Letters in Nonlinear Analysis and its Applications **2** (2024) No. 4, 155-183
Available online at www.lettersinnonlinearanalysis.com
Research Article

Letters in Nonlinear Analysis and its Application
Peer Review Scientific Journal
ISSN: 2958-874x

Meir-Keeler type mappings on b-metric spaces

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Abstract

In this study, we aim to present Meir-Keeler contraction mappings results on b-metric spaces. We collect and combine considerable Meir-Keeler fixed point results on b-metric spaces and generalized b-metric spaces. We have also presented the fixed point studies in b-metric spaces via admissible mappings.

Keywords: Fixed point theory, Meir-Keeler contraction, b-metric, admissible mapping
2010 MSC: 47H10; 54H25

1. Introduction

Fixed point theory in mathematical science and other applied science problems has been a very popular research topic in recent years. Developments in this field provide powerful tools for the existence and uniqueness of fixed points. The Banach contraction mapping principle [7], which Stefan Banach published in 1922, is the cornerstone of fixed point theory. The Banach fixed point theorem is:

A fixed point of a mapping $T : X \to X$ is a point $x \in X$ such that $T(x) = x$.

The famous theorem has been generalized from many directions in various structures. One of these generalizations is the Meir-Keeler contraction mapping. In 1969, Meir and Keeler [33] established a fixed point theorem for self-maps of a metric space

$$(X, d) \text{ which satisfy } \varepsilon \leqslant d(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) < \varepsilon, \text{ where } \varepsilon, \delta > 0.$$

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Received: July 17,2024, Accepted: October 01, 2024, Online: October 06, 2024

This contraction mapping is an important research topic in fixed point theory. Many research papers are published on the Meir-Keeler contraction in different generalizations and in various spaces (see, [29, 30]). Meir-Keeler contraction mappings play a crucial role in fixed point theory within metric spaces and contribute to extending results in b-metric spaces. b-metric spaces, as a generalization of classical metric spaces, introduce new and intriguing challenges in fixed point theory due to their more flexible and adaptable structures.

In this paper, we position our framework within the field of Meir-Keeler's original fixed point results based on a literature review. We aim to recollect and discuss Meir-Keeler fixed point results on b-metric spaces. We present some results in this sense by involving interesting examples. The paper can be a guide for new researchers about the fixed point of the study.

The structure of this study is organized into three main sections. In the first section, we introduce the original Meir-Keeler contraction mapping and provide some essential preliminaries. Section 2 focuses on presenting fixed point results for Meir-Keeler contractions within the context of b-metric spaces. In Section 3, we present Meir-Keeler contraction mappings on b-metric spaces via the concept of admissibility. Prior to each section, we highlight the fundamental characteristics related to results to provide the necessary background for understanding the subsequent material.

1.1. Meir-Keeler Contraction

We start with some notation, basic definitions and well-known fixed point results.

Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers, that is, $\mathbb{N} = \{1, 2, \ldots\}$. We denote by $\mathbb{Z}$ the set of integers, that is, $\mathbb{Z} = \mathbb{N} \cup (-\mathbb{N}) \cup \{0\}$ and $\mathbb{Z}^+ := \mathbb{N}$. The symbols $\mathbb{R}$ the set of all real numbers, $\mathbb{R}_0^+$ denotes the set of all non-negative real numbers, that is, $\mathbb{R}_0^+ := [0, \infty)$ and $\mathbb{R}^+ := (0, \infty)$. Throughout the paper, all considered sets will be presumed nonempty.

In this section, we present the original results of Meir and Keeler's paper [33]. We presume that (X, d) is a complete metric space. For a self mapping T on X, if there is a real number $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leqslant \lambda \cdot d(x, y),$$

for all $x, y \in X$, then T possesses a unique fixed point (i.e., there exists a point $x^* \in X$ such that $Tx^* = x^*$).

Meir and Keeler [33] extended Banach's metric fixed point theorem by replacing contraction condition with "weakly uniformly strict contraction" as follows:

Theorem 1.1. *Let (X, d) is a complete metric space. For a self mapping T on X, if for all $\varepsilon > 0$, there is a $\delta > 0$ so that*

$$\varepsilon \leqslant d(x, y) < \varepsilon + \delta \quad \textit{implies} \quad d(Tx, Ty) < \varepsilon, \tag{1}$$

then T has a unique fix point x^*. Also, we have

$$\lim_{n\to\infty} T^n x = x^*. \tag{2}$$

for any initial point $x \in X$.

Lemma 1.2. *[14] If $T : X \to X$ is a strict contraction and if, for every $x \in X$, $\{T^n(x)\}$ is a Cauchy sequence, then T has a unique fixed point and (2) holds.*

Lemma 1.3. *Condition (1) implies that*

$$\lim d(x_n, x_{n+1}) = 0.$$

The proof is by contradiction. Let $c_n = d(x_n, x_{n+1})$. c_n is decreasing with n. If $c_n \to \varepsilon > 0$, then (1) fails for c_{m+1} where c_m is chosen less than $\varepsilon + \delta$.

Having proved Lemma 1.3, we now suppose that some sequence is not a Cauchy sequence. Then there exists $2\varepsilon > 0$ such that $\limsup d(x_m, x_n) > 2\varepsilon$. By hypothesis, there exists a $\delta > 0$, such that

$$\varepsilon \leqslant d(x,y) < \varepsilon + \delta \quad \text{implies } d(Tx,Ty) < \varepsilon. \tag{3}$$

Formula (3) will remain true with δ replaced by $\delta' = \min(\delta, \varepsilon)$. From Lemma 1.3 we can find M so that $c_M < \delta'/3$. Pick $m, n > M$ so that $d(x_m, x_n) > 2\varepsilon$. For j in $[m, n]$,

$$|d(x_m, x_j) - d(x_m, x_{j+1})| \leqslant c_j < \frac{\delta'}{3}.$$

This implies, since $d(x_m, x_{m+1}) < \varepsilon$ and $d(x_m, x_n) > \varepsilon + \delta'$, that there exists j in $[m, n]$ with

$$\varepsilon + \frac{2\delta'}{3} < d(x_m, x_j) < \varepsilon + \delta' \tag{4}$$

However, for all m and j,

$$d(x_m, x_j) \leqslant d(x_m, x_{m+1}) + d(x_{m+1}, x_{j+1}) + d(x_{j+1}, x_j).$$

So, by (3) and (4),

$$d(x_m, x_j) \leqslant c_m + \varepsilon + c_j < \frac{\delta'}{3} + \varepsilon + \frac{\delta}{3}$$

which contradicts (4). This contradiction proves that x_n must be a Cauchy sequence, and establishes Meir-Keeler theorem.

Edelstein [20] result follows from Meir-Keeler theorem, since in a compact space, any strict contraction $T : X \to X$ is weakly uniformly strict. And, we have the following condition;

$$\inf_{\varepsilon \leqslant d(x,y)} [d(x,y) - d(Tx,Ty)] = \delta(\varepsilon).$$

Since X is compact, this infimum is achieved for some pair of points (a, b) with $d(a, b) \geqslant \varepsilon$. Since T is a strict contraction $\delta(\varepsilon) > 0$.

Rakotch [36] and Boyd and Wong [13] work with a function $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfying the following conditions $\psi(0) = 0, \psi(t) < t$ for all $t > 0$ and ψ is right upper semicontinuous, assume the inequalities

$$d(Tx,Ty) < \psi(d(x,y)) \text{ and } \psi(d) < d \tag{5}$$

The first example confirms that (5) may be violated while the hypothesis (1) of Meir-Keeler theorem is fulfilled.

Example 1.4. Let $X = [0,1] \cup \{3, 4, 6, 7, \ldots, 3n, 3n+1, \ldots\}$ with the Euclidean distance, and let Tx be defined as follows:

$$\begin{array}{lll} Tx = \frac{x}{2} & & 0 \leqslant x \leqslant 1 \\ Tx = 0 & \text{if} & x = 3n \\ Tx = 1 - \frac{1}{n+2} & \text{if} & x = 3n+1. \end{array}$$

Although T satisfies Meir-Keeler hypothesis (1), $\psi(1)$ would have to be 1. Moreover, Rakotch[36] and Boyd and Wong results [13] follow easily from Meir-Keeler theorem.

The second example confirms that (5) may be satisfied in a complete metric space, while the mapping T has no fixed point.

Example 1.5. Let $s_n = \sum_{k=1}^{n}(1 + 1/k)$, and let $X = \{s_n\}$. Let $Ts_n = s_{n+1}$ for all n. Then

$$d(Tx,Ty) \leqslant \psi(d(x,y)) \text{ with } \psi(1 + 1/n) = 1 + 1/(n+1),$$

but there is no fixed point .

We also present this result by Ciric [15].

Definition 1.6. [15] Let (X, d) be a metric space and let T be a mapping of X into itself. A mapping T is called contractive if it satisfies

$$d(Tx, Ty) < d(x, y)$$

for all x, y in X with $x \neq y$.

Theorem 1.7. *[15] Let X be a complete metric space and let T be a mapping of X into itself satisfying the condition.*

Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon < d(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) \leq \varepsilon. \tag{6}$$

Then T has a unique fixed-point u in X and $\lim_{n\to\infty} T^n x = u$ for each x i X.

Example 1.8. [15] Let

$$M = \left\{0, 1, 2, 3, 4 + \frac{1}{2}, \ldots, 3n, 3n + 1 + \frac{1}{n+1}, \ldots\right\}$$

be a subset of reals with the usual metric and let T on X be defined by

$$\begin{aligned} Tx &= 0, \quad \text{if } x = 0, 1, 3, \ldots, 3n, \ldots \\ Tx &= 1, \quad \text{if } x = 4 + \frac{1}{2}, 7 + \frac{1}{3}, \ldots, 3n + 1 + \frac{1}{n+1}, \ldots \end{aligned}$$

Then for each $\varepsilon > 0$ and any $\delta > 0$ the mapping T satisfies (6). However, T does not satisfy (1), as for $\varepsilon = 1$ there is no $\delta > 0$ such that (1) holds. For, if we assume that $\delta(1) > 0$, then we may choose a sufficiently large n such that

$$1 \leq d\left(3n, 3n + 1 + \frac{2}{n}\right) < 1 + \delta$$

Then by (1) this implies $d\left(T(3n), T\left(3n + 1 + \frac{1}{2}\right)\right) < 1$, which is incorrect.

2. Meir-Keeler Contraction Mappings on b-metric Spaces

We present fixed point results for Meir-Keeler contractions on b-metric spaces. Firstly, we recall some b-metric fixed point results.

The idea of b-metric was initiated from Bakhtin [8]. According to Czerwik; "Some problems, particularly the problem of the convergence of measurable functions with respect to a measure, lead to a generalization of concept of a metric." Inspired by this idea, Czerwik [18] introduced an axiom which was weaker than the triangular inequality and formally established a so-called b-metric spaces with a view of generalizing the famous Banach fixed point theorem. We give the definition of a b-metric space.

Definition 2.1. (Bakhtin [8], Czerwik [17, 18]). Let X be a (nonempty) set and let $s \geq 1$ be a given real number. A functional $b : X \times X \to \mathbb{R}_0^+$ is said to be a b-metric if and only if for all $x, y, z \in X$.

