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Implicit Impulsive Conformable Fractional Differential Equations with Infinite Delay

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Abstract

This paper deals with some existence results for a class of conformable implicit fractional differential equations with instantaneous impulses infinite delay. The results are based on Schaefer's fixed point theorem. We illustrate our results by an example in the last section.

Keywords: Conformable fractional integral; conformable fractional order derivative; infinite delay; Schaefer's fixed point theorem; nonlocal conditions
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1. Introduction

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics, and other applied sciences. Considerable attention has been given to the study of initial and boundary value problems for ordinary and fractional differential equations; see the monographs [1, 2, 3, 28, 29, 31], the papers [6, 13, 16, 20, 21, 23, 24, 25, 26, 27] and the references therein.

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Recently in [17], Khalil *et al.* gave a novel definition of fractional derivative which is a natural extension to the standard first derivative. The conformable fractional derivative is natural and it fulfills most of the properties which the classical integral derivative has such as linearity, product rule, quotient rule, power rule, chain rule and it bring us a lot of convenience when it is applied for modeling many physical problems. Indeed, since that time so many articles have been written and numerous equations have been solved with that concept [4, 12, 22].

The study of implicit and impulsive differential equations has received great attention in the last years; see [1, 8, 9, 19, 30].

In [11], Chauhan *et al.* established the existence and uniqueness of solution to an integral boundary-value problem for impulsive fractional functional integro-differential equation with infinite delay of the form:

$$\begin{cases} {}^cD_\theta^\alpha \xi(\theta) = f\left(\theta, \xi_\theta, B\xi(\theta)\right), \quad \theta \in J = [0,T], \theta \neq \theta_k, \\ \Delta\xi\left(\theta_k\right) = Q_k\left(\xi\left(\theta_k^-\right)\right), \quad k = 1,2,\ldots,m, \\ \Delta\xi'\left(\theta_k\right) = I_k\left(\xi\left(\theta_k^-\right)\right), \quad k = 1,2,\ldots,m, \\ \xi(\theta) = \phi(\theta), \quad \theta \in (-\infty, 0], \\ a\xi'(0) + b\xi'(T) = \int_0^T q(\xi(s))ds, \end{cases}$$

where $\alpha \in (1,2)$, ${}^cD_\theta^\alpha$ is the Caputo fractional derivative, the functions $f : J \times \mathfrak{B}_h \times X \to X$ and $q : X \to X$ are given functions that satisfy certain assumptions, where $\mathfrak{B}_h$ is a phase space. The authors employed some well known fixed point theorems to demonstrate their results.

In [5], the authors considered the following fractional impulsive neutral integro-differential systems with infinite delay:

$$\begin{cases} D_\theta^q\left(\xi(\theta) - \chi\left(\theta, \xi_\theta\right)\right) = A(\theta,\xi)\left(\xi(\theta) - \chi\left(\theta,\xi_\theta\right)\right) + f\left(\theta, \xi_\theta, \int_0^\theta h\left(\theta, s, \xi_s\right)ds\right), \\ \qquad \theta \in [0,b], \quad \theta \neq \theta_k, \\ \Delta\xi|_{\theta=\theta_k} = I_k\left(\xi\left(\theta_k^-\right)\right), \quad \theta = \theta_k; k = 1,\ldots,m, \\ \xi(0) + g(\xi) = \phi, \quad \phi \in B_\vartheta, \end{cases}$$

where $0 < q < 1$, D_θ^q is the Caputo fractional derivative and Let $\xi_\theta(.)$ denote $\xi_\theta(\theta) = \xi(\theta + \theta), \theta \in (-\infty, 0]$. The results are obtained by a fixed point theorem.

In [7], by using Burton-Kirk fixed point theorem, Benchohra *et al.* studied the existence of solutions for the following impulsive partial hyperbolic differential equations:

$$\begin{cases} \left({}^cD_{\xi_\jmath}^\vartheta \chi\right)(\xi,\gamma) = \Psi\left(\xi,\gamma,\chi_{(\xi,\gamma)}\right); \text{ if } (\xi,\gamma) \in \Theta_\jmath, \quad \jmath = 0,\ldots,m, \\ \chi\left(\xi_\jmath^+,\gamma\right) = \chi\left(\xi_\jmath^-,\gamma\right) + I_\jmath\left(\chi\left(\xi_\jmath^-,\gamma\right)\right), \quad \text{if } \gamma \in [0,\beta], \jmath = 1,\ldots,m, \\ \chi(\xi,\gamma) = \varsigma(\xi,\gamma); \text{ if } (\xi,\gamma) \in \tilde{\Theta}, \\ \chi(\xi,0) = \varpi(\xi), \; \xi \in [0,\alpha], \; \chi(0,\gamma) = \varkappa(\gamma); \; \gamma \in [0,\beta], \end{cases}$$

where ${}^cD_{\xi_\jmath}^\vartheta$ is the Caputo fractional derivative of order $\vartheta = (\vartheta_1, \vartheta_2) \in (0,1] \times (0,1], \varpi : [0,\alpha] \to \mathbb{R}^n$, $\varkappa : [0,\beta] \to \mathbb{R}^n$ are given continuous functions with $\varpi(\xi) = \varsigma(\xi,0), \varkappa(\gamma) = \varsigma(0,\gamma)$ for each $(\xi,\gamma) \in \Theta$, $0 = \xi_0 < \xi_1 < \cdots < \xi_m < \xi_{m+1} = \alpha, \Psi : \Theta \times \mathcal{G} \to \mathbb{R}^n, I_\jmath : \mathbb{R}^n \to \mathbb{R}^n, \jmath = 1,\ldots,m, \varsigma : \tilde{\Theta} \to \mathbb{R}^n$, are given functions. $\mathcal{G}$ is called a phase space.

