\beta}{\alpha})^{k_0-1}}{(1-b_2(\frac{\beta}{\alpha})^{k_0})}c_0.$$

We notice that

$$c_0=\nu(y_{01},y_{02})=\nu(F_1(x_0),F_2(x_0))\cdot(1+\eta),$$

where $\eta>0$. It is clear that depending on the choice of $y_0=(y_{01},y_{02})$, the number η may be made arbitrarily small. Then we have

$$\varliminf_{w\to\xi}\rho(x_0,w)\le\frac{(b_1)^2Q(b_2\frac{\beta}{\alpha},k_0-1)+b_1(b_2\frac{\beta}{\alpha})^{k_0-1}}{\alpha(1-b_2(\frac{\beta}{\alpha})^{k_0})}\nu(F_1(x_0),F_2(x_0))\cdot(1+\eta)=.$$

$$=\frac{b_1^2Q(b_2\frac{\beta}{\alpha},k_0-1)+b_1(b_2\frac{\beta}{\alpha})^{k_0-1}}{\alpha(1-b_2(\frac{\beta}{\alpha})^{k_0}}\nu(F(x_0),G(x_0))+\varepsilon.$$

Here

$$\varepsilon=\frac{(b_1)^2Q(b_2\frac{\beta}{\alpha},k_0-1)+b_1(b_2\frac{\beta}{\alpha})^{k_0-1}}{\alpha(1-b_2(\frac{\beta}{\alpha})^{k_0})}\nu(F_1(x_0),F_2(x_0))\cdot(1+\eta)-$$

$$-\frac{b_1^2\alpha^{k_0-1}Q(b_2\frac{\beta}{\alpha},k_0-1)+b_1(b_2\beta)^{k_0-1}}{\alpha^{k_0}-b_2\beta^{k_0}}\nu(F(x_0),G(x_0))=$$

$$\frac{(b_1)^2Q(b_2\frac{\beta}{\alpha},k_0-1)+b_1(b_2\frac{\beta}{\alpha})^{k_0-1}}{\alpha(1-b_2(\frac{\beta}{\alpha})^{k_0})}[\nu(F(x_0),G(x_0))\cdot\eta] \tag{14}$$

As it was mentioned above, depending on the choice of $y_0=(y_{01},y_{02})\in(F_1(x_0),F_2(x_0))=F(x_0)$, the number $\eta=\nu(y_{01},y_{02})-\nu(F_1(x_0),F_2(x_0))$ may be made arbitrarily small. Hence, one can see from (14) that the number ε may be also made arbitrarily small.

So, the estimate (9) is proved. We don't consider here the derivation of estimates (10)–(12). □

It should be noticed that the statement similar to theorem 2.10 was given in [10]. Unfortunately there are misprints at the end of its proof given in that paper.

3. (A,B)-search conic functions on a normed-space-valued cone metric space. Basic concepts and results

In this section we introduce a concept of a cone (conic) function with operator coefficients on a cone metric space. The exposition of basic results here follows the author's paper [14]. Below we prove a zero existence theorem for such functions. On this basis, a fixed point theorem for a multivalued self-mapping of a cone metric space is obtained, which generalizes the recent fixed point theorem by E.S. Zhukovskiy and E.A. Panasenko [13], for a contraction multivalued mapping of a cone metric space, with an operator contracting coefficient. In addition, we prove coincidence theorems for two multivalued mappings between cone metric spaces, which generalize the author's previous results on coincidences of two multivalued mappings of usual metric spaces.

Let X be a nonempty set and E be a Banach space, θ be a trivial element of E. Let K be a positive convex closed acute cone, that is a closed convex subset satisfying the following conditions:

$$1)\,\theta\in K;\quad 2)\,\forall a\in K,a\ne\theta\Longrightarrow\forall\mu>0,\mu a\in K,-\mu a\notin K;\quad 3)\,\forall a,b\in K\Longrightarrow\forall t\in[0;1],ta+(1-t)b\in K.$$

The cone K defines a partial order on E. Namely, for any $a,b\in E$ $(a\le_K b)\Longleftrightarrow(b-a\in K)$. Thus, (E,K) is a given partially ordered set.

Definition 3.1. The cone metric on X associated with the cone K is a mapping $d_K:X^2\to E$ satisfying the axioms of a usual metric, that is for any $x,y\in X$, the following conditions hold.

1. $d_K(x, y) \geq_K \theta; (d_K(x, y) = \theta) \iff (x = y)$;
2. $d_K(x, y) = d_K(y, x)$;
3. for every $z \in X$ it is true that $d_K(x, y) \leq_K d_K(x, z) + d_K(z, y)$.

Suppose that the set $X \neq \emptyset$ is equipped with a cone metric d_K as described above. The space (X, d_K) with a cone metric is called a cone metric space. In some previous works by different authors, where cone metric spaces were used, mappings were usually characterized by numerical coefficients, as in usual metric spaces.

