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Polynomial Contractions in G-metric Spaces

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Abstract

In this paper, we introduce these two new classes of polynomial contractions in the setting of G-metric spaces. Our results refine, generalize, and improve several corresponding results in the existing literature. Some examples are presented to validate the originality and applicability of our main results.

Keywords: fixed point, contracting perimeters of triangles, polynomial contractions.
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1. Introduction

It is commonly known that one of the most researched topics of nonlinear functional analysis nowadays is fixed point theory, which focuses on the presence and uniqueness of fixed points. Banach [4] achieved the first notable result in this subject to guarantee the existence and uniqueness of fixed points. Simply put, each contraction mapping has a unique fixed point in a metric space. This outcome is known as the Banach contraction principle. This subject is more important than ever since the Banach principle was originally introduced because of the fixed point theory's endless application potential in a variety of scientific domains, including physics, chemistry, some parts of engineering, economics, and many areas of mathematics.

As a result, numerous authors have looked for further fixed point conclusions using the widely recognized Banach principle and have successfully published new fixed point results that were developed by combining or using two incredibly powerful methodologies. One way to do this is to replace the concept of a metric

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space with a more universal space. Metric space generalizations that might be thought of as replacements include quasi-metric spaces, partial metric spaces, G-metric spaces, fuzzy metric spaces, b-metric spaces etc. Mustafa and Sims [16] established G-metric spaces, which are among the most intriguing of any of these spaces. Consequently, the idea of a G-metric space has garnered a lot of interest from scholars over the past 10 years, particularly from fixed point theorists [1], [3], [11], [14], [15], [17] and [20]. In 2012, Jleli and Samet [9] noted that many fixed point theorems established in G-metric spaces were direct corollaries of existing theorems in standard metric spaces. This realization dampened enthusiasm for further exploration of fixed points in G-metric spaces. Recently, Jleli et al. [7] introduced groundbreaking developments in fixed point theory within G-metric spaces. Their work offers novel versions of the Banach, Kannan, and Reich fixed point theorems, significantly enhancing both the understanding and practical applications of fixed point theory in this advanced mathematical context. The authors also highlighted that the approach used in [9] was not applicable to their paper.

Changing the operator's conditions is the second of these methods. That is, it involves investigating specific conditions that lead to a fixed point being obtained from the contraction mapping. Very recently, Jleli et al. [8] introduced the notions of polynomial and almost polynomial contractions on a metric space, yielding some interesting results utilizing this technique.

In this paper, we aim to bring together the two previously mentioned topics: contractions of polynomial type and G-metric spaces. We will define two classes of single-valued contractions of polynomial type in the context of G-metric spaces. Furthermore, we will establish fixed point results for these classes of contractions. The results obtained will generalize those previously derived by Berinde [5], Jleli et al. [7], Jleli et al. [8], Mustafa and Sims [16] and Perov [18], among others.

Throughout this paper, the following notations are used: $\mathbb{R}^+ = [0, \infty)$, X denotes a nonempty set, $|X|$ denotes the cardinal of X, and for a mapping $T : X \to X$, the set of fixed points of T is denoted by $Fix(T)$.

2. Preliminaries

In this section, we review some fundamental ideas about G-metric spaces in brief. The readers can refer to Mustafa and Sims [16] for more details. Throughout this paper, X denotes a nonempty set and $\mathbb{R}^+ = [0, \infty)$.

Definition 2.1. [16] Let $G : X \times X \times X \to \mathbb{R}^+$ be a given mapping. We say that G is a G-metric on X, if for all $x, y, z, w \in X$, we have

(G1) G(x, y, z) = 0 if and only if $x = y = z$;
(G2) If $x \neq y$, then $G(x, x, y) > 0$;
(G3) G(x, y, z) = G($\sigma(x, y, z)$) for every permutation $\sigma : \{x, y, z\} \to \{x, y, z\}$;
(G4) If $y \neq z$, then $G(x, x, y) \leq G(x, y, z)$;
(G5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$.

If the preceding conditions are satisfied, then (X, G) is called a G-metric space.

Example 2.2. [16] Let (X, d) be a usual metric space, and define G_m and G_s on $X \times X \times X \to R^+$ by

$$G(x, y, z) = d(x, y) + d(y, z) + d(x, z), x, y, z \in X \quad (1)$$
$$G(x, y, z) = \max\{d(x, y) + d(y, z) + d(x, z)\}, x, y, z \in X \quad (2)$$

Then, (X, G_m) and (X, G_s) are G-metric spaces.