Motivated by the above-mentioned papers, in this paper, first we discuss the existence of solutions for the following initial value problem of conformable implicit impulsive fractional differential equations with infinite delay:

$$\begin{cases} \mathcal{T}^r_{\theta_\jmath}\chi(\theta) = \Psi\left(\theta, \chi_\theta, \mathcal{T}^r_\jmath\chi(\theta)\right), \quad \theta \in \Theta_\jmath; \jmath = 0, 1, \ldots, \daleth, \\ \Delta\chi|_{\theta=\theta_\jmath} = \hbar_\jmath(\chi_{\theta_\jmath^-}), \quad \jmath = 1, 2, \ldots, \daleth, \\ \chi(\theta) = \zeta(\theta), \quad \theta \in (-\infty, a], \end{cases} \tag{1}$$

where $0 \leq a = \theta_0 < \theta_1 < \cdots < \theta_\daleth < \theta_{\daleth+1} = \beta < \infty$, $\mathcal{T}^r_{\theta_\jmath}\chi(\theta)$ is the conformable fractional derivative of order $r \in (0,1)$, $\Psi : \Theta \times \mathcal{G} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $\Theta := [a, \beta]$, $\Theta_0 := [a, \theta_1]$, $\Theta_\jmath := (\theta_\jmath, \theta_{\jmath+1}]; \jmath = 1, 2, \ldots, \daleth$, $\zeta : (-\infty, a] \to \mathbb{R}$ and $\hbar_\jmath : \mathcal{G} \to \mathbb{R}$ are given continuous functions, $\mathcal{G}$ is called a phase space that will be determined later and $\chi_{\theta_\jmath^+} = \lim\limits_{\epsilon \to 0^+} \chi(\theta_\jmath + \epsilon)$ and $\chi_{\theta_\jmath^-} = \lim\limits_{\epsilon \to 0^-} \chi(\theta_\jmath - \epsilon)$ represent the right and left hand limits of $\chi(\theta)$ at $\theta = \theta_\jmath$, $\Delta\chi\left|_{\theta=\theta_\jmath}\right. = \chi(\theta_\jmath^+) - \chi(\theta_\jmath^-)$.

For any $\theta \in \Theta$, we define $\chi_\theta \in \mathcal{G}$ by

$$\chi_\theta(\vartheta) = \chi(\theta + \vartheta); \text{ for } \vartheta \in (-\infty, 0].$$

Next, we discuss the existence of solutions for the following initial value problems of conformable implicit nonlocal impulsive fractional differential equations with infinite delay:

$$\begin{cases} \mathcal{T}^r_{\theta_\jmath}\chi(\theta) = \Psi\left(\theta, \chi_\theta, \mathcal{T}^r_a\chi(\theta)\right), \quad \theta \in \Theta_\jmath; \jmath = 0, 1, \ldots, \daleth, \\ \Delta\chi|_{\theta=\theta_\jmath} = \hbar_\jmath(\chi_{\theta_\jmath^-}), \quad \jmath = 1, 2, \ldots, \daleth, \\ \chi(\theta) + \psi(\chi) = \zeta(\theta), \quad \theta \in (-\infty, a], \end{cases} \tag{2}$$

where $\psi : \mathcal{PC}(\Theta, \mathbb{R}) \to \mathbb{R}$ is a continuous function.

The structure of this paper is as follows: Section 2 presents certain notations and preliminaries about the conformable fractional derivatives used throughout this manuscript. In Section 3, we present an existence results for the problem (1) that are based on the Banach contraction principle and Schauder fixed point theorem. Section 4 deals with the existence result of problem (2). In the last section, an illustrative example is provided in support of the obtained results.

2. Preliminaries

Let $C(\Theta)$ be the Banach space of all real continuous functions on Θ with the norm

$$\|\chi\|_\infty = \sup_{\theta \in \Theta} |\chi(\theta)|.$$

By $L^1(\Theta)$ we denote the Banach space of measurable functions $\chi : \Theta \to \mathbb{R}$ with are Lebesgue integrable, equipped with the norm

$$\|\chi\|_{L^1} = \int_0^\beta |\chi(\theta)| dt.$$

$C^n(\Theta)$ denotes the set of mappings having n times continuously differentiable on Θ, where $n \in \mathbb{N}$.
Consider the Banach space

$$\mathcal{PC}(\Theta, \mathbb{R}) = \Bigg\{ \chi : \Theta \to \mathbb{R} : \chi \in C\left(\theta_\jmath, \mathbb{R}\right); \jmath = 0, \ldots, m, \text{ and there exist}$$

$$\chi\left(\theta_\jmath^-\right) \text{ and } \chi\left(\theta_\jmath^+\right); \jmath = 1, \ldots, m, \quad \text{with} \quad \chi\left(\theta_\jmath^-\right) = \chi\left(\theta_\jmath\right) \Bigg\},$$