In 1964 Perov proved a generalization of the contraction mapping principle [11] in spaces with a metric taking values in the cone $\mathbb{R}^n_+$ (see also [12]). A positive linear operator in $\mathbb{R}^n$ with the spectral radius smaller than unity was used as a contraction coefficient. In 2018 this useful natural idea was employed by E.S. Zhukovskiy and E.A. Panasenko [13] (without any reference to A.I. Perov papers) to characterize contraction mappings in a cone metric space.

Here we use the idea of operator coefficients in order to develop and expand the above mentioned method of (α, β)-search functionals to a cone metric space and to demonstrate its applications to the theory of fixed points and coincidences. Assume that the norm in E is monotone with respect to the partial order $\leq_K$, that is $(a \leq_K b) \Longrightarrow (\| a \| \leq \| b \|)$. Note that in this case, K is usualy called *a normal cone* with the normal constant equal to 1. Additionally, we assume that K is a generating cone, that is $E = K - K$. In other words, any element $a \in E$ can be represented as a difference of elements of K.

In the set of all bounded linear operators on E, we consider the subset of linear operators leaving K invariant, that is $\mathcal{L}_+ := \{P \in \mathcal{L}(E) | P(K) \subseteq K\}$. Since K is a generating cone with a monotone norm, it is easy to see that $\mathcal{L}_+$ is also a positive convex closed acute cone in $\mathcal{L}(S)$ (see also [13]). This cone $\mathcal{L}_+$ defines a partial order $\leq$ on the space $\mathcal{L}(E)$. Namely, we say that for $F, G, \in \mathcal{L}(E)$ it is true that $F \leq G$, if $G - F \in \mathcal{L}_+$.

Now we introduce the concept of a cone search function with operator coefficients. Multivalued mappings will be denoted by double arrows $\rightrightarrows$.

Definition 3.2. A multivalued mapping $\varphi : X \rightrightarrows K$ is called *a cone search function* (with operator coefficients $A, B \in \mathcal{L}_+$), or (A, B)-*search cone function* if the following conditions hold:

(i) A and B are bounded linear operators such that A is invertible, $A^{-1} \in \mathcal{L}_+$, and the composition $A^{-1}B : K \to K$ has a spectral radius $\lambda = \lambda(A^{-1}B) < 1$;

(ii) for any $x \in X$ and any $c \in \varphi(x) \subset K$, there exists an element $x' \in X$, such that $d_K(x, x') \leq_K A^{-1}(c)$, and there exists a value $c' \in \varphi(x')$, such that $c' \leq_K A^{-1}B(c)$.

The graph of an (A, B)-cone function φ is denoted by $Graph(\varphi) = \{(x, c) | x \in X, c \in \varphi(x)\} \subseteq X \times E$. The Cauchy property and convergence of sequences in $X \times E$ (in particular, in $Graph(\varphi)$) are considered in the componentwise metric $D = d_K \times \nu$, where $\nu(a, b) :=\| a - b \|$.

Definition 3.3. The graph $Graph(\varphi)$ of an (A, B)-cone function is called θ-*complete* if any Cauchy sequence $\{(x_n, c_n)\} \subset Graph(\varphi)$, where $c_n \to \theta$ as $n \to \infty$, converges to an element of this graph.

$Graph(\varphi)$ is called θ-*closed* if any of its limit elements of the form (ξ, θ) is contained in it.

Theorem 3.4. *Let (X, d_K) be a complete cone metric space, and $\varphi : X \rightrightarrows K$ be a multivalued (A, B)-cone function with operator coefficients $A, B : K \to K$ defined on (X, d_K). Assume that the graph $Graph(\varphi)$ of φ is θ-closed. Then, for any point $x_0 \in X$ and any value $c_0 \in \varphi(x_0)$, there exists a point $x_* = x_*(x_0, c_0) \in X$, such that $\theta \in \varphi(x_*)$ and the cone distance $d_K(x_0, x_*)$ satisfies the estimate*

$$d_K(x_0, x_*) \leq_K A^{-1}(I - A^{-1}B)^{-1}(c_0).$$

Proof. The properties of the multivalued (A, B)-cone function φ imply that, starting from an arbitrary initial point $x_0 \in X$ and any value $c_0 \in \varphi(x_0)$, it is possible to construct a sequence $\{(x_n, c_n)\} \subset Graph(\varphi)$ with the following properties

$$d_K(x_{n-1}, x_n) \leq_K A^{-1}(c_{n-1}), c_n \leq_K A^{-1}B(c_{n-1}). \tag{15}$$