Definition 2.3. [16] A G-metric space (X, G) is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition 2.4. [16] Let (X, G) be a G-metric space. We say that $\{x_n\}$ is

(i) a G-Cauchy sequence if, for any $\epsilon > 0$, there is $N \in \mathbf{N}$ (the set of all positive integers) such that for all $n, m, l \geq N$, $G(x_n, x_m, x_l) < \epsilon$;

(ii) a G-convergent sequence to $x \in X$ if, for any $\epsilon > 0$, there is $N \in \mathbf{N}$ such that for all $n, m \geq N$, $G(x, x_n, x_m) < \epsilon$. A G-metric space (X, G) is said to be complete if every G-Cauchy sequence in X is G-convergent in X.

Definition 2.5. [16] Let $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ is G-convergent to x, if $\lim_{n,m\to\infty} G(x_n, x_m, x) = 0$. The following assertions are equivalent:

1. $\{x_n\}$ is G-convergent to x;
2. $\lim\limits_{n\to\infty} G(x_n, x_n, x) = 0$;
3. $\lim\limits_{n\to\infty} G(x_n, x, x) = 0$.

Recently, the following interesting class of polynomial contractions were introduced by Jleli et al. [8]:

Definition 2.6. [16] Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a polynomial contraction, if there exists $\lambda \in (0, 1)$, a natural number $k \geq 1$ and a family of mappings $a_i : X \times X \to [0, \infty)$, $i = 0, ..., k$, such that

$$\sum_{i=0}^{k} a_i(Tx, Ty)d^i(Tx, Ty) \leq \lambda \sum_{i=0}^{k} a_i(x, y)d^i(x, y)$$

for every $x, y \in X$.

Theorem 2.7. *[16] Let (X, d) be a complete metric space and $T : X \to X$ be a polynomial contraction. Assume that the following conditions hold:*

(i) T is continuous;

(ii) There exist $j \in \{1, - - -, k\}$ and $A_j > 0$ such that

$$a_j(x, y) \geq A_j,$$

$x, y \in X$.

Then, T admits a unique fixed point $z^* \in X$. Moreover, for every $z_0 \in X$, the Picard sequence $\{z_n\} \subseteq X$ defined by $z_{n+1} = Tz_n$ for all $n \geq 0$, converges to z^*.

3. Main Results

The following defines the class of polynomial contractions in G-metric space:

Definition 3.1. Let (X, G) be a G-metric space and $T : X \to X$ be a given mapping. We say that T is a polynomial contraction if there exists $\alpha \in (0, 1)$, a natural number $k \geq 1$ and a family of mappings $f_i : X \times X \times X \to [0, \infty)$, $i = 0, ..., k$, such that for all pairwise distinct points $x, y, z \in X$, we have

$$\sum_{i=0}^{k} f_i(Tx, Ty, Tz)G^i(Tx, Ty, Tz) \leq \alpha \sum_{i=0}^{k} f_i(x, y, z)G^i(x, y, z). \tag{3}$$

Theorem 3.2. *Let (X, G) be a complete G-metric space with $|X| \geq 3$. Let $T : X \to X$ be a polynomial contraction mapping. Assume that the following conditions hold:*

(i) For all $x \in X$, $T(Tx) \neq x$, provided $Tx \neq x$;

(ii) There exists $j \in \{1, ..., k\}$ and $F_j > 0$ such that

$$f_j(x, y, z) \geq F_j,\ x, y, z \in X.$$

Then, $Fix(T) \neq \emptyset$ *and* $|Fix(T)| \leq 2$.