with the norm

$$\|\chi\|_{\mathcal{PC}} = \sup_{\theta \in \Theta} |\chi(\theta)|.$$

Definition 2.1 (The conformable fractional derivative [4])**.** *Let $\Psi : [0,+\infty) \longrightarrow \mathbb{R}$ be a given function, then the conformable fractional derivative of Ψ of order r is defined by*

$$\mathcal{T}^r(\Psi)(\theta) = \lim_{\alpha\to 0} \frac{\Psi\left(\theta + \alpha\theta^{1-r}\right) - \Psi(\theta)}{\alpha},$$

for $\theta > 0$ and $r \in (0,1]$. If Ψ is r-differentiable in some $(0,a)$, $a > 0$, and $\lim_{\theta\to 0+} \mathcal{T}^r(\Psi)(\theta)$ exists, then define $\mathcal{T}^r(\Psi)(0) = \lim_{\theta\to 0+} \mathcal{T}^r(\Psi)(\theta)$. If the conformable fractional derivative of Ψ of order r exists, then we simply say that Ψ is r-differentiable. It is easy to see that if Ψ is differentiable, then $\mathcal{T}^r(\Psi)(\theta) = \theta^{1-r}\Psi'(\theta)$.

Definition 2.2 (The conformable fractional integral [4])**.** *The conformable fractional integral starting from a of the function $\Psi : [a,+\infty) \to \mathbb{R}$ of order $r \in (0,1]$ is defined as*

$$I_a^r\Psi(\theta) = \int_a^\theta (\vartheta - a)^{1-r}\Psi(\theta)d\vartheta.$$

Lemma 2.3 ([4])**.** *Let $r \in (0,1]$ and $\Psi : [a,+\infty) \to \mathbb{R}$ be a continuous function. Then for all $\theta > a$,*

$$\mathcal{T}_a^r I_a^r \Psi(\theta) = \Psi(\theta).$$

Further, if Ψ is differentiable on $(a,+\infty)$, then for all $\theta > a$,

$$I_a^r \mathcal{T}_a^r \Psi(\theta) = \Psi(\theta) - \Psi(a).$$

Lemma 2.4. *Let $\Psi \in L^1(\Theta)$, $a < \beta < +\infty$ and $0 < r \le 1$. Then the initial value problem*

$$\begin{cases} \mathcal{T}_{\theta_\jmath}^r \chi(\theta) = \Phi(\theta), \quad \theta \in \Theta_\jmath; \jmath = 0,1,\ldots,\daleth, \\ \Delta\chi|_{\theta=\theta_\jmath} = \hbar_\jmath(\chi_{\theta_\jmath^-}), \quad \jmath = 1,2,\ldots,\daleth, \\ \chi(\theta) = \zeta(\theta), \quad \theta \in (-\infty, a], \end{cases} \tag{3}$$

has a unique solution defined by

$$\chi(\theta) = \begin{cases} \zeta(a) + \displaystyle\int_a^\theta (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta, \quad if \ \theta \in [a,\theta_1], \\ \\ \zeta(a) + \displaystyle\sum_{\imath=1}^{\jmath} \hbar_\imath(\chi_{\theta_\imath^-}) + \sum_{\imath=1}^{\jmath} \int_{\theta_{\imath-1}}^{\theta_\imath} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta \\ + \displaystyle\int_{\theta_\jmath}^\theta (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta, \quad if \ \theta \in (\theta_\jmath, \theta_{\jmath+1}], \\ \\ \zeta(\theta), \quad \theta \in (-\infty, a], \end{cases} \tag{4}$$

where $\jmath = 1,2,\ldots,\daleth$.

Proof. Applying the conformable fractional integral of order r to both sides the equation $\mathcal{T}_{\theta_\jmath}^r \chi(\theta) = \Phi(\theta)$, and by using Lemma 2.3 and if $\theta \in [a,\theta_1]$, we get

$$\chi(\theta) = \zeta(a) + \int_a^\theta (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta. \tag{5}$$

If $\theta \in (\theta_1, \theta_2]$, then from Lemma 2.3,

$$\chi(\theta) = \chi(\theta_1^+) + \int_{\theta_1}^\theta (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta$$

$$
\begin{aligned}
&= \Delta\chi|_{\theta=\theta_1} + \chi(\theta_1^-) + \int_{\theta_1}^{\theta} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta \\
&= \zeta(a) + \hbar_1(\chi_{\theta_1^-}) + \int_{a}^{\theta} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta + \int_{\theta_1}^{\theta} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta.
\end{aligned}
$$

If $\theta \in (\theta_2, \theta_3]$, then from Lemma 2.3,

$$
\begin{aligned}
\chi(\theta) &= \chi(\theta_2^+) + \int_{\theta_2}^{\theta} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta \\
&= \Delta\chi|_{\theta=\theta_2} + \chi(\theta_2^-) + \int_{\theta_2}^{\theta} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta \\
&= \hbar_2(\chi_{\theta_2^-}) + \left[\zeta(a) + \hbar_1(\chi_{\theta_1^-}) + \int_{a}^{\theta_1} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta + \int_{\theta_1}^{\theta_2} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta\right] \\
&\quad + \int_{\theta_2}^{\theta} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta \\
&= \zeta(0) + \left[\hbar_1(\chi_{\theta_1^-}) + \hbar_2(\chi_{\theta_2^-})\right] + \left[\int_{a}^{\theta_1} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta + \int_{\theta_1}^{\theta_2} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta\right] \\
&\quad + \int_{\theta_2}^{\theta} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta.
\end{aligned}
$$