We show that $\{x_n\}$ is a Cauchy sequence in (X, d_K). Indeed, consider the cone distance $d_K(x_n, x_{n+m})$. Note that the assumption $\lambda(A^{-1}B) < 1$ implies the invertibility of the operator $I - A^{-1}B : E \to E$. Here, the operator $(I - A^{-1}B)^{-1}$ is equal to the sum of the iterative series $\sum\limits_{i=0}^{\infty}(A^{-1}B)^i$. This series consists of the operators $(A^{-1}B)^i \in \mathcal{L}_+$. Since the cone $\mathcal{L}_+$ is closed, it follows that $(I - A^{-1}B)^{-1} \in \mathcal{L}_+$; moreover, for any $N \in \mathbb{N}$, we have $\sum\limits_{i=0}^{N}(A^{-1}B)^i \leq_K (I - A^{-1}B)^{-1}$. Using the inequalities (15) and the properties of the metric d_K one obtains

$$d_K(x_n, x_{n+m}) \leq_K \sum_{j=0}^{m-1} d_K(x_{n+j}, x_{n+j+1}) \leq_K \sum_{j=0}^{m-1} A^{-1}(c_{n+j}) \leq_K \sum_{j=0}^{m-1} A^{-1}(A^{-1}B)^{n+j}(c_0) =$$

$$= A^{-1}\sum_{j=0}^{m-1}(A^{-1}B)^{n+j}(c_0) = A^{-1}(A^{-1}B)^n \sum_{j=0}^{m-1}(A^{-1}B)^j(c_0) \leq_K A^{-1}(I - A^{-1}B)^{-1}(A^{-1}B)^n(c_0).$$

As $A^{-1}B$ has the spectral radius $\lambda = \lambda(A^{-1}B) < 1$, it is true that $(A^{-1}B)^n \underset{n\to\infty}{\longrightarrow} 0$ in $\mathcal{L}_+$. Since the norm is monotone, we have

$$\| d(x_n, x_{n+m}) \| \leq_K \| A^{-1}(I - A^{-1}B)^{-1}(A^{-1}B)^n(c_0) \| \underset{n\to\infty}{\longrightarrow} 0.$$

So, it follows that $\{x_n\}$ is a Cauchy sequence. As the cone metric space (X, d_K) is complete, there exists a limit $\lim\limits_{n\to\infty} x_n = x_* \in X$. The metric d_K is continuous (in the corresponding topology), consequently

$$d_K(x_0, x_*) = \lim_{n\to\infty} d_K(x_0, x_n) \leq_K \lim_{n\to\infty} \sum_{j=0}^{n-1} d_K(x_j, x_{j+1}) \leq_K \lim_{n\to\infty} \sum_{j=0}^{n-1} A^{-1}(c_j) \leq_K$$

$$\leq_K A^{-1} \lim_{n\to\infty} \sum_{j=0}^{n-1}(A^{-1}B)^j(c_0) \leq_K A^{-1}(I - A^{-1}B)^{-1}(c_0).$$

We notice that $c_n \leq_K (A^{-1}B)^n(c_0) \underset{n\to\infty}{\longrightarrow} 0$. By the theorem condition, $Graph(\varphi)$ is $\{\theta\}$-closed, so $(x_*, \theta) \in Graph(\varphi)$ that is $\theta \in \varphi(x_*)$. □

Now we present some consequences of the proved theorem.

Theorem 3.5. *Let (X, d_K) be a complete cone metric space and $F : X \rightrightarrows X$ be a multivalued mapping. Let a mapping $\varphi : X \rightrightarrows K$, where $\varphi(x) = \{d \in K | \exists z \in F(x), d = d_K(x, z)\}$, be an (A, B)-cone function for linear bounded operators $A, B \in \mathcal{L}_+$. Assume that the graph $Graph(\varphi)$ of φ is $\{\theta\}$-closed. Then, for any point $x_0 \in X$ and any $y_0 \in F(x_0)$, there exists a fixed point $\xi = \xi(x_0, y_0)$ of F, i.e., $\xi \in F(\xi)$, and the following estimate holds: $d_K(x_0, \xi) \leq_K A^{-1}(I - A^{-1}B)^{-1}(d_K(x_0, y_0))$.*

Proof. The statement follows from Theorem 3.4. One can notice that the set of zeros of (A, B)-search cone function φ, that is the set of points $x \in X$, such that $0 \in \varphi(x)$, coincides with the fixed point set of F. Taking $c_0 = d_K(x_0, y_0)$ and repeating step by step the proof of Theorem 3.4, we obtain a point x_* such that $(x_*, 0) \in Graph(\varphi)$ that is $0 \in \varphi(x_*)$. It is equivalent to the inclusion $x_* \in F(x_*)$. □

To formulate the next result, we need the following definition.

