Letters in Nonlinear Analysis and its Applications **2** (2024) No. 4, 184-199
Available online at www.lettersinnonlinearanalysis.com
Research Article

Letters in Nonlinear Analysis and its Applications
Peer Review Scientific Journal
ISSN: 2958-874x

The Banach contraction principle: generalizations, extensions and its inverse

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Abstract

In this work, we give several generalizations and extensions of the Banach contraction principle. Its inverse is also studied. The extension is based on weaker topologies. Particularly, we focus on partial metric spaces and b-metric spaces.

Keywords: Fixed point technique, contraction, general topology, (generalized) metric space
2010 MSC: 47H09, 65R20, 54H25

1. Introduction

Fixed point theory examines the existence of an element x belonging to the domain of a mapping T for which $T(x) = x$. Denote by $F(T)$ the set of all fixed points of T. By a fixed point theorem, we mean a statement which asserts that under what conditions a mapping T defined on a set X admits a fixed point.

Example 1.1. (i) A translation has no fixed point.
(ii) A rotation of the plane has a single fixed point (its center).
(iii) If $X = \mathbb{R}$ and $T(x) = x^2 + 6x + 4$, then $F(T) = \{-1, -4\}$.
(iv) If $X = \mathbb{R}$ and $T(x) = x$, then $F(T) = \mathbb{R}$.

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Received: August 26,2024, Accepted: September 29, 2024, Online: October 07, 2024

Fixed Point Theory is divided into the following three major areas:
-Topological fixed point theory;
-Metric fixed point theory;
-Discrete fixed point theory.
Historically, the boundary lines between these three areas was defined by the discovery of three major theorems:

1. Brouwer Fixed Point Theorem (1912).
2. Banach Fixed Point Theorem (1922).
3. Tarski Fixed Point Theorem (1955).

Topological Fixed Point Theory came from Brouwer fixed point theorem in 1912.

Theorem 1.2. *(Brouwer, version 1) Every continuous mapping from a closed ball in $\mathbb{R}^n$ into itself has a fixed point.*

Example 1.3. Any continuous map $T : [a, b] \to [a, b]$ has at least one fixed point.

Another variant of the Brouwer fixed point theorem is as follows:

Theorem 1.4. *(Brouwer, version 2) Let $(X, \|.\|)$ be a finite-dimensional normed space and $K \subset X$ be compact, convex, and nonempty. Any continuous operator $T : K \to K$ has at least one fixed point.*

Example 1.5. The following counter examples show the essentials of each assumption in the Brouwer fixed point theorem (version 2).

1. $K = [0, 1]$ compact, convex and nonempty, but $T : K \to K$ not continuous and the graph $y = T(x)$ does not cross the diagonal $y = x$. No fixed point.

2. $K = \mathbb{R}$ and $T : K \to K$, $T(x) = x + 1$. T is continuous, and K is convex, nonempty, but not compact. No fixed point.

3. Let K be a closed annulus and $T : K \to K$ is a rotation of the annulus around the center. A proper rotation is fixed point free. In this case, K is compact, nonempty, but not convex.

Brouwer fixed point theorem is generalized by Schauder, under the name of Shauder fixed point theorem. There are two versions.

Theorem 1.6. *[24](Schauder, 1930, version 1) Let $(X, \|.\|)$ be a Banach space (over $\mathbb{R}ou$ $\mathbb{C}$) and $K \subset X$ be closed, bounded, convex, and nonempty. Any compact operator $T : K \to K$ has at least one fixed point.*

Theorem 1.7. *(Schauder, 1930, version 2) Let $(X, \|.\|)$ be a Banach space and $K \subset X$ be compact, convex, and nonempty. Any continuous operator $T : K \to K$ has at least one fixed point.*

Remark 1.8. Since a continuous operator on a compact set is always a compact operator, the Schauder fixed point theorem- version 1 yields the Schauder fixed point theorem- version 2.

Further extensions of the Brouwer fixed point theorem are stated as follows:
(Tychonoff, 1935): Every weakly continuous and weakly compact operator from a nonempty closed convex subset of X has a fixed point.
(Kakutani, 1941): Extension of the Brouwer fixed point result to multivalued operators.
(Darbo, 1955): Extension of the Schauder theorem to non-compact operators using the concept of a non-compactness measure.
(Krasnosel'skii, 1955): Giving a fixed point theorem for the sum of two operators by combining the Banach fixed point theorem with the Schauder fixed point theorem in a Banach space.

On the other hand, discrete fixed point theory came from Alfred Tarski [28] in 1955. Tarski fixed point theorem is a discrete fixed point result. Note that a discrete fixed point is a fixed point for functions defined on finite sets.

Theorem 1.9. *[28](Tarski, 1955) If T is a monotone function on a nonempty complete lattice, then the set of fixed points of T forms a nonempty complete lattice.*

After the Tarski fixed point theorem, several works appeared to improve and generalize it in several ways: [Hayashi, 1985], [Heikkila, 1990], [Schroder, 1999], [Jachymski, Gajek, Pokarowski, 2000], [Uhl, 2003], [Ok, 2004]. Tarksi result has many applications in computer science, economics and others. For instance, discrete fixed point theorems have been used to prove the existence of a Nash equilibrium in game theory (The Nash equilibrium of the game is defined as a fixed point of a certain function).
Metric fixed point theory came from the Polish mathematician Stefan Banach (1892-1945). Banach fixed point theorem [2] in 1922 is a useful tool in the study of metric spaces. It ensures the presence and uniqueness of fixed points of particular self-maps of metric spaces and gives a constructive approach to find such fixed points. The concept of metric spaces was introduced in 1906, by the French mathematician Fréchet [8].