(1) $b(x, y) = 0$ if and only if $x = y$;

(2) $b(x, y) = b(y, x)$;

(3) $b(x, z) \leq s[b(x, y) + b(y, z)]$.

The pair (X, b) is called a b-metric space.

Theorem 2.2. *[18] Let (X,d) be e a complete b-metric space and let $T: X \to X$ satisfy*

$$d(T(x),T(y)) \leq \varphi(d(x,y)), x,y \in X, \tag{7}$$

where $\varphi: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is increasing function such that $lim_{n\to\infty}\varphi^n(t) = 0$ for each fixed > 0. Then T has exactly one fixed point u and

$$\lim_{n\to\infty} d\left(T^n(x),u\right) = 0$$

for each $x \in X$.

Remark 2.3. The class of b-metric spaces is effectively larger than that of metric spaces since any metric space is a b-metric space with $s = 1$. b-metric space need not necessarily be a metric space.

Example 2.4. [41] Let $X = [0,1]$ and $d: X \times X \to [0,\infty)$ be defined by

$$d(x,y) = \left|x^2 - y^2\right|, \text{ for all } x,y \in X$$

It is clear that (X,d) is not a metric space, but (X,d) is a b-metric space with coefficient $s \geq 2$.

Example 2.5. [17] Let p be a given real number in the interval $(0,1)$. The space $L_p[0,1]$ of all functions $x: [0,1] \to \mathbb{R}$ such that $\int_0^1 |x(t)|^p dt < 1$, together with the mapping $d: L_p[0,1] \times L_p[0,1] \to \mathbb{R}_0^+$defined by

$$d(x,y) := \left(\int_0^1 |x(t) - y(t)|^p dt\right)^{1/p} \text{ for each } x,y \in L_p[0,1]$$

is a b-metric space with coefficient $s = 2^{1/p}$.

Example 2.6. [42] Let $X = \{x_1,x_2,x_3\}$ and $d: X \times X \to \mathbb{R}_0^+$ such that

$$d(x_1,x_2) = a > 2, d(x_1,x_3) = d(x_2,x_3) = 1 \text{ and } d(x_n,x_n) = 0$$

$d(x_n,x_k) = d(x_k,x_n), d(x_n,x_k) \leq \frac{a}{2}\left[d(x_n,x_i) + d(x_i,x_k)\right], n,k,i = 1,2,3$ Then (X,d) is a b-metric space.

The following lemma is very useful.

Lemma 2.7. *[17] Let (X,d) be a b-metric space.*

(i) Then $D(x,A) \leq s[d(x,y) + D(y,A)]$, for all $x,y \in X, A \subset X$.

(ii) Let $\{x_k\}_{k=0}^n \subset X$. Then:

$$d(x_n,x_0) \leq sd(x_0,x_1) + \ldots + s^{n-1}d(x_{n-2},x_{n-1}) + s^{n-1}d(x_{n-1},x_n).$$

(iii) Let for all $A,B,C \in X$ we have: $H(A,C) \leq s[H(A,B) + H(B,C)]$.

(iv) (1) $A,B \in P_{cl}(X)$. Then for each $\alpha > 0$ and for all $b \in B$ there exists $a \in A$ such that: $d(a,b) \leq H(A,B) + \alpha; v$

(2) $A,B \in P_{cp}(X)$. Then for all $b \in B$ there exists $a \in A$ such that: $d(a,b) \leq sH(A,B).v$.

2.1. Generalized fixed point results in b-metric spaces

Firstly, we present generalized Meir-Keeler contraction in the context of b-metric spaces, which was introduced by Aydi, et al [4]

Lemma 2.8. *Let $(X, b, s > 1)$ be a b-complete b-metric space and $T : X \to X$ such that condition Meir-Keeler[33] holds. If $\{T^n x\}$ is a b-Cauchy sequence for each $x \in X$, then T has a unique fixed point, say $u \in X$ and $T^n x \to u$.*

Theorem 2.9. *Let (X, b) be a complete b-metric space and let T be a self-mapping on X satisfying the following condition.*

Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq b(x, y) < \varepsilon + \delta \text{ implies } s^a b(Tx, Ty) < \varepsilon, \tag{8}$$

where $a > 0$ is given.

Then T has a unique fixed point, say $u \in X$, and, for each

$$x \in X, \lim_{n\to\infty} T^n x = u.$$

Proof. It is clear that, for all $x, y \in X$ with $b(x, y) > 0$, we obtain

$$b(Tx, Ty) \leq \lambda b(x, y) \tag{9}$$

where $\lambda = 1/s^a \in [0, 1)$.

Let $x_0 \in X$ be an arbitrary point. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. If, for some $n, x_n = x_{n+1}$, then x_n is a fixed point of T. From now on, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. From condition (9), we obtain

$$b(x_n, x_{n+1}) \leq \lambda b(x_{n-1}, x_n).$$

The sequence $\{x_n\}$ is b-Cauchy in the b-metric space (X, b). By b-completeness of (X, b), there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u \tag{10}$$

Finally, (9) and (10) imply that $Tu = u$; that is, u is the unique fixed point of T in X. □

Example 2.10. Let $X = \{0, 1, 2\}$ and define $b : X \times X \to [0, +\infty)$ as follows:

$b(x, x) = 0, b(x, y) = b(y, x)$ for all $x, y \in X, b(0, 1) = 1, b(0, 2) = 2.2$, and $b(1, 2) = 1.1$. Then

$(X, b, 22/21 > 1)$ is a b-complete b-metric space, but it is not a metric space. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} 0, & \text{if } x \neq 2 \\ 1, & \text{if } x = 2 \end{cases}$$

We shall check that, for all $x, y \in X$, the contractive condition (9) holds. For this, we distinguish three cases.

(a) $x = 0, y = 1 \Longrightarrow b(T0, T1) = b(0, 0) = 0$. Obviously, condition (9) holds.

(b) $x = 0, y = 2 \Longrightarrow b(T0, T2) = b(0, 1)$. Since $(22/21)^a b(0, 1) \leq b(0, 2)$, i.e., $(22/21)^a \cdot 1 \leq 2.2, a > 0$, which is true, hence, again (9) holds.

(c) $x = 1, y = 2 \Longrightarrow b(T1, T2) = b(0, 1) = 1$. Now, we have $(22/21)^a \cdot 1 \leq 1.1$, i.e., $(22/21)^a \leq 1.1$, which is also true because $a > 0$.

Therefore, condition(9) holds for each $a > 0$. However, condition (8) is not true for $a = 1$. Indeed, for $x = 0$ and $y = 2$, it becomes

$$\varepsilon \leq b(0,2) < \varepsilon + \delta \text{ implies } \frac{22}{21} < \varepsilon,$$

or equivalently

$$\varepsilon \leq 2.2 < \varepsilon + \delta \text{ implies } \frac{22}{21} < \varepsilon.$$

Take $\varepsilon = 1/2$. Then there exists $\delta = \delta(1/2) > 0$ such that $1/2 \leq 2.2 < 1/2 + \delta$ (for example, any $\delta > 17/10$). But $22/21 < 1/2$ is false.

In the sequel, we consider ε-contractive mappings in the context of b-metric spaces.

Definition 2.11. A mapping T of a b-metric space $(X, b, s \geq 1)$ into itself is said to be ε-contractive, if and only if there exists $\varepsilon > 0$ such that

$$0 < b(x,y) < \varepsilon \text{ implies } b(Tx,Ty) < b(x,y).$$

The following results extend ones from standard metric spaces to b-metric spaces, with a continuous b-metric b.

Theorem 2.12. *Let $(X, b, s \geq 1)$ be a b-metric space with continuous b-metric b and T an ε-contractive self-mapping on X. If, for some $x_0 \in X$, the sequence $\{T^n x_0\}$ has a convergent subsequence $\{T^{n_i} x_0\}$ to $u \in X$, then u is a periodic point; that is, there exists a positive integer k such that $T^k u = u$.*

Secondly, we present Meir-Keeler contraction on a complete b-metric established by Lu et al [32].

Theorem 2.13. *(A direct consequence of Theorem 2.1 in [44]). Let (X, b, s) be a complete b-metric space and $T : X \to X$ be a map satisfying for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$,*

$$\varepsilon \leq b(x,y) < \varepsilon + \delta \Rightarrow s \cdot b(Tx,Ty) < \varepsilon \tag{11}$$

Then T has a unique fixed point x^*, and for each $x \in X, \lim_{n\to\infty} T^n x = x^*$.

Note that $s \cdot b(Tx,Ty) < \varepsilon$ in (11) replaces the condition $b(Tx,Ty) < \varepsilon$.

Lemma 2.14. *Let $m, n \in \mathbb{N}$ and $m \leq n$. Then we have*

$$\frac{\sum_{i=m}^{n} \frac{1}{i}}{\sum_{i=m}^{n} \frac{1}{i+1}} \leq 2$$

Remark 2.15. Let (X, b, s) be a b-metric space and $T : X \to X$ be a Meir-Keeler contraction. Then for all $x, y \in X$,

$$b(Tx,Ty) \leq b(x,y)$$

In particular, for all $x \neq y \in X$,

$$b(Tx,Ty) < b(x,y)$$

Indeed, if $x = y$ then $b(Tx,Ty) = b(x,y)$. If $x \neq y$, then $b(x,y) > 0$. By choosing $\varepsilon = b(x,y)$, we get $b(Tx,Ty) < b(x,y)$.

Example 2.16. Let $\tau_n = \sum_{i=1}^{n} \frac{1}{i}$ for all $n \in \mathbb{N}, X = \{\tau_n : n \in \mathbb{N}\}$ and $T : X \to X$ is defined by $T\tau_n = \tau_{n+1}$ for all $n \in \mathbb{N}$. Put

(1) $p_{k,k+n-m} = |\tau_k - \tau_{k+n-m}|$ for $k \in \mathbb{N}$, and let $q_{m,n} \in \mathbb{Z}$ be such that

$$p_{m,n} = p_{m,m+n-m} \in \left[2^{q_{m,n}}, 2^{q_{m,n}+1}\right) \text{ for } m \leq n \text{ in } \mathbb{N}$$

(2) $b_{i,q} = \sum_{k=0}^{i} \frac{2^q - 2^{q-1}}{2^k}$ for $q \in \mathbb{Z}$, and $i \in \mathbb{N} \cup \{0\}$;

(3) let

$$l(m,n) = \max\left\{k \in \mathbb{N} : p_{k,k+n-m} \in \left[2^{q_{m,n}}, 2^{q_{m,n}+1}\right)\right\} + 1$$

and let $i_0(m,n) \in \mathbb{N}$ be such that $p_{l_{m,n},l_{m,n}+n-m} \in \left[b_{i_0(m,n),q_{m,n}}, b_{i_0(m,n)+1,q_{m,n}}\right)$ and $p_{k,k+n-m} \in \left[2^{q_{m,n}}, 2^{q_{m,n}+1}\right)$ for all $m \leq k \leq l_{m,n} - 1$.

For $m = n$, put $b(\tau_n, \tau_m) = 0$, and for $m < n$, put

$$b(\tau_n, \tau_m) = b(\tau_m, \tau_n) = \begin{cases} |\tau_n - \tau_m| & \text{if } m = k_0(m,n) \\ 2^{q_{m,n}} & \text{if } m = k_0(m,n) + 1 \\ & \text{and } p_{k_0(m,n),k_0(m,n)+n-m} \geq b_{1,q_{m,n}+1} \\ b_{i_0(m,n)+l_{m,n}-m,q_{m,n}} & \text{otherwise} \end{cases}$$

where

$$k_0(m,n) = \min\left\{k \in \mathbb{N} : p_{k,k+n-m} \in \left[2^{q_{m,n}}, 2^{q_{m,n}+1}\right)\right\}$$

Then we have

(1) (X, b, s) is a complete b-metric space with $s = 2$.