Definition 3.6. Let (X, d_K) be a cone metric space, $F : X \rightrightarrows X$ be a multivalued mapping, and $Q : K \to K$ be a bounded linear operator with a spectral radius $\lambda(Q) < 1$. The mapping F is called *Q-contraction (contraction with operator coefficient Q)* if for any $x_1, x_2 \in X$ and any $y_1 \in F(x_1)$, there exists $y_2 \in F(x_2)$ for which $d_K(y_1, y_2) \leq Q(d_K(x_1, x_2))$.

The following theorem is contained in [13] and represents a cone metric analogue of the well-known Nadler fixed point theorem for a multivalued mapping. Below we show that this result follows from Theorem 3.5 which follows, in its turn, from theorem 3.4.

Theorem 3.7. *(see [13], Theorem 1). Let (X, d_K) be a complete cone metric space, F be a multivalued Q-contraction mapping with closed images, and $Q \in \mathcal{L}_+$ be an operator contraction coefficient with the spectral radius $\lambda(Q) < 1$. Then, for any $x_0 \in X$ and any $y_0 \in F(x_0)$, the mapping F has a fixed point $\xi \in X$, that is $\xi \in F(\xi)$, and the following estimate holds: $d_K(x_0, \xi) \leq_K (I - Q)^{-1}(d_K(x_0, y_0))$.*

Proof. In fact, this is a particular case of Theorem 3.5. It is easy to see that, under the conditions of Theorem 3.7, the mapping $\varphi : X \rightrightarrows K$, where $\varphi(x) = \{c \in K | \exists y \in F(x), c = d_K(x, y)\}$, is a multivalued (I, Q)-search cone function ($I = id_K$ is the identity self-mapping of K). Indeed, for any point x and any $y \in F(x)$, that is for any value $c = d_K(x, y) \in \varphi(x)$, one can find a point $x' \in X, x' = y \in F(x)$ such that $d_K(x, x') = d_K(x, y) \leq_K c = d_K(x, y)$, and there exists a value $c' \in \varphi(x')$, that is there is an element $y' \in F(x') = F(y)$, such that $c' = d_K(y, y') \leq_K Q(d_K(x, x'))$. We notice that the contraction mapping F is closed, that is its graph is closed. Then the graph of φ is also closed, and in particular it is $\{\theta\}$-closed. Thus, all conditions of Theorem 3.5 are satisfied for the mapping F. Therefore, the statement follows from Theorem 3.5 (which follows, in its turn, from theorem 3.4). □

Now we consider the problem of existence of coincidence points for two multivalued mappings. Let (X, d_K) and (Y, ρ_K) be two cone metric spaces with cone metrics $d_K : X^2 \to K$, $\rho_K : Y^2 \to K$, taking their values in the cone K. We assume that the space (X, d_K) is complete.

Theorem 3.8. *Under the described conditions, let $F, G : X \rightrightarrows Y$ be two multivalued mappings with bounded closed images. Consider the mapping $\psi : X \rightrightarrows K$ defined by the rule $\psi(x) := \{c \in K | \exists y \in F(x), \exists z \in G(x), \rho_K(y, z) = c\}$. Assume that ψ is an (A, B)-search cone function, with respect to some linear operators $A, B \in \mathcal{L}_+$. In addition, assume that $Graph(\psi)$ is $\{\theta\}$-closed. Then, for any initial point $x_0 \in X$ and any pair of values $y_0 \in F(x_0)$, $z_0 \in G(x_0)$, the mappings F, G have a coincidence point $\xi = \xi(x_0, y_0, z_0) \in X$, that is $F(\xi) \cap G(\xi) \neq \emptyset$, and the following estimate holds: $d_K(x_0, \xi) \leq_K A^{-1}(I - A^{-1}B)^{-1}(\rho_K(y_0, z_0))$.*

Proof. It is not difficult to see that the set of coincidence points of the mappings F and G coincides with the set of zeros of the functional ψ. Therefore, the statement follows from Theorem 3.4. Indeed, for an initial point $x_0 \in X$ and points $y_0 \in F(x_0), z_0 \in G(x_0)$, we take $c_0 = \rho_K(y_0, z_0)$ as the initial value of φ. When reasoning similarly to the proof of Theorem 3.4, one obtains a point $\xi \in X$, such that $\varphi(\xi) \ni 0$. It means that $F(\xi) \cap G(\xi) \neq \emptyset$, in other words, $\xi \in Coin(F, G) := \{x \in X | F(x) \cap G(x) \neq \emptyset\}$. □

4. Conclusion and possible prospects

In this paper we have presented some results concerning the development and expanding of our early idea of search for zeros of (α, β)-search functionals in two directions. The first one is the expanding this idea to (b_1, b_2))-quasimetric spaces. The second one is an essential development and expanding it to cone metric spaces with a cone metric taking its values in a cone of a normed space. As one can see, this approach allows to generalize several previous results of other authors.

It seems to be of interest and of some use to continue and expand this activity, to consider the more general classes of spaces and multivalued mappings.

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