Definition 1.10. (Fréchet, 1906) Let X be a nonempty set. A function $d\colon X \times X \to [0,\infty)$ is said to be a metric (distance) if for all $x, y, z \in X$, the following conditions are satisfied:
(i) $d(x,y) = 0 \Longleftrightarrow x = y$ (self-distance);
(ii) $d(x,y) = d(y,x)$ (symmetry);
(iii)$d(x,z) \leq d(x,y) + d(y,z)$ (triangular inequality).

Example 1.11.

1. $X = C([0,1],\mathbb{R})$ and $d_\infty(f,g) = \|f-g\|_\infty = \sup\limits_{t\in[0,1]} |f(t)-g(t)|$.
2. $X = \mathbb{R}^n$. Let $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$.

$$d_1(x,y) = \|x-y\|_1 = |x_1-y_1| + |x_2-y_2| + \cdots + |x_n-y_n|.$$
$$d_2(x,y) = \|x-y\|_2 = \sqrt{|x_1-y_1|^2 + |x_2-y_2|^2 + \cdots + |x_n-y_n|^2}^{\frac{1}{2}}.$$
$$d_\infty(x,y) = \|x-y\|_\infty = \max\{|x_1-y_1|, |x_2-y_2|, \cdots + |x_n-y_n|\}.$$

Now, we state some topological concepts in metric spaces. Let A be a subset of a metric space (X,d).

1. For $a \in X$ and $\varepsilon > 0$, the open ball $B(a,\varepsilon)$ is
$$B(a,\varepsilon) = \{y \in X,\, d(a,y) < \varepsilon\}.$$
The metric d induces a topology τ, where its elements are determined by the open balls of X.

2. A is called open if, for any point x in A, there exists a real number $\varepsilon > 0$ such that $B(x,\varepsilon)$ belongs to A.

3. A is closed if its complement $X\backslash A$ is open in X.

4. Let $\{x_n\}$ be a sequence in X and $x \in X$.
Such a sequence converges to x if $d(x_n, x)$ goes to 0 when n tends to $+\infty$.
Such a sequence is called a Cauchy sequence if $\lim\limits_{n,m\to+\infty} d(x_n, x_m) = 0$.

5. $x \in \overline{A}$ if and only if $d(x,A) =: \inf\{d(x,a), a \in A\} = 0$ if and only if for all $\varepsilon > 0$, $B(a,\varepsilon) \cap A \neq \emptyset$ if and only if there is a sequence $\{x_n\}$ in A such that x is its limit.

6. A is called dense if $\overline{A} = X$.

7. (X, d) is called complete if each Cauchy sequence is convergent.

Definition 1.12. [2](Banach, 1922) Let (X, d) be a metric space, then $T : X \to X$ is said to be a contraction on X if there exists a constant $0 \leq k < 1$ such that

$$d(Tx, Ty) \leq kd(x, y), \quad \text{for all } x, y \in X. \tag{1}$$

The Banach contraction principle (BCP) is as follows:

Theorem 1.13. *[2](Banach, 1922) Let (X, d) be a complete metric space and let T be a contraction on X. Then T has a unique fixed point, say $u \in X$. Moreover, for $x_0 \in X$, define $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ (the Picard iteration), then the sequence $\{x_n\}$ converges to u, the unique fixed point of T. Also, the rate of convergence is controlled as follows:*

$$d(x_n, u) \leq \frac{k^n}{1-k} d(x_0, Tx_0).$$

Remark 1.14. This principle has many applications not only in the various branches of mathematics, but also in economics, chemistry, biology, computer science, engineering, etc.

2. Applications of the BCP

2.1. Newton method for finding roots of equations

Consider the equation $f(x) = x^2 - 3$. Here, $\sqrt{3}$ is one of the two roots of f. Take

$$Tx = x - \frac{f(x)}{f'(x)} = \frac{1}{2}(x + \frac{3}{x}).$$

We see that T is a map from the closed set $[\sqrt{3}, +\infty)$ into itself. Moreover, an element $x \in [\sqrt{3}, +\infty)$ is a fixed point of T if and only if $f(x) = 0$. We also observe that for all $x, y \in [\sqrt{3}, +\infty)$,

$$|T(x) - T(y)| = \frac{1}{2}|x - y||1 - \frac{3}{xy}| \leq \frac{1}{2}|x - y|,$$

that is, T is a contraction mapping. The BCP implies that the Picard iteration (it corresponds to the Newton method which is a fixed point method)

$$x_{n+1} = Tx_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

converges to the root $\sqrt{3}$ for any starting point $x_0 \in [\sqrt{3}, +\infty)$.

Remark 2.1. The most interesting applications of the BCP arise in connection with function spaces.

2.2. Application on integral equations

Consider the linear Fredholm integral equation of second kind:

$$f(x) = g(x) + \lambda \int_a^b h(x, y) f(y) dy, \tag{2}$$

where $g : [a, b] \to [a, b]$ is an unknown function, $k : [a, b] \times [a, b] \to \mathbb{R}$ is a given function (called the kernel) and λ is a parameter. We assume that $g \in C[a, b]$ and the kernel h is continuous on the square $[a, b] \times [a, b]$. Note that a solution $f \in X = C[a, b]$ of (2) is a fixed point of the operator T defined by

$$Tf(x) = g(x) + \lambda \int_a^b h(x, y) f(y) dy. \tag{3}$$

We endow on X the metric d_∞ given as

$$d_\infty(f,g) = \|f-g\|_\infty = \sup_{a\le x\le b} |f(x)-g(x)|, \quad f,g \in X.$$

The continuity of g and h implies that T is a self-operator on X.