Proof. Initially, we establish that $Fix(T) \neq \emptyset$. Suppose $p_0 \in X$ be fixed and $\{p_n\} \subset X$ be the Picard sequence defined by

$$p_{n+1} = Tp_n, n \geq 0$$

The result is proved if $p_n = p_{n+1}$ for some n. Hence, we need to assume that

$$p_n \neq p_{n+1}, n \geq 0.$$

In view of condition (ii), we get $p_n \neq p_{n+2}(= T(Tp_n))$. Therefore, p_n, p_{n+1} and p_{n+2} are pairwise distinct points for every $n \geq 0$. Making use of (3) with $(x, y, z) = (p_0, p_1, p_2)$, we obtain

$$\sum_{i=0}^{k} f_i(Tp_0, Tp_1, Tp_2)G^i(Tp_0, Tp_1, Tp_2) \leq \alpha \sum_{i=0}^{k} f_i(p_0, p_1, p_2)G^i(p_0, p_1, p_2),$$

that is,

$$\sum_{i=0}^{k} f_i(p_1, p_2, p_3)G^i(p_1, p_2, p_3) \leq \alpha \sum_{i=0}^{k} f_i(p_0, p_1, p_2)G^i(p_0, p_1, p_2), \tag{4}$$

Again making use of the equation (3) with $(x, y, z) = (p_1, p_2, p_3)$, we infer

$$\sum_{i=0}^{k} f_i(Tp_1, Tp_2, Tp_3)G^i(Tp_1, Tp_2, Tp_3) \leq \alpha \sum_{i=0}^{k} f_i(p_1, p_2, p_3)G^i(p_1, p_2, p_3),$$

which implies

$$\sum_{i=0}^{k} f_i(p_2, p_3, p_4)G^i(p_2, p_3, p_4) \leq \alpha \sum_{i=0}^{k} f_i(p_1, p_2, p_3)G^i(p_1, p_2, p_3).$$

From Equation (4), we obtain that

$$\sum_{i=0}^{k} f_i(p_2, p_3, p_4)G^i(p_2, p_3, p_4) \leq \alpha^2 \sum_{i=0}^{k} f_i(p_0, p_1, p_2)G^i(p_0, p_1, p_2).$$

Proceeding in the same manner, we derive through induction that

$$\sum_{i=0}^{k} f_i(p_n, p_{n+1}, p_{n+2})G^i(p_n, p_{n+1}, p_{n+2}) \leq \alpha^n \sum_{i=0}^{k} f_i(p_0, p_1, p_2)G^i(p_0, p_1, p_2), n \geq 0. \tag{5}$$

As

$$f_j(p_n, p_{n+1}, p_{n+2})G^j(p_n, p_{n+1}, p_{n+2}) \leq \sum_{i=0}^{k} f_i(p_n, p_{n+1}, p_{n+2})G^i(p_n, p_{n+1}, p_{n+2}).$$

Considering (iii), we obtain that

$$F_j G^j(p_n, p_{n+1}, p_{n+2}) \leq \sum_{i=0}^{k} f_i(p_n, p_{n+1}, p_{n+2})G^i(p_n, p_{n+1}, p_{n+2}).$$

Now, from (5), we infer that

$$G^j(p_n, p_{n+1}, p_{n+2}) \leq \alpha^n \delta_{j,0}, n \geq 0, \tag{6}$$

where

$$\delta_{j,0} = F_j^{-1} \sum_{i=0}^{k} f_i(p_0, p_1, p_2)G^i(p_0, p_1, p_2). \tag{7}$$

Now, we demonstrate that $\{p_n\}$ is G-Cauchy. From $(G3)$, $(G4)$ and using the fact that p_n, p_{n+1} and p_{n+2} are pairwise distinct points, we obtain

$$\begin{aligned} G(p_n, p_{n+1}, p_{n+1}) &= G(p_{n+1}, p_{n+1}, p_n) \\ &\leq G(p_{n+1}, p_n, p_{n+2}) \\ &= G(p_n, p_{n+1}, p_{n+2}). \end{aligned}$$

Hence, in view of (6) we get

$$G^j(p_n, p_{n+1}, p_{n+1}) \leq \alpha^n \delta_{j,0}, n \geq 0. \tag{8}$$

Utilizing $(G5)$ and (8) to obtain for all $n < m$,

$$\begin{aligned} G^j(p_n, p_m, p_m) &\leq G(p_n, p_{n+1}, p_{n+1}) + G(p_{n+1}, p_m, p_m) \\ &\leq G(p_n, p_{n+1}, p_{n+1}) + G(p_{n+1}, p_{n+2}, p_{n+2}) + G(p_{n+2}, p_m, p_m) \\ &. \\ &. \\ &. \\ &\leq G(p_n, p_{n+1}, p_{n+1}) + G(p_{n+1}, p_{n+2}, p_{n+2}) + ... + G(p_{m-1}, p_m, p_m) \\ &\leq \delta_{j,0}(\alpha^n + \alpha^{n+1} + ... + \alpha^{n+m-1}) \\ &\leq \delta_{j,0}\frac{\alpha^n}{1-\alpha}, \end{aligned}$$

which proves that as $n, m \to \infty$,

$$G(p_n, p_m, p_m) \leq (\frac{\delta_{j,0}}{1-\alpha})^{\frac{1}{j}}(\alpha^{\frac{1}{j}})^n \to 0.$$