Continuing this process, we get the solution $\chi(\theta)$ for $\theta \in (\theta_\jmath, \theta_{\jmath+1}]$, where $\jmath = 1, 2, \ldots, \daleth$. Hence

$$
\chi(\theta) = \zeta(a) + \sum_{\imath=1}^{\jmath} \hbar_\imath(\chi_{\theta_\imath^-}) + \sum_{\imath=1}^{\jmath} \int_{\theta_{\imath-1}}^{\theta_\imath} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta + \int_{\theta_\jmath}^{\theta} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta.
$$

Conversely, we can easily show by Lemma 2.3 that if χ verifies equation (4) then it satisfied the problem (3). □

3. Existence of Solutions for the First Problem

This section deals with the existence results for problem (1). Let the space $(\mathcal{G}, \|\cdot\|_{\mathcal{G}})$ be a seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$ given by

$$
\begin{aligned}
\mathcal{G} = \Big\{ &\chi : (-\infty, 0] \to \mathbb{R} : \ \tau \to \chi(\tau) \in C\left((\tau_\jmath, \tau_{\jmath+1}], \mathbb{R}\right); \jmath = 0, \ldots, m, \\
&\text{and there exist } \chi\left(\tau_\jmath^-\right) \text{ and } \chi\left(\tau_\jmath^+\right); \jmath = 1, \ldots, m, \quad \text{with} \\
&\chi\left(\tau_\jmath^-\right) = \chi\left(\tau_\jmath\right) \text{ and } \tau_\jmath = \theta_\jmath - \theta \text{ for each } \theta \in (\theta_\jmath, \theta_{\jmath+1}] \Big\},
\end{aligned}
$$

and verifying the following axioms which were derived from Hale and Kato's originals [14]:

(Ax_1) If $y : (-\infty, \beta] \to \mathbb{R}$, and $\chi_0 \in \mathcal{G}$, then there exist constants $\xi_1, \xi_2, \xi_3 > 0$, such that for each $\theta \in \Theta$, we have:

- (i) χ_θ *is in* $\mathcal{G}$,
- (ii) $\|\chi_\theta\|_{\mathcal{G}} \leq \xi_1\|\chi_1\|_{\mathcal{G}} + \xi_2 \sup_{\vartheta\in[a,\theta]} |\chi(\vartheta)|$,
- (iii) $\|\chi(\theta)\| \leq \xi_3\|\chi_\theta\|_{\mathcal{G}}$.

(Ax_2) For the function $\chi(\cdot)$ *in* (Ax_1), y_θ is a $\mathcal{G}-$ valued continuous function on Θ.

(Ax_3) The space $\mathcal{G}$ is complete.

Consider the space

$$\Omega = \left\{\chi : (-\infty, \beta] \to \mathbb{R},\ \chi|_{(-\infty,a]} \in \mathcal{G},\ \chi|_\Theta \in \mathcal{PC}(\Theta, \mathbb{R})\right\}.$$

Definition 3.1. *By a solution of problem (1), we mean a continuous function $u \in \Omega$*

$$\chi(\theta) = \begin{cases} \zeta(\theta), \quad \theta \in (-\infty, a], \\ \\ \zeta(a) + \sum\limits_{0<\theta_\jmath<\theta} \hbar_\jmath(\chi_{\theta_\jmath^-}) + \sum\limits_{0<\theta_\jmath<\theta} \displaystyle\int_{\theta_{\jmath-1}}^{\theta_\jmath} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta \\ + \displaystyle\int_{\theta_\jmath}^{\theta} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta, \quad \theta \in \Theta, \end{cases} \tag{6}$$

where $\Phi \in C(\Theta)$ such that $\Phi(\theta) = \Psi(\theta, \chi_\theta, \Phi(\theta))$.

Let us introduce the following hypotheses:

(H_1) There exist constants $\varpi_1 > 0$ *and* $0 < \varpi_2 < 1$ such that

$$|\Psi(\theta, \chi, \Im) - \Psi(\theta, \bar{\chi}, \bar{\Im})| \leq \varpi_1 \|\chi - \bar{\chi}\|_{\mathcal{G}} + \varpi_2 |\Im - \bar{\Im}|,$$

for any $\chi, \bar{\chi} \in \mathcal{G},\ \Im, \bar{\Im} \in \mathbb{R}$, and each $\theta \in \Theta$.

(H_2) There exist continuous functions $\varsigma_1, \varsigma_2 : \Theta \to \mathbb{R}_+$ such that

$$|\hbar_\jmath(\chi)| \leq \varsigma_1(\theta)\|\chi\|_{\mathcal{G}} + \varsigma_2(\theta),$$

for each $\chi \in \mathcal{G}$ and $\jmath = 1, 2, \ldots, \gimel$,
where $\varsigma_1{}^* = \sup\limits_{\theta\in\Theta} \varsigma_1(\theta)$, and $\varsigma_2{}^* = \sup\limits_{\theta\in\Theta} \varsigma_2(\theta)$, with $0 < \xi_1 p^* < 1$.

We prove our existence result by using Schaefer's fixed point theorem [15].