The kernel h is continuous, so there must exist $c>0$ so that $h(x,y)\le c$ for all $x,y\in[a,b]$. On the other hand, one obtains for all $f_1, f_2 \in X$,

$$d_\infty(Tf_1, Tf_2) \le c(b-a)|\lambda| d_\infty(f_1,f_2).$$

We choose λ in order that $|\lambda| < \frac{1}{c(b-a)}$. Thus, the operator T is a contraction on X, with a constant $k = c(b-a)|\lambda| < 1$. The BCP yields that T has a unique fixed point, and so the integral equation (2) has a unique solution f in X. Such a solution f is the limit of the Picart iteration (f_n) Defined by

$$f_n(x) = Tf_{n-1}(x) = g(x) + \lambda \int_a^b h(x,y) f_{n-1}(y)dy, \quad n = 1,2,\cdots$$

for any $f_0 \in X$.

2.3. Application on ODE

We consider the following two-point boundary value problem associated with a second-order differential equation:

$$\begin{cases} -x'' = f(t,x(t)), & t\in[0,1] \\ x(0) = x(1) \end{cases} \tag{4}$$

where $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous. Recall that the Green function associated to (4) is given by

$$\begin{cases} G(t,s) = t(1-s), & 0\le t\le s\le 1 \\ G(t,s) = s(1-t), & 0\le s\le t\le 1. \end{cases} \tag{5}$$

Note that $x\in X = C[0,1]$ is a solution of (4) if and only if x is a fixed point of the operator

$$Tx(t) = \int_0^1 G(t,s) f(s,x(s))ds.$$

Assume that for all $\alpha,\beta\in\mathbb{R}$, there is $k<1$ such that

$$|f(s,\alpha) - f(s,\beta)| \le 8k|\alpha-\beta|, \quad \forall\ s\in[0,1].$$

For all $x,y\in X$,

$$|T(x)(t) - T(y)(t)| \le 8k d_\infty(x,y) \int_0^1 G(s,t)ds.$$

As $\int_0^1 G(s,t)ds = \frac{t}{2} - \frac{t^2}{2}$, we get that $\sup_{t\in[0,1]} \int_0^1 G(s,t)ds = \dfrac{1}{8}$. Thus,

$$d_\infty(Tx,Ty) \le k d_\infty(x,y).$$

Applying the BCP, (4) has a unique solution.

The inconvenients of the BCP are:

i) The continuity of T is a necessary condition in BCP.

ii) The completness of X is a necessary condition in BCP.

More precisely,

Question A: Could we find contractive conditions giving existence of a fixed point in a complete metric space without the map is continuous?

Question B: Could we have existence of fixed points without the space X is complete?

Response: Yes for the two questions.
Response to Question A: In 1968, Kannan [11] gave an affirmative answer. He considered this new contraction condition:

$$d(Tx, Ty) \le k[d(x, Tx) + d(y, Ty)], \quad k \in [0, \frac{1}{2}). \tag{6}$$

Example 2.2. Take $X = \mathbb{R}$. Let $T : [0, 2] \to [0, 2]$ be defined by $T(x) = 0$ if $x \in (-\infty, 2]$ and $T(x) = \frac{1}{2}$ if $x > 2$. Here, 0 is the unique fixed point (even T is not continuous, and so T is not a Banach contraction). Namely, the mapping T is a Kannan contraction with $k = \frac{1}{5}$.

Remark 2.3. Kannan contraction opens the door for further new contractive conditions.

Response to Question B: There are incomplete metric spaces on which the Banach contraction mapping has a fixed point.

Example 2.4. Let $X = [0, 1)$ be endowed with the usual distance. Define $T : X \to X$ as $T(x) = \frac{x}{2}$. Here, T is a Banach contraction mapping and has a unique fixed point, even the metric space X is not complete.

Further Questions:

1. How to generalize the Banach contraction?
2. How to extend the BCP to a large class of various spaces?
3. Does the converse of BCP hold?

As mentioned in the response of Question A, Kannan [11] considered the following contractive condition:

$$d(Tx, Ty) \le k[d(x, Tx) + d(y, Ty)], \quad k \in [0, \frac{1}{2}). \tag{7}$$

Theorem 2.5. *[11] Each Kannan contraction mapping on a complete metric space has a unique fixed point.*

Remark 2.6. We note that Kannan fixed point theorem is not an extension of the BCP.

3. Generalizations and extensions of the Banach contraction principle

The BCP has been generalized and extended in several directions:

Case (I): By weakening the contractive nature of the mapping via control function, like altering distance functions [12], implicit functions [21], simulation functions [13] manageable functions [7], α-admissibility [23] ,...).
As an example, we state the definition of a simulation function.

Definition 3.1. [13] A function $\zeta : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ is called a simulation function if
(i): $\zeta(t, r) < r - t$ for all $t, r > 0$;
(ii): For all nonnegative sequences $\{t_n\}$ and $\{r_n\}$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} r_n > 0$, we have $\limsup_{n\to\infty} \zeta(t_n, r_n) < 0$.

Case (II): By weakening the topology (via generalized distances).

3.1. Case (I): Other types of contractions

Some known fixed point results in this direction are given as follows:

Theorem 3.2. *[19] Let (X, d) be a complete metric space, and $T : X \to X$. If for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$x, y \in X, \ \varepsilon \leq d(x, y) < \varepsilon + \delta \quad \textit{implies} \quad d(Tx, Ty) < \varepsilon, \tag{8}$$

then T has a unique fixed point in X.