This demonstrates that the sequence p_n is Cauchy. Now, since (X, G) is complete, there must exist an element $p^* \in X$ such that p_n is G-convergent to p^*. However, this contradicts the idea that p_n, p_{n+1}, and p_{n+2} are pairwise distinct points for every n if there is a k such that for all $n \geq k$, we have $p_n = p^*$. As a result, we may derive a subsequence $\{p_{n(k)}\}_k$ from $\{p_n\}$ such that, for every k, $p_{n(k)} \neq p^*$. We always represent the sequence $\{p_{n(k)}\}_k$ by p_n with $p_n \neq p^*$ for all n, to make writing simpler. We now demonstrate that p^* is a member of $Fix(T)$. From $(G3)$ and $(G5)$, we have

$$\begin{aligned} G^j(p^*, p^*, Tp^*) &\leq G^j(p^*, p_n, p_n) + G^j(p_n, p^*, Tp^*) \\ &= G^j(p^*, p_n, p_n) + G^j(p^*, p_n, Tp^*) \\ &\leq G^j(p^*, p_n, p_n) + G^j(p^*, p_{n+1}, p_{n+1}) + G^j(p_{n+1}, p_n, Tp^*) \\ &= G^j(p^*, p_n, p_n) + G^j(p^*, p_{n+1}, p_{n+1}) + G^j(Tp_n, Tp_{n-1}, Tp^*) \end{aligned}$$

thereby implying in view of (3) and the fact that p_n, p_{n-1} and p^* are pairwise distinct points that

$$\begin{aligned} G^j(p^*, p^*, Tp^*) &\leq G^j(p^*, p_n, p_n) + G^j(p^*, p_{n+1}, p_{n+1}) + \sum_{i=0}^{k} f_j(Tp_n, Tp_{n-1}, Tp^*)G^j(Tp_n, Tp_{n-1}, Tp^*) \\ &\leq G^j(p^*, p_n, p_n) + G^j(p^*, p_{n+1}, p_{n+1}) + \alpha \sum_{i=0}^{k} f_j(p_n, p_{n-1}, p^*)G^j(p_n, p_{n-1}, p^*). \end{aligned} \tag{9}$$

By Definition 2.5, we get

$$\lim_{n\to\infty}[G^j(p^*, p_n, p_n) + G^j(p^*, p_{n+1}, p_{n+1}) + \alpha \sum_{i=0}^{k} f_j(p_n, p_{n-1}, p^*)G^j(p_n, p_{n-1}, p^*)] = 0.$$

Using (9) and $(G1)$, we get that $G(p^*, p^*, Tp^*) = 0$ and $p^* = Tp^*$. This proves that $Fix(T) \neq \emptyset$.
Suppose now that q_i, $i = 1, 2, 3$, are pairwise distinct fixed points of T. Then making use of $(G1)$ and (3), we obtain

$$\begin{aligned} \sum_{i=0}^{k} f_i(Tv_1, Tv_2, Tv_3)G^i(Tv_1, Tv_2, Tv_3) &\leq \alpha \sum_{i=0}^{k} f_i(v_1, v_2, v_3)G^i(v_1, v_2, v_3) \\ \sum_{i=0}^{k} f_i(v_1, v_2, v_3)G^i(v_1, v_2, v_3) &\leq \alpha \sum_{i=0}^{k} f_i(v_1, v_2, v_3)G^i(v_1, v_2, v_3). \end{aligned} \tag{10}$$

In view of Condition (ii), we have

$$\begin{aligned} \sum_{i=0}^{k} f_i(Tv_1, Tv_2, Tv_3)G^i(Tv_1, Tv_2, Tv_3) &\geq f_j(v_1, v_2, v_3)G^j(v_1, v_2, v_3) \\ &\geq F_j G^j(v_1, v_2, v_3). \end{aligned}$$