(2) b is a continuous b-metric.

(3) T is a Meir-Keeler contraction on (X, b, s) that is fixed point free.

Thirdly, we consider Meir-Keeler results with Ćirić theorem ([16]) in b-metric spaces by Pavlovic and Radenovic [34].

Theorem 2.17. *Let (X, d) be a complete metric space and let T be a contractive self-mapping on X satisfying the next condition:*

Given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$

$$\varepsilon \leq d(x,y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) \leq \varepsilon$$

Then T has a unique fixed point, say $u \in X$, and for each $x \in X$, $\lim_{n\to\infty} T^n x = u$.

Theorem 2.18. *Let $(X, b, s > 1)$ be a b-complete b-metric space and let T self-mapping on X satisfy the following condition:*

Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq b(x,y) < \varepsilon + \delta \text{ implies } s^a b(Tx, Ty) < \varepsilon,$$

where $a > 0$ is given.

Then T has a unique fixed point, say $u \in X$, and for each $x \in X$, $\lim_{n\to\infty} T^n x = u$.

Proof. It is clear that for all $x, y \in X$ we obtain

$$b(Tx, Ty) \leq kb(x,y) \tag{12}$$

where $k = \frac{1}{s^a} \in [0, 1)$.

Let $a_0 \in X$ be an arbitrary point. Define the sequence $\{a_n\}$ by $a_{n+1} = Ta_n$ for all $n \geq 0$. If $a_n = a_{n+1}$ for some n, then a_n is a fixed point (unique) of T and the results follows.

So, suppose that $a_n \neq a_{n+1}$ for all $n \geq 0$. From the condition (12), we obtain

$$b(a_n, a_{n+1}) \leq kb(a_{n-1}, a_n).$$

Further, we obtain that $\{a_n\}$ is a b-Cauchy sequence in a b-metric space (X, b). By the b-completeness of (X, b), there exists $u \in x$ such that

$$\lim_{n\to\infty} a_n = u. \tag{13}$$

Finally, (12) and (13) imply that $Tu = u$, i.e. u is a unique fixed point of T in X. □

We give the following result which generalizes Theorem 2.17 in several directions:

Theorem 2.19. *Let $(X, b, s > 1)$ be a b-complete b-metric space and let $T, g : X \to X$ be two self-maps such that $T(X) \subset g(X)$, one of these two subsets of X being b-complete. Suppose the following conditions hold: for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq b(gx, gy) < \varepsilon + \delta \textit{ implies } s^a b(Tx, Ty) < \varepsilon$$

and $Tx = Ty$ whenever $gx = gy$, where $a > 0$ is given.

Then T and g have a unique point of coincidence, say $z \in X$. Moreover, for each $x_0 \in X$, the corresponding Jungck sequence $\{y_n\}$ can be chosen such that $\lim_{n\to\infty} y_n = z$. In addition, if f and g are weakly compatible, then they have a unique common fixed point.

2.2. Multivalued fixed point results in b-metric spaces

We present multivalued Meir-Keeler type operator results in b-metric spaces introduced by Boriceanu et al. [12].

It is known (see Czerwik [17]) that $(P_{cp}(X), H)$ is a complete b-metric space provided (X, b) is a complete b-metric space.

Let (X, b) be a b-metric space. If $T : X \to P(X)$ is a multivalued operator, then we denote by $Tix(T)$ the fixed point set of T, i.e. Fix $(T) := \{x \in X \mid x \in T(x)\}$ and by $SFix(T)$ the strict fixed point set of T, i.e. SF ix $(T) := \{x \in X \mid \{x\} = T(x)\}$. The symbol Graph $(T) := \{(x, y) \in X \times X : y \in T(x)\}$ denotes the graph of T.

Definition 2.20. [38] Let (X, b) be a b-metric space. An operator $T : X \to X$ is, by definition, a Picard operator if:

(i) $Fix(T) = \{x^*\}$;

(ii) $(T^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$, for all $x \in X$.

Lemma 2.21. *Let (X, b) be a b-metric space and $T : X \to P_{cp}(X)$ be a multivalued contractive operator (i.e., $H(T(x), T(y)) < b(x, y)$. for each $x, y \in X$ with $x \neq y$). Then, for any $Y \in P_{cp}(X)$ we have that $T(Y) \in P_{cp}(X)$.*

Question 2.22. Let (X, b) be a complete b-metric space with the b-metric a continuous functional on $X \times X$. Prove a fixed point theorem for a Meir-Keeler type operator on (X, b).

Definition 2.23. If (X, b) is a b-metric space, then $T : X \to P_{cp}(X)$ is a multivalued Meir-Keeler type operator if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in X$,

$$\varepsilon \leq b(x, y) < \varepsilon + \delta \text{ implies } H(T(x), T(y)) < \varepsilon.$$

It is easy to see that any multivalued Meir-Keeler type operator on a b-metric space is contractive and, thus, by Lemma 2.21, we obtain that $T(Y) \in P_{cp}(X)$, for each $Y \in P_{cp}(X)$.

Question 2.24. Let (X, b) be a complete b-metric space with the b-metric a continuous functional on $X \times X$ and let $T_i : X \to P_{cp}(X)$, $i \in \{1, \ldots, m\}$ be a finite family of multivalued Meir-Keeler type operators. Consider the multivalued fractal operator on $P_{cp}(X)$, which, by the above remark, is a self operator on $P_{cp}(X)$, i.e., $T_T : P_{cp}(X) \to P_{cp}(X)$. Prove an existence and uniqueness result for the multivalued fractal generated by the IMS $T = (T_1, \cdots, T_m)$.

Now, we present set-valued Meir-Keeler type fixed point results outlined by Debnath [19].

Theorem 2.25. *Let (X, b, s) be a complete b-metric space and the set-valued map $T : X \to CB(X)$ satisfies the following condition:*

for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq b(x, y) < \varepsilon + \delta \text{ implies } PH(Tx, Ty) < \varepsilon. \tag{14}$$

If we have a Cauchy sequence $\{x_n\}$ in X constructed in such a manner that $x_{n+1} \in Tx_n (n = 0, 1, 2, \ldots)$, where $x_0 \in X$, then $\mathrm{Fix}(T) \neq \phi$.

Our next result may be considered as an alternate form of Theorem 2.25. In Theorem 2.25, a particular Cauchy sequence was assumed in the hypothesis. This assumption may be dropped if we assume the compactness of the subsets of the concerned b-metric space and a stronger condition than (14).

Theorem 2.26. *Let (X, b, s) be a complete b-metric space whose every closed and bounded subset is compact and $T : X \to CB(X)$ be a set-valued map consistent with the following:*

for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq b(x, y) < \varepsilon + \delta \text{ implies } s^a PH(Tx, Ty) < \varepsilon,$$

for all $x, y \in X$, where $a > 0$ is given.

Then $\mathrm{Fix}(T) \neq \phi$.

Example 2.27. Consider the b-metric space (X, b, s), where $b(x, y) = (x - y)^2$ for all $x, y \in X$. Let $T : X \to CB(X)$ be defined by $T(x) = \left\{\frac{1}{8}x^2\right\}$.

Taking some $\varepsilon > 0$ and $a = 1$ (in Theorem 2.26), we have, for all $x, y \in X$ satisfying $\varepsilon \leq b(x, y) < \varepsilon + \delta$,

$$\begin{aligned} s\ PH(Tx, Ty) &= 2PH(Tx, Ty) \\ &= 2PH\left(\left\{\frac{1}{8}x^2\right\}, \left\{\frac{1}{8}y^2\right\}\right) \\ &= 2\max\left\{\sup_{x \in \left\{\left\{\frac{1}{8}y^2\right\}\right.} \Delta\left(x, \left\{\frac{1}{8}x^2\right\}\right), \sup_{y \in \left\{\frac{1}{8}x^2\right\}} \Delta\left(y, \left\{\frac{1}{8}y^2\right\}\right)\right\} \\ &= 2\max\{0, 0\} \\ &= 0 \\ &< \varepsilon \end{aligned}$$

Therefore, all constraints of Theorem 2.26 are fulfilled and $x = 0 \in \mathrm{Fix}(T)$.

3. Meir-Keeler Contraction Mappings on b-metric Spaces via Admissibility

In 2012, Samet *et al.* [39] established the idea of α-admissible mappings and given fixed point results in complete metric spaces. Later this concept is improved in different directions. Now, we present notations of α-admissible, generalized α-admissible, weak α-admissible, triangular α-admissible and α-orbital admissible.

Definition 3.1. Let X be a nonempty set and, let $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}_0^+$.

(i) (Samet *et al.* [39]) We say that T is called α-admissible if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \text{implies that} \quad \alpha(Tx, Ty) \geq 1.$$

(ii) (Aydi *et al.* [5]) Let $S : X \to X$ be a mapping. We say that (T, S) is called a generalized α-admissible pair if for all $x, y \in X$, we have

$$\alpha(x,y) \geq 1 \Longrightarrow \alpha(Tx, Sy) \geq 1 \quad \text{and } \alpha(STx, TSy) \geq 1.$$

(iii) (Sintunavarat [41]) We say that T is called a weak α-admissible mappings, if we have

$$x \in X \text{ with } \alpha(x, Tx) \geq 1 \Rightarrow \alpha\left(Tx, T^2x\right) \geq 1.$$

(iv) (Karapınar *et al.* [27]) We say that T is called triangular α-admissible if

(1) T is α-admissible and

(2) $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$ implies that $\alpha(x, z) \geq 1$, for any $x, y, z \in X$

(v) (Popescu [35]) We say that T is called α-orbital admissible if

$$\alpha(x, Tx) \geq 1 \Rightarrow \alpha\left(Tx, T^2x\right) \geq 1.$$

Also, T is called triangular α-orbital admissible if T is α-orbital admissible and

$$\alpha(x, y) \geq 1 \text{ and } \alpha(y, Ty) \geq 1 \Rightarrow \alpha(x, Ty) \geq 1.$$

In addition, we give examples for admissible mappings.

Example 3.2. [39] We have $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by

(i) Let $X = (0, \infty)$. $T(x) = \ln(x) \quad \forall x \in X$ and $\alpha(x, y) = \begin{cases} 2, & \text{if } x \geq y; \\ 0, & \text{if } x < y \end{cases}$

(ii) Let $X = \mathbb{R}$. $T(x) = \begin{cases} \ln\left(x^2 + 1\right), & \text{if } x > 1 \\ \frac{x}{3}, & \text{if } 0 \leq x \leq 1 \text{ and} \\ x, & \text{otherwise} \end{cases}$

$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1] \\ \ln 1.5, & \text{otherwise} \end{cases}$

Then T is α-admissible.