Remark 3.3. The Meir-Keeler contraction is a weakly uniformly strict contraction. That is,

$$x \neq y \quad \text{implies} \quad d(Tx, Ty) < d(x, y). \tag{9}$$

Each Meir-Keeler mapping is continuous. Each Banach contraction mapping on a metric space is a Meir-Keeler mapping with

$$\delta = \varepsilon(\frac{1}{k} - 1) \quad \text{with } k \in (0, 1). \tag{10}$$

Example 3.4. Let $n \in \mathbb{N}^*$. Endow $X = [0, 1] \cup \{3n, 3n + 1, \cdots\}$ with the Euclidean distance. Consider $T : X \to X$ as follows:

$$\begin{cases} Tx = \frac{x}{2} & \text{if } 0 \leq x \leq 1 \\ Tx = 0 & \text{if } x = 3n \\ Tx = 1 - \frac{1}{n+2} & \text{if } x = 3n + 1. \end{cases}$$

Meir-Keeler contraction holds (T has a unique fixed point, $u = 0$). But, the Banach contraction is not applicable. Indeed, as $n \to +\infty$, we have

$$\frac{d(T(3n), T(3n + 1))}{d(3n, 3n + 1)} = 1 - \frac{1}{n + 2} \to 1.$$

Theorem 3.5. *[5] Let (X, d) be a complete metric space and $T : X \to X$. Then T has a fixed point in X provided that there exists a lower semi-continuous function $\phi : X \to [0, \infty)$ such that*

$$d(x, Tx) \leq \phi(x) - \phi(Tx) \quad \forall\, x \in X. \tag{11}$$

Remark 3.6. The fixed point is not necessarily unique. Each Banach contraction mapping T on a metric space (X, d) is a Caristi mapping with a function

$$\phi(x) = \frac{1}{1 - k} d(x, Tx), \quad \forall\, x \in X, \quad \text{with } k \in (0, 1). \tag{12}$$

One of the applications of the Caristi fixed point theorem is the variational principle of Ekeland. The only known proof of Caristi theorem was based on a discrte technique: Zorn Lemma. There were some trials of finding a pure metric proof of Caristi theorem without success so far, till the work of Wei-Shih Du in the year 2016.

Example 3.7. Let $X = [0, 1]$ and d be the usual metric. Choose $T : X \to X$ as $Tx = \sqrt{x}$. Take $\phi(x) = 1-x$. We have

$$d(x, Tx) \leq \phi(x) - \phi(Tx), \quad \text{for all } x \in X.$$

Using the Caristi fixed point result, T has two fixed points, $u = 0, 1$. Here, we could not apply the BCP because

$$d(T0, T1) > d(0, 1).$$

Theorem 3.8. *[4] Let (X,d) be a complete metric space, and suppose that $T : X \to X$ satisfies*

$$d(Tx,Ty) \leq \psi(d(x,y)) \quad \text{for all } x,y \in X, \tag{13}$$

where $\psi : [0,\infty) \to [0,\infty)$ is upper semi-continuous from the right [i.e., for any sequence $t_n \downarrow t \geq 0 \implies \limsup_{n\to\infty} \psi(t_n) \leq \psi(t)$] and satisfies $\psi(t) < t$ for all $t > 0$. Then T has a unique fixed point.

Remark 3.9. Matkowski [17] considered the same contraction of Boyd-Wong with a nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ such that $\lim_{n\to+\infty} \psi^n(t) = 0$ for all $t > 0$ [Here, $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$.

Example 3.10. Let $X = [0,1] \cup \{2,3,\cdots\}$. We endow on X the complete metric

$$d(x,y) = \begin{cases} |x-y| & \text{if } x,y \in [0,1] \\ x+y & \text{otherwise.} \end{cases}$$

Consider $T : X \to X$ as

$$Tx = \begin{cases} x - \frac{1}{2}x^2 & \text{if } x \in [0,1] \\ x-1 & \text{if } x = 2,3,\cdots. \end{cases}$$

Take

$$\psi(t) = \begin{cases} t - \frac{1}{2}t^2 & \text{if } t \in [0,1] \\ t-1 & \text{if } t > 1. \end{cases}$$

Here, the Boyd-Wong contraction holds (T has a unique fixed point, $u = 0$.). But, as $n \to \infty$, $\frac{d(Tn,T0)}{d(n,0)} \to 1$. That is, the Banach contraction is not applicable.

Theorem 3.11. *[26] Let (X,d) be a complete metric space and $T : X \to X$. Define a nonincreasing function $\theta : [0,1) \to (1/2,1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ (1-r)r^{-2} & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq 2^{-1/2}, \\ (1+r)^{-1} & \text{if } 2^{-1/2} \leq r < 1. \end{cases}$$

Assume that there exists $r \in [0,1)$ such that

$$x,y \in X, \ \theta(r)d(x,Tx) \leq d(x,y) \implies d(Tx,Ty) \leq rd(x,y). \tag{14}$$

Then T has a unique fixed point.

Remark 3.12. Each Banach contraction mapping is a Suzuki mapping.

Definition 3.13. [29] Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be an $F-$contraction if there exists $\tau > 0$ such that for $x,y \in X$;

$$d(Tx,Ty) > 0 \implies \tau + F\big(d(Tx,Ty)\big) \leq F\big(d(x,y)\big) \tag{15}$$

where $F : (0,+\infty) \to \mathbb{R}$ is a function satisfying the following conditions:

(F_1) F is increasing;

(F_2) for each sequence $\{\alpha_n\} \subseteq (0,+\infty)$, we have $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F_3) there exists $0 < k < 1$ such that $\lim_{n\to 0^+} \alpha^k F(\alpha) = 0$.