Theorem 3.2. *Assume that the hypotheses (H_1) and (H_2) hold. Then problem (1) has at least one solution on $(-\infty, \beta]$.*

Proof. Consider the operator $\aleph : \ \Omega \to \Omega$ defined by:

$$(\aleph\chi)(\theta) = \begin{cases} \zeta(\theta), \quad \theta \in (-\infty, a], \\ \\ \zeta(a) + \sum\limits_{0<\theta_\jmath<\theta} \hbar_\jmath(\chi_{\theta_\jmath^-}) + \sum\limits_{0<\theta_\jmath<\theta} \displaystyle\int_{\theta_{\jmath-1}}^{\theta_\jmath} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta \\ + \displaystyle\int_{\theta_\jmath}^{\theta} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta, \quad \theta \in \Theta, \end{cases} \tag{7}$$

where $\Phi \in C(\Theta)$ such that $\Phi(\theta) = \Psi(\theta, \chi_\theta, \Phi(\theta))$.
Let $\varkappa_1 : (-\infty, \beta] \to \mathbb{R}$ be a function given by

$$\varkappa_1(\theta) = \begin{cases} \zeta(\theta), & \theta \in (-\infty, a], \\ \zeta(a), & \theta \in \Theta. \end{cases}$$

Then $\varkappa_{10} = \zeta$. For each $\varkappa_2 \in C(\Theta)$, with $\varkappa_2(0) = 0$, le $\overline{\varkappa_2}$ be the function given by

$$\overline{\varkappa_2} = \begin{cases} 0, & \theta \in (-\infty, a], \\ \varkappa_2(\theta), & \theta \in \Theta. \end{cases}$$

If $\chi(\cdot)$ satisfies the integral equation

$$\chi(\theta) = \zeta(a) + \sum_{0<\theta_\jmath<\theta} \hbar_\jmath(\chi_{\theta_\jmath^-}) + \sum_{0<\theta_\jmath<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta + \int_{\theta_\jmath}^{\theta} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta.$$

we can decompose $\chi(\cdot)$ as $\chi(\theta) = \overline{\varkappa_2}(\theta) + \varkappa_1(\theta)$; for $\theta \in \Theta$, which implies that $\chi_\theta = \overline{\varkappa_2}_\theta + \varkappa_{1\theta}$ for every $\theta \in \Theta$, and the function $\varkappa_2(\cdot)$ satisfies

$$\varkappa_2(\theta) = \sum_{0<\theta_\jmath<\theta} \hbar_\jmath(\chi_{\theta_\jmath^-}) + \sum_{0<\theta_\jmath<\theta} \int_{\theta_{\imath-1}}^{\theta_\imath} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta + \int_{\theta_\jmath}^{\theta} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta,$$

where

$$\Phi(\theta) = \Psi(\theta, \overline{\varkappa_2}_\theta + \varkappa_{1\theta}, \Phi(\theta)); \ \theta \in \Theta.$$

Set

$$C_0 = \{\varkappa_2 \in C(\Theta); \ \varkappa_{20} = 0\},$$

and let $\|\cdot\|_\beta$ be the norm in C_0 defined by

$$\|\varkappa_2\|_\beta = \|\varkappa_{20}\|_{\mathcal{G}} + \sup_{\theta\in\Theta} |\varkappa_2(\theta)| = \sup_{\theta\in\Theta} |\varkappa_2(\theta)|; \ \varkappa_2 \in C_0,$$

where C_0 is a Banach space with norm $\|\cdot\|_\beta$. Define the operator $\mathcal{K} : C_0 \to C_0$ by

$$(\mathcal{K}\varkappa_2)(\theta) = \sum_{0<\theta_\jmath<\theta} \hbar_\jmath(\chi_{\theta_\jmath^-}) + \sum_{0<\theta_\jmath<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta + \int_{\theta_\jmath}^{\theta} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta, \tag{8}$$

where

$$\Phi(\theta) = \Psi(\theta, \overline{\varkappa_2}_\theta + \varkappa_{1\theta}, \Phi(\theta)); \ \theta \in \Theta.$$

For each given $\delta > 0$, we define the ball

$$\Omega_\delta = \{\varkappa_1 \in C_0, \ \|\varkappa_1\|_\beta \leq \delta\}.$$

Step 1. $\mathcal{K}$ is continuous.
Let $\varkappa_{2n}$ be a sequence such that $\varkappa_{2n} \to \varkappa_2$ in C_0. For each $\theta \in \Theta$, we have

$$\begin{aligned} |(\mathcal{K}\varkappa_{2n})(\theta) - (\mathcal{K}\varkappa_2)(\theta)| \leq & \sum_{0<\theta_\jmath<\theta} |\hbar_\jmath(\varkappa_{2n\theta_\jmath^-}) - \hbar_\jmath(\varkappa_{2\theta_\jmath^-})| \\ & + \sum_{0<\theta_\jmath<\theta} \int_{\theta_{\imath-1}}^{\theta_\imath} (\vartheta - a)^{r-1}|\Phi_n(\vartheta) - \Phi(\vartheta)|d\vartheta \\ & + \int_{\theta_\jmath}^{\theta} (\vartheta - a)^{r-1}|\Phi_n(\vartheta) - \Phi(\vartheta)|d\vartheta, \end{aligned} \tag{9}$$

where $\Phi_n, \ \Phi \in C(\Theta)$ such that

$$\Phi_n(\theta) = \Psi(\theta, \overline{\varkappa_2}_{n\theta} + \varkappa_{1\theta}, \Phi_n(\theta)) \quad \textit{and} \quad \Phi(\theta) = \Psi(\theta, \overline{\varkappa_2}_\theta + \varkappa_{1\theta}, \Phi(\theta)).$$