Example 3.3. [27] Let $X = \mathbb{R}$,

(i) $Tx = \sqrt[3]{x}$ and $\alpha(x, y) = e^{x-y}$. Indeed, if $\alpha(x, y) = e^{x-y} \geq 1$ then $x \geq y$ which implies $Tx \geq Ty$. That is, $\alpha(Tx, Ty) = e^{Tx-Ty} \geq 1$. Also, if $\left\{ \begin{array}{l} \alpha(x,z) \geq 1, \\ \alpha(z,y) \geq 1 \end{array} \right.$ then $\left\{ \begin{array}{l} x - z \geq 0, \\ z - y \geq 0 \end{array} \right.$ That is, $x - y \geq 0$ and so $\alpha(x, y) = e^{x-y} \geq 1$.

(ii) Let $X = \mathbb{R}, Tx = x^3 + \sqrt[7]{x}$ and $\alpha(x, y) = x^5 - y^5 + 1$.

Then T is a triangular α-admissible mapping.

The simulation function was introduced by Khojasteh et al. in [31] as follows.

Definition 3.4. (Khojasteh et al. [31]) A simulation function is a mapping $\zeta : [0, \infty)\times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(0, 0) = 0$;

(ζ_2) $\zeta(t,s) < s - t$ for all $t, s > 0$;

(ζ_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$$

Here, we present Meir-Keeler α-contractive fixed results by [1].

Definition 3.5. [1] Let (X,d) be a metric space and $T : X \to X$ be a self-mapping, $\alpha : X \times X \to [0,\infty)$ be a mapping. Then T is called Meir-Keeler α-contractive if, given an $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\varepsilon \leq d(x,y) < \varepsilon + \delta \text{ implies } \alpha(x,y)d(Tx,Ty) < \varepsilon.$$

Definition 3.6. [1] Let (X,d) be a metric space and $T : X \to X$ be a self-mapping, $\alpha : X \times X \to [0,\infty)$ be a mapping. Then T is called generalized Meir-Keeler α-contractive if, given an $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\varepsilon \leq M_T(x,y) < \varepsilon + \delta \text{ implies } \alpha(x,y)d(Tx,Ty) < \varepsilon,$$

where

$$M_T(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$$

Definition 3.7. [1] Let (X,d) be a metric space and $T, g : X \to X$ be self-mappings, $\alpha : X \times X \to [0,\infty)$ be a mapping. Then the pair (T,g) is called generalized Meir-Keeler α-contractive if, given an $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\varepsilon \leq M_{(T,g)}(x,y) < \varepsilon + \delta \quad \text{implies} \quad \alpha(x,y)d(Tx,gy) < \varepsilon,$$

where

$$M_{(T,g)}(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,gy), \frac{d(x,gy)+d(y,Tx)}{2}\right\}$$

We write $M_T(x,y) = M_{(T,T)}(x,y)$

We also give $\alpha-\psi$-Meir-Keeler contractive mappings results via a triangular $\alpha-$admissible mapping by [27].

Denote with Ψ the family of nondecreasing functions $\psi : [0,+\infty) \to [0,+\infty)$ continuous in $t = 0$ such that

- $\psi(t) = 0$ if and only if $t = 0$,
- $\psi(t+s) \leq \psi(t) + \psi(s)$.

Definition 3.8. [27] Let (X,d) be a metric space and $\psi \in \Psi$. Suppose that $T : X \to X$ is a triangular α-admissible mapping satisfying the following condition: for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq \psi(d(x,y)) < \varepsilon + \delta \text{ implies } \alpha(x,y)\psi(d(Tx,Ty)) < \varepsilon \tag{15}$$

for all $x, y \in X$. Then T is called an $\alpha-\psi$-Meir-Keeler contractive mapping.

Remark 3.9. [27] Let T be an $\alpha-\psi$-Meir-Keeler contractive mapping. Then

$$\alpha(x,y)\psi(d(Tx,Ty)) < \psi(d(x,y))$$

for all $x, y \in X$ when $x \neq y$. Also, if $x = y$ then $d(Tx,Ty) = 0$. i.e.,

$$\alpha(x,y)\psi(d(Tx,Ty)) \leq \psi(d(x,y))$$

for all $x, y \in X$.

Theorem 3.10. *[27] Let (X, d) be a complete metric space. Suppose that T is a continuous $\alpha - \psi$ Meir-Keeler contractive mapping and that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.*

Theorem 3.11. *[27] Let (X, d) be a complete metric space and let T be a α - ψ-Meir-Keeler contractive mapping. If the following conditions hold:*

(i) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,

(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$ for all n.

Then T has a fixed point.

3.1. Fixed point results for admissible mappings in generalized b-metric spaces

In this section, we give fixed point results for generalized (α, ψ)-Meir-Keeler type contractions in generalized b-metric spaces introduced in the work of Karapınar, et al. [26].

Aydi and Czerwik [6] itroduced generalized b-metric space, as follow.

Definition 3.12. Let X be a set and let $s \geq 1$ be a given real number. A functional $d_b : X \times X \to [0, \infty]$ is said to be a generalized b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d_b(x, y) = 0$ if and only if $x = y$;

(2) $d_b(x, y) = d_b(y, x)$;

(3) $d_b(x, z) \leq s[d_b(x, y) + d_b(y, z)]$.

A pair (X, d_b) is called a generalized b-metric space.

Berinde [9] characterized comparison functions to define the contraction mappings in the setting of b-metric spaces.

Definition 3.13. Let $s \geq 1$ be a real number. A function $\phi : [0, \infty) \to [0, \infty)$ is called a (c)-comparison function if

(1) ϕ is increasing;

(2) there exist $k_0 \in \mathbb{N}, a \in (0, 1)$, and a convergent nonnegative series $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\phi^{k+1}(t) \leq as^k\phi^k(t) + v_k$, for $k \geq k_0$ and any $t \geq 0$.

Denote Ψ as the set of (c)-comparison functions.

Lemma 3.14. *(see [9, 10, 37]) For a (c)-comparison function $\varphi : [0, +\infty) \to [0, +\infty)$, the following statements hold:*

(1) The series $\sum_{k=0}^{\infty} s^k\varphi^k(t)$ converges for any $t \in [0, +\infty)$.

(2) The function $b_s : [0, +\infty) \to [0, +\infty)$ defined by $b_s(t) = \sum_{k=0}^{\infty} s^k\varphi^k(t), t \in [0, \infty)$, is increasing and continuous at 0 .

(3) Each iterate φ^k of φ for $k \geq 1$ is also a (c)-comparison function.

(4) φ is continuous at 0 .

(5) $\varphi(t) < t$ for any $t > 0$.

Inspired by Popescu [35], we introduce the concept of generalized α-orbital admissible mappings.

Definition 3.15. Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty]$ be a function. We say that T is a generalized α-orbital admissible if

$$\alpha(x, Tx) \geq 1 \Longrightarrow \alpha\left(Tx, T^2x\right) \geq 1$$
$$\alpha(x, Tx) < \infty \Rightarrow \alpha\left(Tx, Tx^2\right) < \infty$$

Notice that each α-orbital admissible mapping [35] is generalized α-orbital admissible.

Definition 3.16. For an arbitrary constant $s \geq 1$, let T be a self mapping defined on a generalized b-metric space (X, d_b, s). Then T is called an (α, ψ)-Meir-Keeler contractive mapping if there exist two auxiliary mappings $\alpha : X \times X \to [0, \infty]$ and $\psi \in \Psi$ such that

$$\varepsilon \leq \psi(d_b(x, y)) < \varepsilon + \delta \text{ implies } \alpha(x, y)d_b(Tx, Ty) < \varepsilon, \forall x, y \in X. \tag{16}$$

Remark 3.17. For $x \neq y$ and $d_b(x, y) < \infty$ with $\alpha(x, y) < \infty$, from (16) we derive that

$$\alpha(x, y)d_b(Tx, Ty) < \psi(d_b(x, y)). \tag{17}$$

Theorem 3.18. *Let $s \geq 1$ be a fixed constant and (X, d_b, s) be a complete generalized b-metric space. Suppose that a self mapping $T : X \to X$ is an (α, ψ)-Meir-Keeler type contraction. Assume also that*

(i) T is generalized α-orbital admissible;

(ii) there exists $x \in X$ such that $1 \leq \alpha(x, Tx) < \infty$;

(iii) T is continuous.

Then for such x, one of the following statements holds:
(A) For every $n \in \mathbb{N} \cup \{0\}$,

$$d_b\left(T^nx, T^{n+1}x\right) = \infty$$
$$\text{or } \alpha\left(T^nx, T^{n+1}x\right) = \infty$$

(B) There exists $k \in \mathbb{N} \cup \{0\}$ such that $d_b\left(T^kx, T^{k+1}x\right) < \infty$ and $\alpha\left(T^kx, T^{k+1}x\right) < \infty$. In this case, there exists $u \in X$ such that $Tu = u$.

Proof. On account of assumption (ii), there exists $x \in X$ such that $\alpha(x, Tx) \geq 1$. We suppose that case (A) is not satisfied. Consequently, we have to examine case (B). Consequently, there exists $k \in \mathbb{N} \cup \{0\}$ such that $d_b\left(T^kx, T^{k+1}x\right) < \infty$ and $\alpha\left(T^kx, T^{k+1}x\right) < \infty$. If $T^kx = T^{k+1}x$, the proof is completed. Assume that $d_b\left(T^kx, T^{k+1}\right) > 0$. By property of ψ and Remark 3.17, we have

$$\alpha\left(T^kx, T^{k+1}x\right) d_b\left(T^{k+1}x, T^{k+2}x\right) < \psi\left(d_b\left(T^kx, T^{k+1}x\right)\right) < d_b\left(T^kx, T^{k+1}x\right) < \infty \tag{18}$$

Since T is a generalized α-orbital admissible mapping, by (ii), we derive that

$$1 \leq \alpha(x, Tx) < \infty \Longrightarrow 1 \leq \alpha\left(Tx, T^2x\right) < \infty.$$

Recursively, we obtain that

$$1 \leq \alpha\left(T^{k+n}x, T^{k+n+1}x\right) < \infty \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{19}$$

Applying (19) in (18), we get

$$d_b\left(T^{k+1}x, T^{k+2}x\right) < \psi\left(d_b\left(T^kx, T^{k+1}x\right)\right) < \infty$$

Thus

$$d_b\left(T^{k+n}x, T^{k+n+1}x\right) < \infty \quad \forall n \in \mathbb{N} \cup \{0\} \tag{20}$$

Again, on account of (19) and (20) in (18), by induction, one gets

$$d_b\left(T^{k+n}x, T^{k+n+1}x\right) < \psi^n\left(d_b\left(T^k x, T^{k+1}x\right)\right) \tag{21}$$

Consequently, for $n, v \in \mathbb{N} \cup \{0\}$, by (21)) we have

$$\begin{aligned}
d_b\left(T^{k+n}x, T^{k+n+v}x\right) \leq & s d_b\left(T^{k+n}x, T^{k+n+1}x\right) + \cdots \\
& + s^{v-1} d_b\left(T^{k+n+v-2}x, T^{k+n+v-1}x\right) \\
& + s^v d_b\left(T^{k+n+v-1}x, T^{k+n+v}x\right) \\
< & s\psi^n\left(d_b\left(T^k x, T^{k+1}x\right)\right) + \cdots \\
& + s^{v-1}\psi^{n+v-2}\left(d_b\left(T^k x, T^{k+1}x\right)\right) \\
& + s^v \psi^{n+v-1}\left(d_b\left(T^k x, T^{k+1}x\right)\right) \\
& \leq s \sum_{m=0}^{\infty} s^m \psi^{n+m}\left(d_b\left(T^k x, T^{k+1}x\right)\right) \\
& \leq s \sum_{m=0}^{\infty} s^m \psi^{m}\left(d_b\left(T^k x, T^{k+1}x\right)\right)
\end{aligned}$$

Finally,

$$d_b\left(T^{k+n}(x), T^{k+n+v}x\right) \leq s \sum_{m=0}^{\infty} s^m \psi^m\left(d_b\left(T^k x, T^{k+1}x\right)\right) \tag{22}$$

for all $n, v \in \mathbb{N} \cup \{0\}$. By (22) and the fact that $\psi \in \Psi$, it follows that $\{T^n x\}$ is a Cauchy sequence of elements of X.