Since $F_j > 0$ and $G(v_1, v_2, v_3) > 0$, we get that

$$\sum_{i=0}^{k} f_i(Tv_1, Tv_2, Tv_3)G^i(Tv_1, Tv_2, Tv_3) > 0.$$

Now, dividing (10) by $\sum_{i=0}^{k} f_i(Tv_1, Tv_2, Tv_3)G^i(Tv_1, Tv_2, Tv_3)$, we arrive at a contradiction with $\alpha \in (0, 1)$. So, we conclude that $|Fix(T)| \leq 2$. Hence proved. □

The following result is an immediate consequence of Theorem 3.2:

Corollary 3.3. *Let (X, G) be a complete G-metric space and $T : X \to X$ be a given mapping. Assume that there exists $\alpha \in (0, 1)$, a natural number $k \geq 1$ and a finite sequence $\{f_i\}_{i=1}^{k} \subset (0, \infty)$ such that*

$$\sum_{i=1}^{k} f_i G^i(Tx, Ty, Tz) \leq \alpha \sum_{i=1}^{k} f_i G^i(x, y, z), \tag{11}$$

for every $x, y, z \in X$. Then, T admits a unique fixed point $p^* \in X$. Moreover, for every $p_0 \in X$, the Picard sequence $\{p_n\} \subset X$ defined by $p_{n+1} = Tp_n$ for all $n \geq 0$, converges to p^*.

Remark 3.4. Observe that Corollary 3.3 recovers the main result of Jleli et al. [7]. Indeed, taking $k = 1$ and $f_1 = 1$, the inequality (11) reduces to

$$G(Tx, Ty, Tz) \leq \alpha G(x, y, z).$$

Now, we provide an example demonstrating that condition (i) cannot be eliminated from the statement of Theorem 3.2.

Example 3.5. Let $X = \{a, b, c, d\} \subseteq \mathbb{R}$. Let us define the mappings $T_1, T_2 : X \to X$ defined by

$$T_1 a = a, T_1 b = c, T_1 c = d, T_1 d = a.$$

and

$$T_2 a = b, T_2 b = a, T_2 c = d, T_2 d = c.$$

Consider the G-metric on X given by

$$G(x, y, z) = d(x, y) + d(y, z) + d(z, x),$$

where $x, y, z \in X$ and d is the discrete metric on X. Let the mapping $f_0 : X \times X \times X \to [0, \infty)$ defined by

$$f_0(a,b,c) = f_0(a,c,b) = f_0(b,a,c) = f_0(b,c,a) = f_0(c,a,b) = f_0(c,b,a)$$
$$f_0(a,b,c) = 7, f_0(a,c,d) = 1, f_0(c,d,a) = 2, f_0(b,c,d) = 8,$$
$$f_0(d,a,b) = 17, f_0(b,d,c) = 7, f_0(a,c,b) = 9, f_0(c,a,d) = 2, f_0(d,b,a) = 3, f_0(b,a,d) = 1.$$

It can be easily verified that

$$f_0(Tx,Ty,Tz) + G(Tx,Ty,Tz) \leq \frac{1}{2}(f_0(x,y,z) + G(x,y,z) \tag{12}$$

for every $x, y, z \in X$, that is, T is a polynomial contraction in the given G-metric space with $k = 1$, $f_1 \equiv 1$ and $\alpha = 12$. All the conditions of Theorem 3.2 are satisfied ((ii) is satisfied with $F_1 = 1$). Also, $Fix(T_1) = \{a\}$. On the other hand, the mapping T_2 satisfies the inequality (3) for all $x, y, z \in X$. But, we have $T_2 a \neq a$ and $T_2(T_2 a) = T_2 b = a$, which proves that condition (i) of the Theorem 3.2 is not satisfied. Also, we have $Fix(T_2) = \emptyset$, which shows that in the absence of Condition (i), the result of Theorem 3.2 is not true.

Remark 3.6. Since in the above example,

$$\frac{G(T_1 a, T_1 b, T_1 c)}{G(a,b,c)} = \frac{G(a,c,d)}{G(a,b,c)} = 1.$$

Therefore, Banach's fixed point theorem in G-metric spaces proved by Jleli et al. [7] is not applicable.

We provide here the class of almost polynomial contractions in the setting of G-metric spaces, inspired by Berinde [5].