Since $\|\varkappa_{2n} - \varkappa_2\|_\beta \to 0$ as $n \to \infty$ and Ψ, Φ and Φ_n are continuous, then

$$\|\mathcal{K}(\chi_n) - \mathcal{K}(\chi)\|_\beta \to 0 \quad \text{as } n \to \infty.$$

Hence, $\mathcal{K}$ is continuous.

Step 2. $\mathcal{K}(\Omega_\delta)$ is bounded.
Let $\varkappa_2 \in \Omega_\delta$, for each $\theta \in \Theta$, we have

$$\begin{aligned}|\Phi(\theta)| &\leq |\Psi(\theta, \overline{\varkappa_2}_\theta + \varkappa_{1\theta}, \Phi(\theta))| \\ &\leq \varpi_1\|\overline{\varkappa_2}_\theta + \varkappa_{1\theta}\|_{\mathcal{G}} + \varpi_2|\Phi(\theta)| + \Psi^* \\ &\leq \varpi_1\left[\|\overline{\varkappa_2}_\theta\|_{\mathcal{G}} + \|\varkappa_{1\theta}\|_{\mathcal{G}}\right] + \varpi_2\|\Phi\|_\infty + \Psi^* \\ &\leq \varpi_1\xi_2\delta + \varpi_1\xi_1\|\varphi\|_{\mathcal{G}} + \varpi_2\|\Phi\|_\infty + \Psi^*,\end{aligned}$$

where $\Psi^* = \sup\limits_{\theta\in\Theta}\Psi(\theta, 0, 0)$. Then

$$\|\Phi\|_\infty \leq \frac{\Psi^* + \varpi_1\xi_2\delta + \varpi_1\xi_1\|\varphi\|_{\mathcal{G}}}{1 - \varpi_2}.$$

Thus,

$$\begin{aligned} rlll|(\mathcal{K}\varkappa_2)(\theta)| &\leq \sum_{0<\theta_\jmath<\theta} |\hbar_\jmath(\varkappa_{2\theta_\jmath^-})| \\ &\quad + \sum_{0<\theta_\jmath<\theta}\int_{\theta_{\imath-1}}^{\theta_\imath}(\vartheta - a)^{r-1}|\Phi(\vartheta)|d\vartheta \\ &\quad + \int_{\theta_\jmath}^{\theta}(\vartheta - a)^{r-1}|\Phi(\vartheta)|d\vartheta \\ &\leq \daleth({\varsigma_1}^*\delta + {\varsigma_2}^*) + \frac{(\daleth + 1)(\beta - a)^r(\Psi^* + \varpi_1\xi_2\delta + \varpi_1\xi_1\|\varphi\|_{\mathcal{G}})}{r(1 - \varpi_2)} \\ &:= \tilde{\ell}.\end{aligned}$$

Hence

$$\|\mathcal{K}(\varkappa_2)\|_\beta \leq \tilde{\ell}.$$

Consequently, $\mathcal{K}$ maps bounded sets into bounded sets in C_0.

Step 3. $\mathcal{K}(\Omega_\delta)$ is equicontinuous.
For $1 \leq \theta_1 \leq \theta_2 \leq \beta$, and $\varkappa_2 \in \Omega_\delta$, we have

$$\begin{aligned}&|\mathcal{K}(\chi)(\theta_2) - \mathcal{H}(\chi)(\theta_1)| \\ &\quad\leq \sum_{0<\theta_\jmath<\theta_2-\theta_1}|\hbar_\jmath(\varkappa_{2\theta_\jmath^-})| + \left|\int_{\theta_\jmath}^{\theta 1}(\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta - \int_{\theta_\jmath}^{\theta_2}(\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta\right| \\ &\quad\leq (\theta_2 - \theta_1)({\varsigma_1}^*\delta + {\varsigma_2}^*) + \frac{\Psi^* + \varpi_1\xi_2\delta + \varpi_1\xi_1\|\varphi\|_{\mathcal{G}}}{r(1 - \varpi_2)}\left|(\theta_2 - a)^r - (\theta_1 - a)^r\right|.\end{aligned}$$

As $\theta_2 \to \theta_1$ then $|\mathcal{K}(\chi)(\theta_1) - \mathcal{K}(\chi)(\theta_2)| \to 0$. We conclude that $\mathcal{K}$ maps bounded sets into equicontinuous sets in C_0. Thus, $\mathcal{K} : C_0 \to C_0$ is completely continuous.