Example: $F(t) = \ln(t)$, $F(t) = t - \ln(t), \cdots$.

Theorem 3.14. *[29] Each $F-$contraction mapping on a complete metric space has a unique fixed point.*

Now, we consider the cyclic case.

Theorem 3.15. *[14] Let A_1 and A_2 be two nonempty closed subsets of a complete metric space (X, d). Suppose that a cyclic mapping $T : A_1 \cup A_2 \to A_1 \cup A_2$ satisfies the following:*

$$d(Tx, Ty) \leq kd(x, y), \quad \textit{for all } x \in A_1,\ y \in A_2, \tag{16}$$

where $0 \leq k < 1$. Then T has a unique fixed point in $A_1 \cap A_2$.

Remark 3.16. 1. Note that T is cyclic if $T(A_1) \subset A_2$ and $T(A_2) \subset A_1$.
2. If $A_1 = A_2 = X$ in above theorem, we get the Banach fixed point theorem.
3. Above theorem could be extended to a partition $\{A_1, A_2, \cdots, A_p\}$ (p is an integer such that $p \geq 3$).

Example 3.17. Let $A_1 = [-1, 0]$ and $A_2 = [0, 1]$. Define $Tx = -x$. Here, T is cyclic.

Example 3.18. Consider $A_1 = [0, \frac{1}{2}]$, $A_2 = [\frac{1}{4}, \frac{3}{4}]$ and $A_3 = [\frac{1}{6}, 1]$. Let

$$Tx = \begin{cases} \frac{2}{5} & \text{if } x \in [0, \frac{1}{2}] \\ \frac{2}{3}(1-x) & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

We have

$$T(A_1) = \{\frac{2}{5}\} \subset A_2,\ T(A_2) = [\frac{1}{6}, \frac{2}{5}] \subset A_3,\ T(A_3) = [0, \frac{2}{5}] \subset A_1.$$

That is, the map T is cyclic.

The analogue of the BCP for monotone mappings in partially ordered sets is

Theorem 3.19. *[22] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that the metric space (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping w.r.t. $\preceq$. Suppose that the following assertions hold:*

(i) There exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$ with $x \preceq y$;
(ii) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
(iii) T is continuous.

Then T has a fixed point (not necessarily unique).

Remark 3.20. The contractive nature of the mapping in (i) is restricted to the comparable elements of $(X, \preceq)$, not to the entire set X. BCP is not applicable.

Ran and Reurings [22] ensured the existence of a positive definite solution to the linear matrix equation of the type

$$U - A_1^* U A_1 - \cdots - A_m^* U A_m = Q \tag{17}$$

where Q is a positive definite matrix and $A_1, \cdots, A_m$ are arbitrary $n \times n$ matrices.

Let $H(n)$ be the set of Hermitian $n \times n$ matrices and $P(n)$ be the set of positive definite hermitian matrices. Consider the operator $T : X = H(n) \to H(n)$ given as

$$T(U) = A_1^* U A_1 + \cdots + A_m^* U A_m + Q.$$

Note that a fixed point of T is a solution of the matrix equation. For $U, V \in H(n)$, $U \leq V$ means that $(V - U)$ is positive semi-definite. Then the set $H(n)$ is partially ordered. Mention that $X = H(n)$ equipped with the metric induced by $\|.\|_{1,Q}$ is a complete metric space for any positive definite Q, where

$$\|U\|_{1,Q} = \|Q^{\frac{1}{2}} U Q^{\frac{1}{2}}\|_1.$$

Here, $\|B\|_1 = \sum_{j=1}^n s_j(B)$, where $s_j(B)$, $j = 1, \cdots, n$ are the singular values of B. We could ensure there is a fixed point of T, which is a solution of the matrix equation. Observe that T maps $P(n)$ into the set $\{U \in H(n),\ U \geq Q\}$. So the solution must be in this set, and hence it is positive definite.

Theorem 3.21. *[20] Above theorem remains true if we replace the continuity of T by the following hypothesis (A proper condition on X):*
(H): If $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ for all n and $x_n \to x \in X$ as $n \to \infty$,

then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.

Remark 3.22. The uniqueness of the fixed point is guaranteed if we assume the additional hypothesis:
(U): All $x, y \in F(T)$ are comparable.

In 2008, Jachymski [10] initiated the language of graph theory and introduced the concept of G- contractions on a metric space endowed with a graph and obtained some fixed point results which unified most of the previous results concerning the partial ordering (generalizing the partial order concept).

The notion of α-admissibility was initiated in 2012 by Samet et al. [23].

Definition 3.23. [23] Let X be nonempty set. Given $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$. A self-mapping T is said α-admissible if

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1.$$

Some examples :

- Take $X = (0, \infty)$, $Tx = \ln(x)$ and

$$\alpha(x, y) = \begin{cases} 2 & \text{if } x \geq y \\ 0 & \text{if } x < y. \end{cases}$$

- Take $X = [0, \infty)$, $Tx = \sqrt{x}$ and

$$\alpha(x, y) = \begin{cases} e^{x-y} & \text{if } x \geq y \\ 0 & \text{if } x < y. \end{cases}$$

Let Ψ be the set of functions $\psi : [0, \infty) \to [0, \infty)$ such that

i) ψ is nondecreasing;

(ii) $\sum_{n=1}^{+\infty} \psi^n < \infty$ for all $t > 0$.