Since X is complete, there exists $u \in X$ with

$$\lim_{n\to\infty} d_b\left(T^n x, u\right) = 0$$

Since T is continuous, we get

$$u = \lim_{n\to\infty} T^{n+1}x = T\left(\lim_{n\to\infty} T^n x\right) = Tu$$

and u is a fixed point of T, which ends the proof. □

Definition 3.19. Let $s \geq 1$ be a fixed constant. We say that a generalized b-metric space (X, d_b, s) is regular if $\{x_n\}$ is a sequence in X such that

$1 \leq \alpha\left(x_n, x_{n+1}\right) < \infty$ for all n and $x_n \to x \in X$ as $n \to \infty$;

then there exists a subsequence $\left\{x_{n(k)}\right\}$ of $\{x_n\}$ such that

$1 \leq \alpha\left(x_{n(k)}, x\right) < \infty$ and $0 < d_b\left(x_{n(k)}, x\right) < \infty$ for all k.

Theorem 3.20. *Let $s \geq 1$ be a fixed constant and (X, d_b, s) be a complete generalized b-metric space. Suppose that a selfmapping $T : X \to X$ is an (α, ψ)-Meir-Keeler type contraction. Assume also that*

(i) T is a generalized α-orbital admissible mapping;

(ii) there exists $x \in X$ such that $1 \leq \alpha(x, Tx) < \infty$;

(iii) (X, d_b, s) is regular.

Then for such x, one of the following statements holds:

(A) For every $n \in \mathbb{N} \cup \{0\}$,

$$d_b\left(T^n x, T^{n+1} x\right) = \infty$$
$$\text{or } \alpha\left(T^n x, T^{n+1} x\right) = \infty.$$

(B) There exists $k \in \mathbb{N} \cup \{0\}$ such that $d_b\left(T^k x, T^{k+1} x\right) < \infty$ and $\alpha\left(T^k x, T^{k+1} x\right) < \infty$. In this case, there exists $u \in X$ such that $Tu = u$.

For the uniqueness of a fixed point of an (α, ψ)-Meir Keeler type contraction mapping T in $Y = \{t \in X : d_b\left(T^k x, t\right) < \infty$, we shall consider the following condition:

(U) For all $x, y \in \operatorname{Fix}(T)$, we have $1 \leq \alpha(x, y) < \infty$, where $\operatorname{Fix}(T)$ denotes the set of fixed points of T.

Theorem 3.21. *By adding condition (U) to the hypotheses of Theorem 3.18 (resp., Theorem 3.20), T has at most one fixed point in*

$$Y := \left\{t \in X : d_b\left(T^k x, t\right) < \infty\right\}.$$

Proof. Let T be an (α, ψ)-Meir-Keeler type contraction. Owing to Theorem 3.18 (resp., Theorem 3.20), T has a fixed point $u \in X$.

Now, we shall show that T has at most one fixed point in Y. We argue by contradiction. For this, assume that there exist two distinct fixed points u_1 and u_2 of T, where $u_1, u_2 \in Y$; that is,

$$d_b\left(T^k x, u_i\right) < \infty \quad \forall i = 1, 2.$$

We deduce

$$d_b\left(u_1, u_2\right) \leq s d_b\left(u_1, T^k x\right) + s d_b\left(T^k x, u_2\right) < \infty.$$

By condition $(U), 1 \leq \alpha\left(u_1, u_2\right) < \infty$ and since $0 < d_b\left(u_1, u_2\right) < \infty$, in view of (17), one writes

$$d_b\left(u_1, u_2\right) = d_b\left(T u_1, T u_2\right) \leq \alpha\left(u_1, u_2\right) d_b\left(T u_1, T u_2\right) < \psi\left(d_b\left(u_1, u_2\right)\right) < d_b\left(u_1, u_2\right)$$

which is a contradiction, so $u_1 = u_2$. This completes the proof. □

By letting $\alpha(x, y) = 1$ and $\phi(t) = t/2s$, we get the following result.

Theorem 3.22. *Let (X, d_b, s) be a generalized complete b-metric space and $T : X \to X$ satisfy the following: given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq d_b(x, y) < \varepsilon + \delta \text{ implies } d_b(Tx, Ty) < \frac{\varepsilon}{2s}, x, y \in X.$$

Let $x \in X$. Then one of the following alternatives holds:

(A) For every $n \in \mathbb{N} \cup \{0\}$,

$$d_b\left(T^n x, T^{n+1} x\right) = \infty.$$

(B) There exists $k \in \mathbb{N} \cup \{0\}$ such that $d_b\left(T^k x, T^{k+1} x\right) < \infty$.

In case (B), we assert the following:

(i) The sequence $\{T^m x\}$ is Cauchy in X.

(ii) There exists a point $u \in X$ such that $Tu = u$ and $\lim_{n \to \infty} d_b\left(T^n x, u\right) = 0$.

(iii) u is the unique fixed point of T in $B := \left\{t \in X : d_b\left(T^k x, t\right\} < \infty\right\}$.

(iv) For every $t \in B$,

$$\lim_{n \to \infty} d_b\left(T^n t, u\right) = 0$$

Remark 3.23. If (X, d_b) is a metric space, we do not get the result of Meir-Keeler.

3.2. Fixed point results for wt-distance in b-metric spaces

We present Meir-Keeler type contractions in the wt-distances over b-metric spaces proposed by Karapınar, et al. [25].

Definition 3.24. Let $s \in [1, \infty)$ and b be a distance function on X. If the following inequality holds for all $x, z, y \in X$,

$$b(x, y) \leq s[b(x, z) + b(z, y)]$$

then b is called b-metric over constant s.

In short, (X, b, s) (respectively, (X^*, b, s)) denotes a b-metric space over s.

Theorem 3.25. *Let $q : X \times X \to [0, \infty)$ be a wt-distance on (X^*, b, s), and T be a self-mapping on X. We presume that*

(i) for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq \mu(b(x, z)) < \varepsilon + \delta \textit{ implies } \sigma(\beta(x, z)q(T(x), T(z)), \varepsilon) \geq 0, \textit{ for all } x, z \in X$$

where $\sigma \in \Sigma, \mu \in \mathcal{B}$ with $\mu(t) < \frac{t}{s}$, for all $t > 0$,

(ii) T is triangular β-orbital admissible;

(iii) there exists $x_0 \in X$ such that $\beta(x_0, T(x_0)) \geq 1$;

(iv) T is continuous, or,

(iv') for all $x \in X$, with $\beta(x, T(x)) \geq 1$, $\quad \inf\{q(x, z) + q(x, T(x))\} > 0$
for all $x \in X, x \neq T(x)$.

Then, T has a fixed point.

3.3. Fixed point results for admissible mappings in b-metric-like spaces

We present generalized Meir-Keeler contraction mappings results in the b-metric-like spaces by Gholamian and Khanehgir [22]. Alghamdi [3] introduced concept of b-metric-like space, as follows.

Definition 3.26. [3] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $\sigma_b : X \times X \to \mathbb{R}_0^+$is a b-metric-like if, for all $x, y, z \in X$, the following conditions are satisfied:

($\sigma_b 1$) $\quad \sigma_b(x, y) = 0$ implies $x = y$,
($\sigma_b 2$) $\quad \sigma_b(x, y) = \sigma_b(y, x)$,
($\sigma_b 3$) $\quad \sigma_b(x, y) \leq s\,[\sigma_b(x, z) + \sigma_b(z, y)]$.

A b-metric-like space is a pair (X, σ_b) such that X is a nonempty set and σ_b is a b-metriclike on X. The number s is called the coefficient of (X, σ_b).

Proposition 3.27. *[24] Let (X, σ) be a metric-like space and $\sigma_b(x, y) = [\sigma(x, y)]^p$, where $p > 1$. Then σ_b is a b-metric-like with coefficient $s = 2^{p-1}$.*

We introduced an extension of the Meir-Keeler contractions defined in [33].

Definition 3.28. Suppose that (X, σ_b) is a b-metric-like space with coefficient s. A triangular α-admissible mapping $T : X \to X$ is said to be generalized Meir-Keeler contraction if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq \beta\left(\sigma_b(x, y)\right)\sigma_b(x, y) < \varepsilon + \delta \quad \text{implies} \quad \alpha(x, y)\sigma_b(Tx, Ty) < \varepsilon \tag{23}$$

for all $x, y \in X$ where $\beta : [0, \infty) \to \left(0, \frac{1}{s}\right)$ is a given function.

Remark 3.29. Let T be a generalized Meir-Keeler contractive mapping. Then it is intuitively clear that

$$\alpha(x,y)\sigma_b(Tx,Ty) < \beta\left(\sigma_b(x,y)\right)\sigma_b(x,y)$$

for all $x, y \in X$ when $x \neq y$.

We introduced generalized Meir-Keeler contraction of type (I) and type (II) on b-metric-like spaces.

Definition 3.30. Let (X, σ_b) be a b-metric-like space with coefficient s. A triangular α-admissible mapping $T : X \to X$ is said to be generalized Meir-Keeler contraction of type (I) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq \beta\left(\sigma_b(x,y)\right) M(x,y) < \varepsilon + \delta \quad \text{implies} \quad \alpha(x,y)\sigma_b(Tx,Ty) < \varepsilon \tag{24}$$

where

$$M(x,y) = \max\left\{\sigma_b(x,y), \sigma_b(Tx,x), \sigma_b(Ty,y)\right\} \tag{25}$$

for all $x, y \in X$.

Definition 3.31. Let (X, σ_b) be a b-metric-like space with coefficient s. A triangular α-admissible mapping $T : X \to X$ is said to be generalized Meir-Keeler contraction of type (II) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq \beta\left(\sigma_b(x,y)\right) N(x,y) < \varepsilon + \delta \quad \text{implies} \quad \alpha(x,y)\sigma_b(Tx,Ty) < \varepsilon \tag{26}$$

where

$$N(x,y) = \max\left\{\sigma_b(x,y), \frac{1}{2}\left[\sigma_b(Tx,x) + \sigma_b(Ty,y)\right]\right\} \tag{27}$$

for all $x, y \in X$.

Remark 3.32. Suppose that $T : X \to X$ is a generalized Meir-Keeler contraction of type (I) (respectively, type (II)). Then

$$\alpha(x,y)\sigma_b(Tx,Ty) < \beta\left(\sigma_b(x,y)\right) M(x,y) \quad (\text{ respectively, } \beta\left(\sigma_b(x,y)\right) N(x,y))$$

for all $x, y \in X$ when $M(x,y) > 0$ (respectively, $N(x,y) > 0$).