Step 4. The priori bounds.
We prove that the set

$$\mathcal{E} = \{\chi \in C_0 : \Im = \lambda\mathcal{K}(\chi);\ for\ some\ \lambda \in (0, 1)\}$$

is bounded. Let $\varkappa_2 \in C_0$. Let $u \in C_0$, such that $\varkappa_2 = \lambda \mathcal{K}(\varkappa_2)$; *for some* $\lambda \in (0,1)$. Then for each $\theta \in \Theta$, we have

$$\begin{aligned}\varkappa_2(\theta) &= \lambda(\mathcal{K}\varkappa_2)(\theta)\\ &= \lambda\left[\sum_{0<\theta_\jmath<\theta} \hbar_\jmath(\chi_{\theta_\jmath^-}) + \sum_{0<\theta_\jmath<\theta}\int_{\theta_{\jmath-1}}^{\theta_\jmath}(\vartheta-a)^{r-1}\Phi(\vartheta)d\vartheta + \int_{\theta_\jmath}^{\theta}(\vartheta-a)^{r-1}\Phi(\vartheta)d\vartheta\right].\end{aligned}$$

By (H_1) we have

$$\begin{aligned}|\Phi(\theta)| &\le |\Psi(\theta, \overline{\varkappa_{2}}_\theta + \varkappa_{1\theta}, \Phi(\theta))|\\ &\le \varpi_1\|\overline{\varkappa_{2}}_\theta + \varkappa_{1\theta}\|_{\mathcal{G}} + \varpi_2|\Phi(\theta)| + \Psi^*\\ &\le \varpi_1\left[\|\overline{\varkappa_{2}}_\theta\|_{\mathcal{G}} + \|\varkappa_{1\theta}\|_{\mathcal{G}}\right] + \varpi_2\|\Phi\|_\infty + \Psi^*\\ &\le \varpi_1\xi_2\|\varkappa_2\|_\beta + \varpi_1\xi_1\|\varphi\|_{\mathcal{G}} + \varpi_2\|\Phi\|_\infty + \Psi^*.\end{aligned}$$

This gives

$$\begin{aligned}\|\Phi\|_\infty &\le \frac{\varpi_1\xi_2\|\varkappa_2\|_\beta + \varpi_1\xi_1\|\varphi\|_{\mathcal{G}} + \Psi^*}{1-\varpi_2}\\ &:= \eta.\end{aligned}$$

Thus, for each $\theta \in \Theta$, we obtain

$$\begin{aligned}|\varkappa_2(\theta)| &\le \sum_{0<\theta_\jmath<\theta}|\hbar_\jmath(\varkappa_{2\theta_\jmath^-})| + \sum_{0<\theta_\jmath<\theta}\int_{\theta_{\imath-1}}^{\theta_\imath}(\vartheta-a)^{r-1}|\Phi(\vartheta)|d\vartheta + \int_{\theta_\jmath}^{\theta}(\vartheta-a)^{r-1}|\Phi(\vartheta)|d\vartheta\\ &\le \daleth({\varsigma_1}^*\delta + {\varsigma_2}^*) + \frac{(\daleth+1)(\beta-a)^r\eta}{r}\\ &:= \eta'.\end{aligned}$$

Hence

$$\|\varkappa_2\|_\beta \le \eta'.$$

This shows that the set $\mathcal{E}$ is bounded. Thus, by Scheafer's fixed point theorem [15], $\mathcal{K}$ has a fixed point which is a solution of problem (1). $\square$

4. Existence of Solutions for the Second Problem

In this section, we establish some existence results for problem (2).

Let us introduce the following hypotheses:

(H_3) There exist a constant $\varsigma > 0$ such that

$$|\psi(\chi) - \psi(\bar{\chi})| \le \varsigma|\chi(t) - \bar{\chi}(t)|,$$

for each $\chi, \bar{\chi} \in \mathcal{G}$.

Theorem 4.1. *Assume that the hypotheses* (H_1)-(H_3) *hold. Then problem* (2) *has at least one solution on* $(-\infty, \beta]$.

Proof. Consider the operator $\aleph' :\ \Omega \to \Omega$ defined by:

$$(\aleph'\chi)(t) = \begin{cases} \zeta(\theta) - \psi(\chi);\ \theta \in (-\infty, a], \\ \\ \zeta(a) - \psi(\chi) + \sum\limits_{0<\theta_\jmath<\theta} \hbar_\jmath(\chi_{\theta_\jmath^-}) + \sum\limits_{0<\theta_\jmath<\theta} \int_{\theta_{\jmath-1}}^{\theta_\jmath} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta \\ + \int_{\theta_\jmath}^{\theta} (\vartheta - a)^{r-1}\Phi(\vartheta)d\vartheta;\ \theta \in \Theta, \end{cases} \tag{10}$$

where $\Phi \in C(\Theta)$ such that $\Phi(\theta) = \Psi(\theta, \chi_\theta, \Phi(\theta))$.
Clearly, the fixed points of the operator $\aleph'$ are solution of the problem (2). By repeating the same process of Theorem 3.2, we can easily show all the conditions of Theorem 4.1 are satisfied by $\aleph'$. Since the proof is standard, we omit it here. □