Definition 3.24. [23] Let (X, d) be a metric space and $T : X \to X$ be a given mapping. We say that T is $\alpha - \psi$-contractive if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)), \text{ for all } x, y \in X. \tag{18}$$

Particular cases:
1. Consider $\alpha(x, y) = 1$ and $\psi(t) = kt$ where $k \in (0, 1)$. Then, we get the Banach contraction.
2. Standard fixed point results: Consider the Boyd-Wong contraction:

$$d(Tx, Ty) \leq \psi(d(x, y)) \quad \text{for all } x, y \in X. \tag{19}$$

It suffices to take $\alpha(x, y) = 1$ in (18). Thus, Boyd-Wong contraction is a particular case of Samet, Vetro and Vetro contraction.

Also, we may consider particular cases for ψ.
3. Fixed point results with a partial order: Consider the contraction

$$d(Tx,Ty) \leq \psi(d(x,y)) \quad \text{for all } \ x \preceq y. \tag{20}$$

Define the mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 \text{ if } x \preceq y, \\ 0 \text{ otherwise.} \end{cases}$$

We get that (20) becomes (18).
The generalizations of Ran-Reurings (2004) and Nieto-Lopez (2005) contractions are particular cases of Samet et al. [23] contraction. 4. Cyclic fixed point results: Consider A_1 and A_2 as closed subsets of a metric space (X,d). Take $Y = A_1 \cup A_2$ and $T : Y \to Y$ such that $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$. Take the cyclic contraction of [14]:

$$d(Tx,Ty) \leq \psi(d(x,y)) \quad \text{for all } (x,y) \in A_1 \times A_2. \tag{21}$$

Define the mapping $\alpha : Y \times Y \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 \text{ if } (x,y) \in A_1 \times A_2, \\ 0 \text{ otherwise.} \end{cases}$$

Then, (21) becomes (18). Thus, Kirk et al. [14] contraction is a simple consequence of Samet et al [23] contraction.

Theorem 3.25. *[23] Let (X,d) be a complete metric space and $T : X \to X$ be an $\alpha-\psi$-contractive mapping. Suppose that*

(i) T is $\alpha-$admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous.

Then T has a fixed point.

Remark 3.26. Above theorem remains true if we replace the continuity of T by the following hypothesis: (H) : If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Remark 3.27. The uniqueness of the fixed point is guaranteed if we assume the additional hypothesis: For all $x, y \in F(T)$, $\alpha(x,y) \geq 1$.

3.2. Case (II): Generalized distances

The metric setting has been extended to some generalized metrics. Among them, we state some ones:
1. Partial metric space [Matthews, 1994], $d(x,x) \neq 0$: study of denotational semantics of dataflow networks (computer science).
2. Cone metric spaces [Perov, 1964 (K-metric), Zhang-Song, 2007], replace $[0,\infty)$ by a real Banach space ordered by a cone: $d : X \times X \to E$.
3. G-metric space [Sims-Mustafa, 2004], $G : X \times X \times X \to [0,\infty)$: depending on three variables.
4. Generalized metric space [Brianciari, 2000], $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$: rectangular inequality.
5. b-metric space [Czerwik, 1993], $d(x,y) \leq s[d(x,z) + d(z,y)]$ with $s \geq 1$: modified triangular inequality.
6. Quasi-metric space, $d(x,y) \neq d(y,x)$. Lack of symmetry.

Remark 3.28. *It is known that the limit of a convergent sequence in either a Branciari metric space, or a partial metric space is not necessarily unique (not Huasdorff). The b-metric, the partial metric and the Branciari metric are all not continuous. A convergent sequence in a Branciari metric space is not necessary a Cauchy sequence.*

We will focus on the partial metric and the b-metric. Namely, we study their topological concepts.
A) Partial metric:
In 1994, Matthews [18] extended the concept of a metric to a partial metric in order to obtain appropriate mathematical models in the theory of computation and, in particular, to give the improvement of the BCP.

Definition 3.29. Let X be a nonempty set. A function $p\colon X \times X \to [0,\infty)$ is said to be a partial metric if for all $x, y, z \in X$ the following conditions are satisfied:
(i) $p(x,x) = p(y,y) = p(x,y) = 0 \Longleftrightarrow x = y$;
(ii) $p(x,y) = p(y,x)$;
(iii) $p(x,x) \leq p(x,y)$;
(iv) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

Remark 3.30. We have $p(x,y) = 0$ implies $x = y$. But, $p(x,x) \neq 0$ (non zero self-distance).

Lemma 3.31. *Let (X,p) a partial metric space. Consider $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$. Then (X,d_p) is a metric space.*

Example 3.32. The classical example: $X = [0,+\infty)$ and $p(x,y) = \max\{x,y\}$.

Example 3.33. Let $S = S_F \cup S_I$, where S_F is the set of all finite sequences and S_I is the set of all infinite sequences. Take $p : S \times S \to [0,+\infty)$ as

$$p(x,y) = 2^{-\sup\{n,\forall\, i<n \text{ such that } x_i = y_i\}}.$$

For the finite sequence $x = (x_1, x_2, ..., x_m)$ (m is a positive integer), the self-distance is $p(x,x) = 2^{-m} \neq 0$. In computer science programming, using finite sequences is more reasonable and affective in case of taking the termination of the program into account. Roughly speaking, one can declare that programming with infinite sequences may leads to infinite loops in running and has a problem of termination and hence getting an output.

Example 3.34. Let $X = \{[a,b] : a \leq b\}$. Take $p(A = [a,b], B = [c,d]) = \max\{b,d\} - \min\{a,c\}$. Here, (X,p) is a partial metric space.