Remark 3.33. It is readily verified that $N(x,y) \leq M(x,y)$ for all $x, y \in X$, where $M(x,y)$ and $N(x,y)$ are defined in (25) and (27), respectively.

We establish a fixed point theorem for generalized Meir-Keeler type contractions via a rational expression inspired by Samet et al [40].

Theorem 3.34. *Let (X, σ_b) be a complete b-metric-like space and $T : X \to X$ be a triangular α-admissible mapping. Suppose that the following conditions hold:*

(i) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1, \alpha(Tx_0, x_0) \geq 1$,

(ii) if $\{x_n\}$ is a sequence in X such that $x_n \to z$ as $n \to \infty$ and $\alpha(x_n, x_m) \geq 1$ for all $n, m \in \mathbb{N}$, then $\alpha(x_n, z) \geq 1$ for all $n \in \mathbb{N}$,

(iii) for each $\varepsilon > 0$, there exists $\delta > 0$ satisfying the following condition:

$$\begin{aligned} &2s\varepsilon \leq \sigma_b(y,Ty)\frac{1+\sigma_b(x,Tx)}{1+M(x,y)} + N(x,y) < s(2\varepsilon+\delta) \quad \textit{implies} \\ &\alpha(x,y)\sigma_b(Tx,Ty) < \varepsilon \end{aligned} \tag{28}$$

Then T has a fixed point in X.

The following example reveals the usefulness of Theorem 3.34.

Example 3.35. Let $X = \{0, 1, 2, 3\}$. Define $\sigma_b : X \times X \to \mathbb{R}_0^+$ as follows:

$$\sigma_b(x,y) = \begin{cases} 4, & x = y = 0 \text{ or } 2 \text{ or } 3, \\ 0, & x = y = 1 \\ 1, & x \neq y \end{cases}$$

Clearly, (X, σ_b) is a complete b-metric-like space with $s = 2$. Consider $T : X \to X$ defined by $T0 = 0, T1 = 1, T2 = 2$, and $T3 = 1$. Also, define $\alpha : X \times X \to \mathbb{R}_0^+$ as follows:

$$\alpha(x,y) = \begin{cases} \frac{1}{5}, & x + y = 1 \text{ or } 3 \\ 0, & x = y = 0 \\ 1, & x = y = 1 \\ \frac{1}{x+y+1}, & \text{otherwise} \end{cases}$$

It easily can be shown that T is triangular α-admissible. In order to check the condition (28), we choose $\delta = 4\varepsilon$ so that

$$4\varepsilon \leq \sigma_b(y, Ty)\frac{1 + \sigma_b(x, Tx)}{1 + M(x, y)} + N(x, y) < 8\varepsilon \text{ which implies } \alpha(x, y)\sigma_b(Tx, Ty) < \varepsilon.$$

Note that $\alpha(1, T1) \geq 1, \alpha(T1, 1) \geq 1$. All conditions of Theorem 3.34 are satisfied and so T has a fixed point.

Corollary 3.36. *Let (X, σ_b) be a complete b-metric-like space and $T : X \to X$ be a mapping. Let φ be a locally integrable function from $\mathbb{R}_0^+$ into itself such that*

$$\int_0^t \varphi(s)ds > 0 \text{ for all } t > 0.$$

Assume that conditions (i) and (ii) of Theorem 3.34 hold and also T fulfills the following condition for all $x, y \in X$:

$$\int_0^{2\alpha(x,y)\sigma_b(Tx,Ty)} \varphi(t)dt \leq c \int_0^{\frac{1}{s}\sigma_b(y,Ty)\frac{1+\sigma_b(x,Tx)}{1+M(x,y)} + \frac{1}{s}N(x,y)} \varphi(t)dt$$

where $c \in \left(0, \frac{1}{2s}\right)$ is a constant. Then T has a fixed point.

Definition 3.37. Let (X, σ_b) be a b-metric-like space, and let T be a self-mapping on X. T is called orbitally continuous whenever

$$\lim_{n\to\infty} \sigma_b(T^n x, z) = \sigma_b(z, z) \Rightarrow \lim_{n\to\infty} \sigma_b(TT^n x, Tz) = \sigma_b(Tz, Tz)$$

for each $x, z \in X$. It is clear that continuous mappings are orbitally continuous. But the converse may not be true. To show this, let $([0, 1], \sigma_b)$ be the b-metric-like space, where $\sigma_b(x, y) = [\max\{x, y\}]^q$ ($q \geq 1$). Consider $T : X \to X$ defined by

$$T(x) = \begin{cases} \frac{x}{2}, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$$

Clearly T is not continuous but it is orbitally continuous.

Theorem 3.38. *Let (X, σ_b) be a complete b-metric-like space with coefficients and $T : X \to X$ be a mapping. Suppose that the following conditions hold:*

(i) T is an orbitally continuous generalized Meir-Keeler contraction of type (I),

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1, \alpha(Tx_0, x_0) \geq 1$,

(iii) if $\{x_n\}$ is a sequence in X such that $x_n \to z$ as $n \to \infty$ and $\alpha(x_n, x_m) \geq 1$ for all $n, m \in \mathbb{N}$, then $\alpha(z, z) \geq 1$,

(iv) $s > 1$ or β is a continuous function.

Then T has a fixed point in X.

By Remark 3.33 we know $N(x, y) \leq M(x, y)$, so a slight change in the proof of Theorem 3.38 shows actually the following theorem holds.

Theorem 3.39. *Let (X, σ_b) be a complete b-metric-like space, $T : X \to X$ be a mapping. Suppose that the following conditions hold:*

(i) T is an orbitally continuous generalized Meir-Keeler contraction of type (II),

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1, \alpha(Tx_0, x_0) \geq 1$,

(iii) if $\{x_n\}$ is a sequence in X such that $x_n \to z$ as $n \to \infty$ and $\alpha(x_n, x_m) \geq 1$ for all $n, m \in \mathbb{N}$, then $\alpha(z, z) \geq 1$,

(iv) $s > 1$ or β is a continuous function.

Then T has a fixed point in X.

Example 3.40. Let (X, σ_b) and α be as in Example 3.35. Consider $T : X \to X$ defined by $T0 = T2 = 0$ and $T1 = 2$. Also, define $\beta : [0, +\infty) \to \left(0, \frac{1}{s}\right)$ as follows:

$$\beta(x) = \begin{cases} \frac{1}{x}, & x = 3, 4, 8 \\ \frac{5}{9(x+1)}, & \text{otherwise} \end{cases}$$

In order to check the condition (24), we choose $\delta = \varepsilon$ so that

$$\varepsilon \leq \beta(\sigma_b(x, y)) M(x, y) < \varepsilon + \delta = 2\varepsilon, \text{ which implies } \alpha(x, y)\sigma_b(Tx, Ty) < \varepsilon.$$

Therefore, the map T is a generalized Meir-Keeler contraction of type (I). Note that T is continuous with respect to τ_{σ_b} and $\alpha(0, T0) \geq 1, \alpha(T0, 0) \geq 1$. Now, all conditions of Theorem 3.38 are satisfied and so T has a fixed point.

3.4. Fixed point results for admissible mappings in Branciari b-metric Spaces

Firstly, α-Meir-Keeler and generalized α-Meir-Keeler contractions on Branciari b-metric spaces are introduced by Gülyaz et al. [23].

Branciari [11] defined Branciari (rectangular) metric spaces as follows:

Definition 3.41. [11] Let X be a nonempty set and let $d : X \times X \to [0, \infty)$ be a function such that for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y, the following conditions are satisfied:

(BM1) $d(x, y) = 0$ if and only if $x = y$;

(BM2) $d(x, y) = d(y, x)$;

(BM3) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$.

Combining these two types of metric makes it possible to introduce a new metric as in the following definition.

We recall definition of Branciari b-metric spaces

Definition 3.42. [21] Let X be a nonempty set and let $b_b : X \times X \to [0,\infty)$ be a function such that for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y, the following conditions are satisfied:

(BM_b1) $b_b(x,y) = 0$ if and only if $x = y$;

(BM_b2) $b_b(x,y) = b_b(y,x)$;

(BM_b3) $b_b(x,y) \leq s[b_b(x,u) + b_b(u,v) + b_b(v,y)]$ for some real number $s \geq 1$.

The map b_b is called a Branciari b-metric and the pair (X, b_b) is called a Branciari b-metric space (BM_bS).

Now, we will generalize the classical Meir-Keeler contraction mappings by inserting the α admissibility and replacing the metric $b_b(x,y)$ in the definition by a more general term.

Definition 3.43. Let (X, b_b) be a Branciari b-metric space with a constant $s \geq 1$. Let $T : X \to X$ be an α-admissible mapping. Suppose that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq b_b(x,y) < \varepsilon + \delta \text{ implies } \alpha(x,y) b_b(Tx,Ty) < \frac{\varepsilon}{s},$$

for all $x, y \in X$. Then T is called α-Meir-Keeler contraction.

Remark 3.44. If T is an α-Meir-Keeler contraction, then

$$\alpha(x,y) b_b(Tx,Ty) \leq \frac{b_b(x,y)}{s}$$

for all $x, y \in X$, where the equality holds for $x = y$.

Further generalization on Meir-Keeler mappings can be done as follows.

Definition 3.45. Let (X, b_b) be a Branciari b-metric space with a constant $s \geq 1$. Let $T : X \to X$ be an α-admissible mapping. If for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq M(x,y) < \varepsilon + \delta \text{ implies } \alpha(x,y) b_b(Tx,Ty) < \frac{\varepsilon}{s}$$

where

$$M(x,y) = \max\{b_b(x,y), b_b(Tx,x), b_b(Ty,y)\},$$

for all $x, y \in X$, then T is called generalized α-Meir-Keeler contraction.

Remark 3.46. Let $T : X \to X$ be a generalized α - Meir Keeler contraction. Then

$$\alpha(x,y) b_b(Tx,Ty) \leq \frac{M(x,y)}{s}$$

for all $x, y \in X$, where the equality may hold only when $x = y$.

We will first prove the following lemma which will be used in the proof of existence and uniqueness theorems.

Lemma 3.47. *Let (X, b_b) be a Branciari b-metric space with a constant $s \geq 1$. Let $\{x_n\}$ be a sequence in X satisfying*

(1) $x_m \neq x_n$ *for all* $m \neq n, m, n \in \mathbb{N}$*,*

(2) $b_b(x_n, x_{n+1}) \leq \frac{1}{s} b_b(x_{n-1}, x_n)$*, for all* $n \in \mathbb{N}$*,*

(3) $\lim_{n\to\infty} b_b(x_n, x_{n+2}) = 0$.

Then $\{x_n\}$ *is a Cauchy sequence in* (X, b_b)*.*

Here, we give generalized α-Meir-Keeler contractions on Branciari b-metric spaces.

Theorem 3.48. *Let* (X, b_b) *be a complete Branciari b-metric space with a constant* $s \geq 1$ *and* $T : X \to X$ *be a continuous generalized* α*-Meir-Keeler contraction, that is, T satisfies the conditions of Definition 3.45. If* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$ *for some* $x_0 \in X$*, then T has a fixed point in X.*

In order to provide the uniqueness of the fixed point of α-admissible mappings, extra condition is required. There are different versions of the uniqueness condition, two of which are given below.

(U1) For every pair x and y of fixed points of $T, \alpha(x, y) \geq 1$.