5. An Example

Consider the following problem:

$$\begin{cases} \chi(\theta) = \theta, \quad \theta \in (-\infty, 0], \\ \Delta\chi|_{\theta=\frac{1}{2}} = \dfrac{\chi_{\theta_\jmath^-}}{30 + \chi_{\theta_\jmath^-}}, \\ (\mathcal{T}_0^{\frac{1}{2}}\chi)(\theta) = \frac{\chi_\theta e^{-\varrho\theta+\theta}}{180\left(e^\theta - e^{-\theta}\right)(1+\|\chi_\theta\|)} + \frac{1+\chi(\theta)e^{-\varrho\theta+\theta}}{60\left(e^\theta - e^{-\theta}\right)\left(1+|(\mathcal{T}_0^{\frac{1}{2}}\chi)(\theta))|\right)}, \quad \theta \in \Theta_0 \cup \Theta_1. \end{cases} \tag{11}$$

where $\Theta_0 = [0, \frac{1}{2}]$,$\Theta_1 = (\frac{1}{2}, 1]$, $\theta_0 = 0$ and $\theta_1 = 1$. Let ϱ be a positive real constant and

$$B_\varrho = \Bigg\{ \chi : (-\infty, 0] \to \mathbb{R} :\ \tau \to \chi(\tau) \in C\left((\tau_\jmath, \tau_{\jmath+1}], \mathbb{R}\right); \jmath = 0, \dots, m,$$
$$\text{and there exist } \chi\left(\tau_\jmath^-\right) \text{ and } \chi\left(\tau_\jmath^+\right); \jmath = 1, \dots, m, \text{ with } \chi\left(\tau_\jmath^-\right) = \chi\left(\tau_\jmath\right),$$
$$\tau_\jmath = \theta_\jmath - \theta \text{ for each } \theta \in (\theta_\jmath, \theta_{\jmath+1}] \text{ and } \lim_{\tau\to-\infty} e^{\varrho\tau}\chi(\tau) \text{ exists in } \mathbb{R} \Bigg\},$$

The norm of B_ϱ is given by

$$\|\chi\|_\varrho = \sup_{\tau\in(-\infty,1]} e^{\varrho\tau}|\chi(\tau)|.$$

Let $\chi : (-\infty, 0] \to \mathbb{R}$ be such that $\chi_0 \in B_\varrho$. Then

$$\begin{aligned} \lim_{\tau\to-\infty} e^{\varrho\tau}\chi_\theta(\tau) &= \lim_{\tau\to-\infty} e^{\varrho\tau}\chi(\theta + \tau - 1) = \lim_{\tau\to-\infty} e^{\varrho(\tau-\theta+1)}\chi(\tau) \\ &= e^{\varrho(-\theta+1)} \lim_{\tau\to-\infty} e^{\varrho(\tau)}\chi_1(\tau) < \infty. \end{aligned}$$

Hence $\chi_\theta \in B_\varrho$. Now, we demonstrate that

$$\|\chi_\theta\|_\varrho \le \xi_1\|\chi_1\|_\varrho + \xi_2 \sup_{\vartheta\in[0,\theta]} |\chi(\vartheta)|,$$

where $\xi_1 = \xi_2 = 1$ and $\xi_3 = 1$. We have

$$\|\chi_\theta(\tau)\| = |\chi(\theta + \tau)|.$$

If $\theta+\tau\leq 1$, we get

$$\|\chi_\theta(\xi)\|\leq \sup_{\vartheta\in(-\infty,0]}|\chi(\vartheta)|.$$

For $\theta+\tau\geq 0$, then we have

$$\|\chi_\theta(\xi)\|\leq \sup_{\vartheta\in[0,\theta]}|\chi(\vartheta)|.$$

Thus for all $\theta+\tau\in\Theta$, we get

$$\|\chi_\theta(\xi)\|\leq \sup_{\vartheta\in(-\infty,0]}|\chi(\vartheta)|+\sup_{\vartheta\in[0,\theta]}|\chi(\vartheta)|.$$

Then

$$\|\chi_\theta\|_\varrho\leq\|\chi_0\|_\varrho+\sup_{\vartheta\in[0,\theta]}|\chi(\vartheta)|.$$

It is clear that $(B_\varrho,\|\cdot\|)$ is a Banach space. We can conclude that B_ϱ a phase space.
Set

$$\Psi(\theta,\chi,\Im)=\frac{e^{-\varrho\theta+\theta}}{180\left(e^\theta-e^{-\theta}\right)\left(1+\|\chi\|_{B_\varrho}\right)}+\frac{1+e^{-\varrho\theta+\theta}}{60\left(e^\theta-e^{-\theta}\right)\left(1+|\Im|\right)},$$

where $\theta\in[0,1],\ \chi\in B_\varrho,\Im\in\mathbb{R}$.
For any $\chi,\bar\chi\in B_\varrho,\ \Im,\bar\Im\in\mathbb{R}$ and $\theta\in[0,1]$, we have

$$|\Psi(t,\chi,\Im)-\Psi(t,\bar\chi,\bar\Im)|\leq\frac{1}{180}\|\chi-\bar\chi\|_{B_\varrho}+\frac{1}{60}|\Im-\bar\Im|.$$

Hence hypothesis (H_1) is satisfied with

$$\varpi_1=\frac{1}{180}\quad \textit{and}\quad \varpi_2=\frac{1}{60}.$$

For each $\chi\in B_\varrho$, we have

$$|\hbar_\jmath(\chi)|\leq\frac{1}{30}\|\chi\|_{B_\varrho}+1$$

Hence hypothesis (H_2) is satisfied with

$$\varsigma_1(\theta)=\frac{\theta}{30}\quad \textit{and}\quad \varsigma_2(\theta)=\theta.$$

Simple computations show that all conditions of Theorem 3.2 are satisfied. It follows that problem (11) has at least one solution defined on $(-\infty,1]$.

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