Definition 3.7. Let (X, G) be a G-metric space and $T : X \to X$ be a given mapping. We say that T is an almost polynomial contraction, if there exists $\alpha \in (0, 1)$, a natural number $k \geq 1$, finite sequences $\{S_i\}_{i=0}^k, T_i\}_{i=0}^k \subset (0, \infty)$ and a family of mappings $f_i : X \times X \times X \to [0, \infty)$, $i = 0, 1, ..., k$, such that for all pairwise distinct points $x, y, z \in X$, we have

$$\sum_{i=0}^{k} f_i(Tx,Ty,Tz)G^i(Tx,Ty,Tz) \leq \alpha \sum_{i=0}^{k} f_i(x,y,z)[G^i(x,y,z) + S_i G^i(y,y,Tx) + T_i G^i(z,z,Ty)] \tag{13}$$

Theorem 3.8. *Let (X, G) be a complete G-metric space with $|X| \geq 3$. Let $T : X \to X$ be an almost polynomial contraction mapping. Assume that the following conditions hold:*

(i) For all $x \in X$, $T(Tx) \neq x$, provided $Tx \neq x$;

(ii) There exists $j \in \{1, ..., k\}$ and $F_j > 0$ such that

$$f_j(x,y,z) \geq F_j,\ x,y,z \in X.$$

Then, $Fix(T) \neq \emptyset$.

Proof. Assume $p_0 \in X$ be fixed and $\{p_n\} \subseteq X$ be the Picard sequence defined by

$$p_{n+1} = Tp_n, n \geq 0.$$

Utilizing (13) with $(x, y, z) = (p_0, p_1, p_2)$, we get

$$\sum_{i=0}^{k} f_i(Tp_0,Tp_1,Tp_2)G^i(Tp_0,Tp_1,Tp_2) \geq \alpha \sum_{i=0}^{k} f_i(p_0,p_1,p_2)[G^i(p_0,p_1,p_2) + S_i G^i(p_2,p_2,Tp_1)]$$

that is,

$$\begin{aligned} \sum_{i=0}^{k} f_i(p_1,p_2,p_3)G^i(p_1,p_2,p_3) &\leq \alpha \sum_{i=0}^{k} f_i(p_0,p_1,p_2)[G^i(p_0,p_1,p_2) + S_i G^i(p_2,p_2,p_2)], \\ &\leq \alpha \sum_{i=0}^{k} f_i(p_0,p_1,p_2)[G^i(p_0,p_1,p_2) \end{aligned} \tag{14}$$

Once again, applying inequality (13) with $(x, y, z) = (p_1, p_2, p_3)$, and considering equation (14), we obtain

$$\begin{aligned}\sum_{i=0}^{k} f_i(Tp_1, Tp_2, Tp_3)G^i(Tp_1, Tp_2, Tp_3) &\leq \alpha \sum_{i=0}^{k} f_i(p_1, p_2, p_3)[G^i(p_1, p_2, p_3) + S_i G^i(p_3, p_3, Tp_2)] \\ &\leq \alpha^2 \sum_{i=0}^{k} f_i(p_0, p_1, p_2)G^i(p_0, p_1, p_2)\end{aligned}$$

Following this procedure, we learn through induction that

$$\sum_{i=0}^{k} f_i(p_n, p_{n+1}, p_{n+2})G^i(p_n, p_{n+1}, p_{n+2}) \leq \alpha^n \sum_{i=0}^{k} f_i(p_0, p_1, p_2)G^i(p_0, p_1, p_2), n \geq 0.$$

owing to (ii) we get

$$G^j(p_n, p_{n+1}, p_{n+2}) \leq \alpha^n \sigma_{j,0}, n \geq 0,$$

where $\sigma_{j,0}$ is provided by (7). Next, by following the steps in the Theorem 3.2 proof, we can determine that $\{z_n\}$ is a Cauchy sequence and the following inequality holds:

$$G^j(p^*, p^*, Tp^*) \leq G^j(p^*, p_n, p_n) + G^j(p^*, p_{n+1}, p_{n+1}) + G^j(Tp_n, Tp_{n-1}, Tp^*)$$