(U2) For every pair x and y of fixed points of T, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$. We give uniqueness theorem employing the condition (U1).

Theorem 3.49. *If the condition (U1) is added to the conditions of Theorem 3.48, then the mapping T has a unique fixed point.*

In order to weaken the conditions for existence of a fixed point often the continuity of the mapping T is being replaced by the so-called α-regularity condition of the space. The α-regularity on a Branciari b-metric space is defined as follows.

Definition 3.50. A Branciari b-metric space (X, b_b) is called α-regular iffor any sequence $\{x_n\}$ such that $\lim_{n\to\infty} b_b(x_n, x) = 0$ and satisfying $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Combining the Theorems 3.48 and 3.49 with the condition of the Definition 3.50, we state another theorem.

Theorem 3.51. *Let* (X, b_b) *be a complete Branciari b-metric space with a constant* $s \geq 1$ *and* $T : X \to X$ *be a generalized* α*-Meir-Keeler contraction. Assume that,*

(1) There exists $x_0 \in X$ *for which* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$*,*

(2) Either T is continuous or (X, b_b) *is* α*-regular. Then T has a fixed point in X.*

If, in addition, the condition (U1) holds, the fixed point of T is unique.

Corollary 3.52. *Let* (X, b_b) *be a complete Branciari b-metric space with a constant* $s \geq 1$*. Let* $T : X \to X$ *be an* α*-admissible mapping satisfying the following conditions.*

(1) For every $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\varepsilon \leq N(x, y) < \varepsilon + \delta \text{ implies } \alpha(x, y) b_b(Tx, Ty) < \frac{\varepsilon}{s},$$

where

$$N(x, y) = \max\left\{b_b(x, y), \frac{1}{2}[b_b(Tx, x) + b_b(Ty, y)]\right\},$$

for all $x, y \in X$*.*

(2) There exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$*.*

If T is continuous or (X, b_b) *is* α*-regular, then T has a fixed point. If, in addition, T satisfies the condition (U1), then the fixed point is unique.*

Corollary 3.53. *Let (X, b_b) be a complete Branciari b-metric space with a constant $s \geq 1$. Let $T : X \to X$ be an α-Meir-Keeler contraction, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq b_b(x, y) < \varepsilon + \delta \text{ implies } \alpha(x, y)b_b(Tx, Ty) < \frac{\varepsilon}{s},$$

for all $x, y \in X$.

If T is continuous or (X, b_b) is α-regular, then T has a fixed point. If, in addition, T satisfies the condition (U1), then the fixed point is unique.

Some more consequences are concluded from the main result given in Theorem 3.51 by taking $\alpha(x, y) = 1$.

Corollary 3.54. *Let (X, b_b) be a complete Branciari b-metric space with a constant $s \geq 1$. Let $T : X \to X$ be a continuous mapping satisfying the following:*

For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq M(x, y) < \varepsilon + \delta \text{ implies } b_b(Tx, Ty) < \frac{\varepsilon}{s}$$

where

$$M(x, y) = \max\{b_b(x, y), b_b(Tx, x), b_b(Ty, y)\},$$

for all $x, y \in X$. Then T has a unique fixed point.

Corollary 3.55. *Let (X, b_b) be a complete Branciari b-metric space with a constant $s \geq 1$. Let $T : X \to X$ be a continuous mapping. Assume that for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq N(x, y) < \varepsilon + \delta \text{ implies } b_b(Tx, Ty) < \frac{\varepsilon}{s},$$

where

$$N(x, y) = \max\left\{b_b(x, y), \frac{1}{2}[b_b(Tx, x) + b_b(Ty, y)]\right\},$$

for all $x, y \in X$. Then T has a unique fixed point.

The last consequence is the classical Meir-Keeler contraction on Branciari b-metric spaces.

Corollary 3.56. *Let (X, b_b) be a complete Branciari b-metric space with a constant $s \geq 1$. Let $T : X \to X$ be a continuous Meir-Keeler contraction mapping, that is, given $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq b_b(x, y) < \varepsilon + \delta \text{ implies } b_b(Tx, Ty) < \frac{\varepsilon}{s},$$

for all $x, y \in X$. Then T has a unique fixed point.

The general nature of α-admissible mappings makes it possible to deduce fixed point theorems for cyclic mappings and mappings defined on partially ordered spaces. Assume that a partial ordering $\preceq$ is defined on a Branciari b-metric space (X, b_b) with a constant $s \geq 1$. Let $T : X \to X$ be an increasing mapping. Then, by choosing the function α as

$$\alpha(x, y) = \begin{cases} 1 & \text{if} \quad x \preceq y \text{ or } y \preceq x \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

the following fixed point theorems easily follow from the main theorems.

Corollary 3.57. *Let (X, b_b) be a complete Branciari b-metric space with a constant $s \geq 1$ on which a partial ordering $\preceq$ is defined. Let $T : X \to X$ be an increasing mapping satisfying the following condition for all comparable pairs $x, y \in X$:*

Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq M(x,y) < \varepsilon + \delta \text{ implies } b_b(Tx,Ty) < \frac{\varepsilon}{s'},$$

where

$$M(x,y) = \max\{b_b(x,y), b_b(Tx,x), b_b(Ty,y)\}.$$

Assume that $x_0 \preceq Tx_0$ and $x_0 \preceq T^2x_0$ for some $x_0 \in X$. Then T has a fixed point. If, in addition, any two fixed points of T are comparable then T has a unique fixed point.

Corollary 3.58. *Let (X, b_b) be a complete Branciari b-metric space with a constant $s \geq 1$ on which a partial ordering $\preceq$ is defined. Let $T : X \to X$ be an increasing mapping satisfying the following condition for all comparable pairs $x, y \in X$:*

Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq N(x,y) < \varepsilon + \delta \text{ implies } b_b(Tx,Ty) < \frac{\varepsilon}{s},$$

where

$$N(x,y) = \max\left\{b_b(x,y), \frac{1}{2}[b_b(Tx,x) + b_b(Ty,y)]\right\}.$$

Assume that $x_0 \preceq Tx_0$ and $x_0 \preceq T^2x_0$ for some $x_0 \in X$. Then T has a fixed point. If, in addition, any two fixed points of T are comparable then T has a unique fixed point.

Corollary 3.59. *Let (X, b_b) be a complete Branciari b-metric space with a constant $s \geq 1$ on which a partial ordering $\preceq$ is defined. Let $T : X \to X$ be an increasing mapping satisfying the following condition for all comparable pairs $x, y \in X$:*

Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq b_b(x,y) < \varepsilon + \delta \text{ implies } b_b(Tx,Ty) < \frac{\varepsilon}{s}.$$

Assume that $x_0 \preceq Tx_0$ and $x_0 \preceq T^2x_0$ for some $x_0 \in X$. Then T has a fixed point. If, in addition, any two fixed points of T are comparable then T has a unique fixed point.

3.5. Fixed point results for hybrid contraction via admissibility in b-metric spaces

Firstly, we introduce the notion of generalized (α, ϕ)-Meir–Keeler hybrid contractive mappings of type I and II via simulation function based on Mamud and Tola [2].

Karapinar et al. [28] introduced the class of hybrid contraction mappings of type I and II and studied fixed point results for such mappings.

Definition 3.60. [28] Let T be a self-mapping on a metric space (X, d) and $\zeta \in Z_w$. Suppose that $p : X \times X \to \mathbb{R}_0^+$ is a function that satisfies only $\left(P^1 p : M\right)$. Then T is called a hybrid contraction of type I if the following conditions are fulfilled:

(1) For any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $x \neq y$ and $p(x,y) < \epsilon + \delta(\epsilon)$ imply $d(Tx,Ty) \leq \epsilon$;

(2) $x \neq y$ and $p(x,y) > 0$ imply $\zeta(\alpha(x,y)d(Tx,Ty), p(x,y)) \geq 0$.

Let a mapping $N : X \times X \to \mathbb{R}_0^+$be defined as follows:

$$N(x,y) = \max\left\{d(y,Ty)\frac{1 + d(x,Tx)}{1 + d(x,y)}, d(x,y)\right\}$$

where T is a self-mapping defined on a metric space (X, d). We notice that, for any $x, y \in X$ with $x = y$, we have $0 = d(Tx,Ty) \leq N(x,y)$. Moreover, if $x \neq y$, then $N(x,y) > 0$.

Definition 3.61. [28] Let T be a self-mapping on a metric space (X,d) and $\zeta \in Z_w$. Suppose that $p : X \times X \to \mathbb{R}_0^+$ is a function that satisfies $\left(P^1 p : N\right)$ and $\left(P^2 p : c\right)$, for all $c \in [0,1)$. Then T is called a hybrid contraction of type II if the following conditions are satisfied:

(1) For any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $x \neq y$ and $p(x,y) < \epsilon + \delta(\epsilon)$ imply $d(Tx,Ty) \leq \epsilon$;

(2) $x \neq y$ and $p(x,y) > 0$ imply $\zeta(\alpha(x,y)d(Tx,Ty), p(x,y)) \geq 0$.

Now, we introduce generalized (α,ϕ)-Meir-Keeler hybrid contractive mapping of type I in the setting of b-metric spaces and prove fixed point results for such mappings.

Remark 3.62. We denote the class of mappings Ψ by

$$\Psi = \left\{\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+ : \phi \text{ is continuous, monotone nondecreasing, } \phi(t) = 0 \text{ if } t = 0\right\}$$

Let (X,b) be a b-metric space with $s \geq 1$ and $T : X \to X$ be a self-mapping. We define a mapping $M_s : X \times X \to \mathbb{R}_0^+$by

$$M_s(x,y) = \max\left\{b(x,y), b(x,Tx), b(y,Ty), \frac{b(x,Ty)+b(y,Tx)}{2s}\right\}$$

Let also $p : X \times X \to \mathbb{R}_0^+$ be a mapping. The following conditions are used:

$$\left(P_p^1 : M_s\right) x \neq y \text{ and } b(x,Tx) \leq b(x,y) \text{ imply } p(x,y) \leq M_s(x,y)$$

$$\left(P_p^2 : sc\right) x_n \neq y, \lim_{n\to\infty} b\left(x_n, y\right) = 0 \text{ and } \lim_{n\to\infty} b\left(x_n, Tx_n\right) = 0 \text{ imply } \limsup_{n\to\infty}\left(sb\left(x_n\right.\right.$$
$$\left.\left. y\right)\right) \leq cb(y,Ty), \text{ where } c \in [0,1)$$

Definition 3.63. Let (X,b) be a b-metric space with $s \geq 1, T : X \to X, \alpha : X \times X \to \mathbb{R}_0^+, p : X \times X \to \mathbb{R}_0^+$ satisfy $\left(P_p^1 : M_s\right)$, and $\phi \in \Psi$. Then the mapping T is said to be a generalized (α,ϕ)-Meir-Keeler hybrid contractive mapping of type I if it satisfies, for all $x,y \in X$, the following conditions:

(1) For any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $x \neq y$ and $p(x,y) < \epsilon + \delta(\epsilon)$ imply $b(Tx,Ty) \leq \frac{\epsilon}{s}$;

(2) $x \neq y$ and $p(x,y) > 0$ imply $\zeta(\alpha(x,y)\phi(b(Tx,Ty)), \phi(p(x,y))) \geq 0$.