Each partial metric p on X generates a T_0 topology τ_p on X with a base, the family of open balls $\{B(x,\varepsilon), x \in X, \varepsilon > 0\}$, where

$$B(x,\varepsilon) = \{y \in X, p(x,y) < p(x,x) + \varepsilon\}.$$

Remark 3.35. $y \in B(x,\varepsilon)$ does not imply $x \in B(y,\varepsilon)$.

Example 3.36. Let $X = [0,+\infty)$ and $p(x,y) = \max\{x,y\}$. For $\alpha \in X$, observe that

$$B(\alpha, r > 0) = [0, \alpha + r).$$

By taking $y = 0.9$, $x = 1.05$ and $\varepsilon = 0.1$, the precedent remark holds, that is, $y \in B(x,\varepsilon)$, but $x \in B(y,\varepsilon)$.
Also,
1. $A = [0,\beta[$ (for each $\beta > 0$) is an open subset in (X,p).
2. $A =]\gamma,\beta[$ (for all $\gamma < \beta(> 0)$) is not an open subset in (X,p).
One can write

$$\tau_p = \{[0,t),\ 0 \leq t \leq +\infty\}.$$

Clearly, (X,τ_p) has T_0 property, but not having T_1 property (and so T_2 property does not hold, i.e., the space is not Hausdorff).

Now, we give some separation axioms for a partial metric space (general case) due to Han et al. [9].

Theorem 3.37. *[9]Let (X,p) be a partial metric space.*
1. (X,τ_p) has T_1 property if and only if $\forall x \neq y \in X$, $p(x,x) < p(x,y)$.
2. (X,τ_p) has T_2 property (Hausdorff) if and only if $\forall x \neq y \in X$, $\beta_{xy} := \inf_{z\in X}(p(x,z) - p(x,x) + P(y,z) - p(y,y)) > 0$.
3. (X,τ_p) has T_2 property if and only if every convergent sequence has a unique limit in X.

Definition 3.38. Let $\{x_n\}$ be a sequence in a partial metric space (X,p) and $x \in X$.
1. The sequence $\{x_n\}$ converges to x if and only if $\lim\limits_{n\to\infty} p(x_n,x) = p(x,x)$.
2. $\{x_n\}$ is Cauchy if and only if $\lim\limits_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.
3. (X,p) is a complete partial space if each Cauchy sequence $\{x_n\}$ in X is convergent to some $x \in X$ such that $\lim\limits_{n,m\to\infty} p(x_n,x_m) = p(x,x)$.

Remark 3.39. As we said, most of partial metric spaces are not Hausdorff. The next example confirms this assertion. Namely, we haven't uniqueness of limit of a convergent sequence.

Example 3.40. Let $X = [0,\infty)$ and $p(x,y) = \max\{x,y\}$. The convergent sequence $\{\frac{1}{n}\}_{n\geq 1}$ has an infinite number of limits in (X,p).

Proposition 3.41. *1. A sequence $\{x_n\}$ in a partial metric space (X,p) is Cauchy if and only if $\{x_n\}$ is Cauchy in the metric space (X,d_p).*
2. A partial metric space (X,p) is complete if and only if the metric space (X,d_p) is complete.

Lemma 3.42. *Let (X,p) be a partial metric space and A be a subset in X. We have*

$$x \in \bar{A} \Longleftrightarrow p(x,A) =: \inf\{p(x,y),\ y \in A\} = p(x,x).$$

Example 3.43. Let $X = [0,+\infty)$ and $p(x,y) = \max\{x,y\}$. Take $A = [0,1]$. We have $\bar{A} = X$. Moreover,

$$\overline{\{\frac{1}{n}, n \geq 1\}} = [0,+\infty) = X.$$

That is, (X,p) is separable.

B) b-metric:
A b- metric [6] corresponds to a modified triangular inequality: $d(x,y) \leq s[d(x,z) + d(z,y)]$ with $s \geq 1$.

Example 3.44. Let (X,d) be a metric space, and $\rho(x,y) = (d(x,y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.

Example 3.45. Let $0 < p < 1$. Consider $l_p = \{x = (x_n),\ \sum\limits_{n=1}^{+\infty} |x_n|^p < \infty\}$. Take $d(x,y) = (\sum\limits_{n=1}^{+\infty} |x_n - y_n|^p)^{\frac{1}{p}}$. Here, d is a b-metric on l_p with $s = 2^{\frac{1}{p}}$.

Example 3.46. Let $0 < p < 1$. Take $X = L^p([0,1])$ and $d(f,g) = (\int_0^1 |f(x) - g(x)|^p dx)^{\frac{1}{p}}$. Again, d is a b-metric on L^p with $s = 2^{\frac{1}{p}}$.

The family of all open balls $B(x,r) = \{y \in X,\ \ d(x,y) < r\}$ does not form a base for any topology on a b-metric space (X,d) (there exists an open ball $B(a,r)$ which is not open).

Example 3.47. [16]Let $X = \{0, 1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots\}$ and

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \in \{0,1\} \\ |x-y| & \text{if } x \neq y \in \{0\} \cup \{\frac{1}{2n}, n = 1, 2, \cdots\} \\ 4 & \text{otherwise.} \end{cases}$$

Then we have
(1) d is a b-metric on X with $s = \frac{8}{3}$.
(2) d is not a metric on X.
(3) d is not continuous in each variable.
(4) $0 \in B(1,2) = \{0,1\}$, but $B(0,r) \nsubseteq B(1,2)$ for each $r > 0$.

Remark 3.48. On a b-metric space, we consider the topology induced by its convergence. Note that Convergence, Cauchyness, Completness in b-metric spaces are the same as in metric spaces.