In view of condition that p_n, p_{n-1} and p^* are pairwise distinct points and from (3.7), we obtain that

$$\begin{aligned}G^j(p^*, p^*, Tp^*) &\leq G^j(p^*, p_n, p_n) + G^j(p^*, p_{n+1}, p_{n+1}) + \sum_{i=0}^{k} f_j(Tp_n, Tp_{n-1}, Tp^*)G^j(Tp_n, Tp_{n-1}, Tp^*) \\ &\leq G^j(p^*, p_n, p_n) + G^j(p^*, p_{n+1}, p_{n+1}) + \alpha \sum_{i=0}^{k} f_j(p_n, p_{n-1}, p^*)[G^j(p_n, p_{n-1}, p^*) \\ &\quad + S_i G^j(p_{n-1}, p_{n-1}, Tp_n) + T_i G^j(p^*, p^*, Tp_{n-1})]. \qquad (15)\end{aligned}$$

The Definition of G-convergence and Definition 2.5 implies that

$$\lim_{n\to\infty} [G^j(p^*, p_n, p_n) + G^j(p^*, p_{n+1}, p_{n+1}) + \alpha \sum_{i=0}^{k} f_j(p_n, p_{n-1}, p^*)[G^j(p_n, p_{n-1}, p^*) + S_i G^j(p_{n-1}, p_{n-1}, Tp_n) + T_i G^j(p^*, p^*, Tp_{n-1})]] = 0.$$

Utilizing the above inequality and (G_1), we infer that $G(p^*, p^*, Tp^*) = 0$ and $p^* = Tp^*$. Hence, we proved that $Fix(T) \neq \emptyset$.

□

Now, we show an example that illustrates Theorem 3.8.

Example 3.9. Let $X = [0, 1]$ and $T : X \to X$ be the mapping defined by

$$T(x) = \begin{cases} \frac{1}{8} & \text{if } \ 0 \leq x < 1, \\ 0 & \text{if } \ x = 1. \end{cases}$$

Let G be the metric defined by

$$G(x, y, z) = d(x, y) + d(y, z) + d(x, z),$$

where $x, y, z \in X$ and d be the standard metric on X. Let us define the mapping $f_0 : X \times X \times X \to [0, \infty)$ by

$$f_0(x, y, z) = \left|6x^2 - 5x + \frac{17}{32}\right| + \left|6y^2 - 5y + \frac{17}{32}\right| + \left|6z^2 - 5z + \frac{17}{32}\right|,$$

$x, y, z \in X$. Now, we prove that T is an almost polynomial contraction in G-metric spaces with $k = 1$, $f_1 \equiv 1$, $S_0 = S_1 = 1$, $T_0 = T_1 = 2$ and $\alpha = \frac{1}{4}$. Putting these values in Definition 3.7, we obtain

$$f_0(Tx, Ty, Tz) + G(Tx, Ty, Tz) \leq \frac{1}{4}\left(4f_0(x, y, z) + G(x, y, z) + G(y, y, Tx) + 2G(z, z, Ty)\right) \tag{16}$$

There will be following six cases:
Case 1: $0 \leq x, y, z < 1$. Here, we have

$$f_0(Tx, Ty, Tz) + G(Tx, Ty, Tz) = 0$$

Hence, equation (16) holds.
Case 2: $0 \leq x, y < 1$, $z = 1$

$$\begin{aligned} f_0(Tx, Ty, Tz) + G(Tx, Ty, Tz) = \frac{17}{32} + \frac{1}{4} &= \frac{25}{32} \\ &\leq G(z, z, Ty) \\ &= \frac{7}{4} \\ &\leq \frac{1}{4}\left(4f_0(x, y, z) + G(x, y, z) + G(y, y, Tx) + 2G(z, z, Ty)\right) \end{aligned}$$

Hence, equation (16) holds.
Case 3: $0 \leq x, z < 1$, $y = 1$

$$\begin{aligned} f_0(Tx, Ty, Tz) + G(Tx, Ty, Tz) = \frac{17}{32} + \frac{1}{4} &= \frac{25}{32} \\ &\leq G(y, y, Tx) \\ &= \frac{7}{4} \\ &\leq \frac{1}{4}\left(4f_0(x, y, z) + G(x, y, z) + G(y, y, Tx) + 2G(z, z, Ty)\right) \end{aligned}$$

Hence, equation (16) holds.
Case 4: $0 \leq y, z < 1$, $x = 1$

$$\begin{aligned} f_0(Tx, Ty, Tz) + G(Tx, Ty, Tz) = \frac{17}{32} + \frac{1}{4} &= \frac{25}{32} \\ &\leq \frac{49}{32} \\ &= \frac{1}{4}[4f_0(x, y, z)] \\ &\leq \frac{1}{4}\left(4f_0(x, y, z) + G(x, y, z) + G(y, y, Tx) + 2G(z, z, Ty)\right) \end{aligned}$$

Hence, equation (16) holds.
Therefore, equation (16) holds for all $x, y, z \in X$. Also, condition (i) of Theorem 3.8 is clearly satisfied. Regarding condition (ii), it is valid for $j = 1$ and $F_1 = 1$. Consequently, Theorem 3.8 holds. Clearly, $p = \frac{1}{8}$ is a fixed point of T.