Theorem 3.64. *Let (X,b) be a complete b-metric space with $s \geq 1, T : X \to X, \alpha : X \times X \to \mathbb{R}_0^+$ be mappings, and $\phi \in \Psi$. Suppose the following conditions hold:*

(i) T is generalized (α,ϕ)-Meir-Keeler hybrid contractive mapping of type I;

(ii) T is a triangular α-orbital admissible mapping;

(iii) There exists $x_0 \in X$ such that $\alpha\left(x_0, Tx_0\right) \geq 1$;

(iv) T is b-continuous.

Then T has a fixed point z. Moreover, $\{T^n x\}$ converges to z for all $x \in X$.

We introduce generalized (α,ϕ)-Meir-Keeler hybrid contractive mapping of type II and study fixed point results for such mappings.

Definition 3.65. Let (X,b) be a b-metric space with $s \geq 1, T : X \to X, \alpha : X \times X \to \mathbb{R}_0^+$, $\zeta \in Z_w, \phi \in \Psi$, and suppose $p : X \times X \to \mathbb{R}_0^+$ is a function that satisfies $\left(P_p^1 : N_s\right)$ and $\left(P_p^2 : sc\right)$. The mapping T is said to be a generalized (α,ϕ)-Meir-Keeler hybrid contractive mapping of type II if it satisfies for all $x,y \in X$ the following conditions:

(1) For any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $x \neq y$ and $p(x,y) < \epsilon + \delta(\epsilon)$ imply $b(Tx,Ty) \leq \frac{\epsilon}{s}$;

(2) $x \neq y$ and $p(x,y) > 0$ imply $\zeta(\alpha(x,y)\phi(b(Tx,Ty)), \phi(p(x,y))) \geq 0$

We define a mapping $N_s : X \times X \to \mathbb{R}_0^+$ by

$$N_s(x,y) = \max\Big\{b(x,y), b(x,Tx), b(y,Ty), \frac{b(y,Ty)[1+b(x,Tx)]}{1+b(x,y)}, \frac{b(x,Tx)[1+b(y,Ty)]}{1+b(Tx,Ty)}\Big\}.$$

We note that, for any $x, y \in X$ with $x = y$, we have $0 = b(Tx,Ty) \leq N_s(x,y)$. Moreover, if $x \neq y$, then $N_s(x,y) > 0$.

Theorem 3.66. *Let (X,b) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be a generalized (α,ϕ)-Meir-Keeler hybrid contractive mapping of type II satisfying the following conditions:*

(i) T is a triangular α-orbital admissible mapping;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) either T is continuous;

(iv) or T^2 is continuous and $\alpha(z,Tz) \geq 1$;

(v) or (X,b) is regular.

Then T has a fixed point z. Moreover, $\{T^n x\}$ is convergent to z for all $x \in X$.

Example 3.67. Let $X = [0,4]$ and $b : X \times X \to \mathbb{R}_0^+$ be defined by $b(x,y) = |x-y|^2$ for all $x, y \in X$. Then (X,b) is a complete b-metric space with $s = 2$ which is not a metric space. Let $T : X \to X$ be defined by

$$T(x) = \begin{cases} 1 & \text{if } x \in [0,2) \\ \frac{x}{2} & \text{if } x \in [2,4] \end{cases}$$

Also, we define $\alpha : X \times X \to \mathbb{R}_0^+$, $q : X \times X \to \mathbb{R}_0^+$ and $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ as follows:

$$\alpha(x,y) = \begin{cases} 2 & \text{if } x, y \in [0,2) \\ 1 & \text{if } x, y \in [2,4] \\ 0 & \text{otherwise} \end{cases}$$

$$q(x,y) = \max\left\{b(x,y), \frac{b(x,Tx)b(y,Ty)}{1+b(x,y)}, \frac{b(x,Tx)b(y,Ty)}{1+b(Tx,Ty)}\right\}$$

and $\phi(t) = \frac{t^2}{2}$. First, we note that q satisfies condition $\left(P_q^1 : N_s\right)$ and $q(x,y) > 0$ for all $x \neq y$. Since, for $x = 0$ we have $T0 = 1$ and $\alpha(0,T0) = \alpha(0,1) = 2 > 1$, assumption (ii) of Theorem 3.66 is satisfied. Also, it is easy to see that T is triangular α-orbital admissible. Let $\zeta \in Z_w$ be is given by $\zeta(t,s) = \frac{2}{3}s - t$.

Secondly, we present fixed points theorems for generalized (α,ϕ)-Meir-Keeler Gregus quadratic type hybrid contraction mappings via simulation function in b-metric spaces introduced by Tiwari and Sharma [43]

Lemma 3.68. *Let (X,b) be a b-metric space with $s \geq 1$ and $T : X \to X$ be a self-mapping. We define a mapping $M_s : X \times X \to \mathbb{R}_0^+$ by*

$$M_s(x,y) = ab^2(x,y) + (1-a)\max\left\{b^2(x,y), b(x,Tx)\cdot b(y,Ty), \frac{b^2(x,Ty)+b^2(y,Tx)}{2s}\right\}$$

And let $p : X \times X \to \mathbb{R}_0^+$ be a mapping satisfies the following conditions:

(1) $\left(P_p^1 : M_s\right)$ *$x \neq y$ and $b(x,Tx) \leq b(x,y) \Longrightarrow p(x,y) \leq M_s(x,y)$;*

(2) $\left(P_p^2 : sc\right)$ *$x_n \neq y, \lim_{n\to\infty} b(x_n,y) = 0$, and $\lim_{n\to\infty} b(x_n,Tx_n) = 0 \Longrightarrow \limsup_{n\to\infty} b(x_n,y) \leq cb(y,Ty)$, where $c \in [0,1)$.*

Definition 3.69. Let (X,b) be a b-metric space with $s \geq 1, T : X \to X, \alpha : X \times X \to \mathbb{R}_0^+ p : X \times X \to \mathbb{R}_0^+$ satisfy $\left(P^1 p : M_s\right)$ and $\phi \in \psi$. Then the mapping T is said to be a generalized (α,ϕ)-MeirKeeler Gregus quadratic type hybrid contractive mapping of type I if it satisfies, for all $x,y \in X$, the following conditions:

(1) For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $x \neq y$ and $p(x,y) < \varepsilon + \delta(\varepsilon) \Longrightarrow b(Tx,Ty) \leq \frac{\varepsilon}{s}$;

(2) $x \neq y$ and $p(x,y) > 0 \Longrightarrow \xi(\alpha(x,y)b(Tx,Ty), p(x,y)) \geq 0$.

Theorem 3.70. *Let (X,b) be a complete b-metric space with $s \geq 1, T : X \to X, \alpha : X \times X \to \mathbb{R}_0^+$ be mappings, and $\phi \in \psi$. Suppose the following conditions hold:*

(i) T is generalized $(\alpha,\phi)MKGq$ type hybrid contractive mapping of type I;

(ii) T is a triangular α-orbital admissible mapping;

(iii) There exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$;

(iv) T is b-continuous. Then T has a fixed point z. Moreover, $\{T^n x\}$ converges to z for all $x \in X$.

Further we introduce generalized (α,ϕ) - MKGq type hybrid contractive mapping of type II and study fixed point results.

Definition 3.71. Suppose (X,b) be a b-metric space with $s \geq 1, T : X \to X, \alpha : X \times X \to \mathbb{R}_0^+$, $\xi \in Z_w, \phi \in \psi$, and suppose $p : X \times X \to \mathbb{R}_0^+$ is a function that satisfies $\left(P_p^1 : N_s\right)$ and $\left(P_p^2 : sc\right)$. The mapping T is said to be a generalized (α,ϕ)-MKGq type hybrid contractive mapping of type II if it satisfies for all $x,y \in X$ the following conditions:

(1) For any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $x \neq y$ and $p(x,y) < \varepsilon + \delta(\varepsilon) \Longrightarrow b(Tx,Ty) \leq \frac{\varepsilon}{s}$;

(2) $x \neq y$ and $p(x,y) > 0$ imply
$$\xi(\alpha(x,y)b(Tx,Ty), p(x,y)) \geq 0.$$

We define a mapping $N_s : X \times X \to \mathbb{R}_0^+$ by

$$N_s(x,y) = ab^2(x,y) + (1-a)\max\left\{\begin{array}{c} b^2(x,y), b(x,Tx)\cdot b(y,Ty), \\ \frac{b^2(y,Ty)\left[1+b^2(x,Tx)\right]}{1+b^2(x,y)}, \\ \frac{b^2(x,Tx)\left[1+b^2(y,Ty)\right]}{1+b^2(Tx,Ty)} \end{array}\right\}.$$

We note that, for any $x,y \in X$ with $x = y$, we have $0 = b^2(Tx,Ty) \leq N_s(x,y)$. Moreover, if $x \neq y$, then $N_s(x,y) > 0$.

Theorem 3.72. *Let (X,b) be a complete b-metric space with $s \geq 1$ and let $T : X \to X$ be a generalized $(\alpha,\phi) - MKGq$ type hybrid contractive mapping of type II satisfying the following conditions:*

(i) T is a triangular α-orbital admissible mapping;

(ii) There exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$;

(iii) either T is continuous;

(iv) or T^2 is continuous and $\alpha(z,Tz) \geq 1$;

(v) or (X, b) is a regular.

Then T has a fixed point of z, Moreover, $\{T^n x\}$ is convergent to z for all $x \in X$.

Example 3.73. Let $X = [0, 6]$ and $b : X \times X \to \mathbb{R}_0^+$ be defined by $b(x, y) = |x - y|$ for all $x, y \in X$. Then (X, b) is a complete b-metric space with $s = 4$ which is not a metric space. Let $T : X \to X$ be defined by

$$T(x) = \begin{cases} 1 & \text{if } x \in [0, 3) \\ \frac{x}{3} & \text{if } x \in [3, 6]. \end{cases}$$

Also, we define $\alpha : X \times X \to \mathbb{R}_0^+, q : X \times X \to \mathbb{R}_0^+$, and $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ as follows:

$$\alpha(x, y) = \begin{cases} 3 \text{ if } x, y \in [0, 3) \\ 2 \text{ if } x, y \in [3, 6] \\ 0 \text{ otherwise} \end{cases}$$

$$q(x, y) = \max \left\{ b^2(x, y), \frac{b(x, Tx) \cdot b(y, Ty)}{1 + b^2(x, y)}, \frac{b(x, Tx) \cdot b(y, Ty)}{1 + b^2(Tx, Ty)} \right\}$$

and $\phi(t) = \frac{t}{3}$. First we note that q satisfies the condition $\left(p_q^1 : N_s\right)$ and $q(x, y) > 0$ for all $x \neq y$. Since, for $x = 0$ we have $T0 = 2$ and $\alpha(0, T0) = \alpha(0, 1) = 3 > 1$, assumption (ii) of Theorem 3.72 is satisfied. Also, it is easy to see that T is a triangular α-orbital admissible. Suppose $\xi \in Z_w$ be given by $\xi(t, s) = \frac{9}{16}s - t$.

4. Conclusion

We present a review by collecting and combining fixed point studies on the Meir-Keeler contraction mapping in b-metric spaces. Also, some interesting examples are considered. It seems that the presented results that fixed point results in b-metric spaces are easily obtained with the Meir-Keeler contractions. Thus, these contraction results are an important research area for the fixed point literature.

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