Lemma 3.49. *Let (X,d) be a b-metric space and $x, y \in X$ such that $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(y_n, y) = 0$. Then*

$$s^{-2}d(x,y) \le \liminf_{n\to\infty} d(x_n, y_n) \le \limsup_{n\to\infty} d(x_n, y_n) \le s^2 d(x,y),$$

and for all $z \in x$,

$$s^{-1}d(x,z) \le \liminf_{n\to\infty} d(x_n, z) \le \limsup_{n\to\infty} d(x_n, z) \le s d(x,z).$$

Remark 3.50. A b-metric is not continuous. That is, if $x_n \to x$ and $y_n \to y$ as $n \to \infty$, we haven't $d(x_n, y_n) \to d(x,y)$. Also, we haven't $d(x_n, z) \to d(x,z)$.

Remark 3.51. [27] $\sum_{n=1}^{+\infty} d(x_n, x_{n+1}) < \infty$ does not imply (in general) the Cauchyness on sequences. Such a property is one of characteristics of a b-metric space.

3.3. On functional inequalities

For $\rho > 0$, let us denote by L_ρ the set of functions $\ell : [0,\infty) \to [0,\infty)$ satisfying

$$c_\rho := \inf_{t>0} \frac{\ell(t)}{t^\rho} > 0. \tag{22}$$

In 2024, Aydi et al. [1] ensured the existence of fixed points for mappings satisfying functional inequalities.

Theorem 3.52. *[1] Let $C \subset [0,\infty)$, $C \neq \emptyset$, and $f : C \to C$. Assume that the following conditions hold:*

(i) *C is a closed subset;*

(ii) *For some $\rho > 0$, $\ell \in L_\rho$, and $\alpha, \sigma \in (0,1)$,*

$$\ell(\alpha f(t) + (1-\alpha) f(s)) \le \sigma \ell(\alpha t + (1-\alpha)s) \tag{23}$$

for every $t, s \in C$ with $f(t) \neq f(s)$;

(iii) *For every $t, s \in C$, if $\lim_{n\to\infty} f^n(t) = s$, then $\{f^n(t)\}$ admits a subsequence $\{f^{n_q}(t)\}$ such that $\lim_{q\to\infty} f(f^{n_q}(t)) = f(s)$.*

Then, f possesses a unique fixed point. Moreover, for every $t_0 \in C$, the sequence $\{f^n(t_0)\}$ converges to this fixed point.

Remark 3.53. As particular cases when ℓ is a convex or concave function, and making use of Hermite-Hadamard inequalities, we deduce several new fixed point theorems.

Example 3.54. Let $C = \{0,1,3,5\}$ and $f : C \to C$ be the function defined by

$$f(0) = 0, \; f(1) = 5, \; f(5) = 3, \; f(3) = 0.$$

Observe that the Banach fixed point theorem is not applicable. We now introduce the function $\ell : [0,\infty) \to [0,\infty)$ defined by

$$\ell(t) = \begin{cases} 8t & \text{if} \quad 0 \leq t \leq \dfrac{1}{2}, \\ \dfrac{2}{3}t & \text{if} \quad \dfrac{1}{2} < t \leq \dfrac{3}{2}, \\ 3t & \text{if} \quad \dfrac{3}{2} < t \leq 2, \\ \dfrac{4}{5}t & \text{if} \quad 2 < t \leq \dfrac{5}{2}, \\ 3t & \text{if} \quad \dfrac{5}{2} < t \leq 3, \\ \dfrac{7}{4}t & \text{if} \quad t > 3. \end{cases}$$

Note that $\ell(t) \geq \frac{2}{3}t$ and so $\ell \in L_1$. Here, f has a fixed point, which is $u = 0$.

4. Characterization of metric completeness (Inverse of the BCP)

The famous BCP holds in complete metric spaces, but as said before in the response of question B, completeness is not a necessary condition. Indeed, we said there are incomplete metric spaces on which every contraction mapping has a fixed point ($Tx = \frac{x}{2}$ and $X = [0, 1)$). On the other hand, contractions on incomplete metric spaces may fail to have fixed points.

Example 4.1. Let $X = (0, 1]$ with the usual distance. Define the same $T : X \to X$ as $T(x) = \frac{x}{2}$.

This principle does not characterize the metric completeness of a metric space. Characterizations of metric completeness have been discussed in several works, that is, to present various circumstances in which fixed point results imply completeness. The first result is due to Bessaga [3].

Theorem 4.2. *[3] Let T be a self-mapping on a nonempty set X such that each iterate T^n has a unique fixed point. Let $q \in (0, 1)$, then there exists a complete metric d on X such that T is contractive, and q is the contraction constant.*

Later, Kirk [15] showed that Caristi's theorem characterizes the metric completeness.

Theorem 4.3. *[15] A metric space (X, d) is complete if and only if every Caristi mapping on (X, d) has a fixed point.*

Going in same direction, Kannan theorem characterizes the metric completeness.

Theorem 4.4. *[25] A metric space (X, d) is complete if and only if every Kannan mapping on (X, d) has a fixed point.*

Finally, Suzuki theorem characterizes the metric completeness.

Theorem 4.5. *[27] A metric space (X, d) is complete if and only if every Suzuki mapping on (X, d) has a fixed point.*

Remark 4.6. Suzuki contraction and Kannan contraction are independent, but both types of contractions characterize the metric completeness of the underlying spaces.

Data availability

No data is associated with this study.

Conflicts of interest

The authors declare that they have no conflicts of interest.

Author's contributions

All authors contributed equally and significantly in writing this article.

Funding

Not applicable.

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