By taking $f_0 \equiv 0$ and f_i is constant for all $i \in \{1, 2, ..., k\}$, we deduce the following result from Theorem 3.8:

Corollary 3.10. *Let (X, G) be a complete G-metric space with $|X| \geq 3$. Let $T : X \to X$ be a given mapping. Assume that the following conditions hold:*

(i) For all $x \in X$, $T(Tx) \neq x$, provided $Tx \neq x$;

(ii) there exists $\alpha \in (0,1)$, *a natural number* $k \geq 1$ *and three finite sequences* $\{f_i\}_{i=1}^k$, $\{S_i\}_{i=1}^k$ *and* $\{T_i\}_{i=1}^k \subseteq (0,\infty)$ *such that*

$$\sum_{i=1}^{k} f_i G^i(Tx,Ty,Tz) \leq \alpha \sum_{i=1}^{k} f_i[G^i(x,y,z) + S_i G^i(y,y,Tx) + T_i G^i(z,z,Ty)]$$

Then $Fix(T) \neq \emptyset$.

We now introduce a version of Berinde's Theorem within the framework of G-metric spaces.

Corollary 3.11. *Let* (X,G) *be a complete* G*-metric space with* $|X| \geq 3$. *Let* $T : X \to X$ *be a given mapping. Assume that the following conditions hold:*

(i) For all $x \in X$, $T(Tx) \neq x$, *provided* $Tx \neq x$;
(ii) there exists $\alpha \in (0,1)$, *a natural number* $k \geq 1$ *and three finite sequences* $\{f_i\}_{i=1}^k$, $\{S_i\}_{i=1}^k$ *and* $\{T_i\}_{i=1}^k \subseteq (0,\infty)$ *such that*

$$G(Tx,Ty,Tz) \leq \alpha[G(x,y,z) + lG(y,y,Tx) + mG(z,z,Ty)]$$

Then $Fix(T) \neq \emptyset$ *and* $|Fix(T)| \leq 2$.

Proof. By choosing $k = 1$, $f_1 = 1$, $S_1 = l$, and $T_1 = m$, with $l, m > 0$, we obtain the required proof. □

In 2023, a very interesting class of mappings were introduced by Perov [18] which can be identified as mappings contracting perimeters of triangles.

Definition 3.12. (Perov [18]) Let (X,d) be a metric space with $|X| \geq 3$. We shall say that $T : X \to X$ is a mapping contracting perimeters of triangles on X, if there exists $\lambda \in (0,1)$ such that the inequality

$$d(Tx,Ty) + d(Ty,Tz) + T(Tz,Tx) \leq \lambda[d(x,y) + d(y,z) + d(z,x)], \tag{17}$$

holds for all three pairwise distinct points $x, y, z \in X$.

The following outcome attributed to Petrov [18] is a direct outcome of our Theorem 3.2.

Corollary 3.13. *Let* (X,d), $|X| \geq 3$, *be a complete metric space and let the mapping* $T : X \to X$ *satisfies the following two conditions:*

(I) For all $x \in X$, $T(Tx) \neq x$, *provided* $Tx \neq x$;
(II) T *is a mapping contracting perimeters of triangles on* X.

Then, $Fix(T) \neq \emptyset$ *and* $|Fix(T)| \leq 2$.

4. Conclusion

Recently, Jleli et al. [8] introduced two novel classes of single-valued contractions of polynomial type in metric spaces. Inspired by their work, this paper presents two new classes of single-valued contractions of polynomial type within the framework of G-metric spaces. Motivated by the advancements in fixed point theory in G-metric spaces by Jleli et al. [7], we have extended and generalized the existing results to this setting. Our findings broaden the scope of the results established by Jleli et al. [7], Jleli et al. [8] and Perov [18]. Future research could explore potential applications of these generalized results.

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