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Coupled fixed point results in ordered partial metric spaces and application to the Volterra type integral equations

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Abstract

This study aims to demonstrate coupled fixed point theorems for contractive type conditions with control functions in partially ordered partial metric spaces. Furthermore, some consequences of the established conclusions and illustrative instances to back up the findings are discussed. An application to the Volterra type integral equation is also shown, followed by an illustration.

Keywords: coupled fixed point, contractive type condition, partial metric space, partially ordered set, control function.
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1. Introduction

The Banach contraction mapping principle is the most famous and widely known fixed point theorem. It has been expanded and improved by other mathematicians. A partial metric space is a more generalized

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version of metric space. *Matthews* [17, 18] proposed the concept of partial metric space, which does not need each object to have a zero distance from itself. Partial metric spaces are generally acknowledged as crucial in creating models in computation theory (e.g., [10], [15]). Later, *Matthews* demonstrated the partial metric version of the Banach fixed point theorem [4]. Following this finding, many researchers have performed additional studies on fixed point theorems and their topological features in the same class of spaces (see e.g. ([2, 3, 11, 13, 14, 16, 20]).
Bhashkar and Lakshmikantham [5] proved a few coupled fixed point theorems on ordered metric spaces in 2006 and provided an application in determining whether a periodic boundary value problem has a unique solution or not (see, also [9]). Later, *Lakshmikantham* [7] and *Ćirić* looked into a few more coupled fixed point theorems in partially ordered sets. In addition, a large number of scholars have found coupled fixed point solutions for mappings under different contractive conditions within the context of generalized metric spaces and metric spaces. In [1], Abdeljawad et al. considered a new Meir-Keeler type coupled fixed point results os ordered partial metric space. Later, Bilgili et al. [6] remarked that many coupled fixed point result could be derived from fixed point results as well. Using more general contraction condition, Karapınar et al. [12] discussed the similar type of results. In [22], Roldan et al. discussed multidimensional fixed point results in ordered partial metric spaces under (ψ, φ)-contractive condition. More detail on fixed point results in generalized metric spaces are discuused in the book [14].
In the context of partial metric spaces, *Saluja* [23] recently proved a few linked fixed point theorems for contractive conditions involving rational terms. Furthermore, he provided some implications and applications of the established results (see, also [19]).
In this paper, we prove coupled fixed point theorems for contractive type conditions with control functions in partially ordered partial metric spaces. Furthermore, we provide some concrete cases to support the established findings. An application of the Volterra integral equation is also provided. Our findings expand and generalize several previously published results from the literature.

2. Preliminaries

In this section, we give some basic definitions and lemmas which are useful for main results in this paper.

Definition 2.1. ([5]) Let $(U, \leq)$ be a partially ordered set. The mapping $F\colon U \times U \to U$ is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in U$,

$$x_1, x_2 \in U, \;\; x_1 \leq x_2 \;\Rightarrow\; F(x_1, y) \leq F(x_2, y),$$

and

$$y_1, y_2 \in U, \;\; y_1 \leq y_2 \;\Rightarrow\; F(x, y_1) \geq F(x, y_2).$$

Definition 2.2. ([5, 7]) An element $(x, y) \in U \times U$ is said to be a coupled fixed point of the mapping $F\colon U \times U \to U$ if $F(x, y) = x$ and $F(y, x) = y$.

Example 2.3. ([2]) Let $U = [0, +\infty)$ and $F\colon U \times U \to U$ be defined by $F(x, y) = \frac{x+y}{3}$ for all $x, y \in U$. Then one can easily see that F has a unique coupled fixed point $(0, 0)$.

Example 2.4. ([2]) Let $U = [0, +\infty)$ and $F\colon U \times U \to U$ be defined by $F(x, y) = \frac{x+y}{2}$ for all $x, y \in U$. Then we see that F has two coupled fixed point $(0, 0)$ and $(1, 1)$, that is, the coupled fixed point is not unique.

Definition 2.5. ([18]) Let $U \neq \emptyset$ be a set. A partial metric on U is a function $p\colon U \times U \to [0, +\infty)$ such that for all $x, y, t \in U$ the followings are satisfied:

$(PM1)$ $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
$(PM2)$ $p(x, x) \leq p(x, y)$,
$(PM3)$ $p(x, y) = p(y, x)$,
$(PM4)$ $p(x, y) \leq p(x, t) + p(t, y) - p(t, t)$.

Then p is called a partial metric on U and the pair (U, p) is called a partial metric space (in short PMS).

Remark 2.6. ([2]) It is clear that if $p(x,y)=0$, then from $(PM1)$, $(PM2)$, and $(PM3)$, $x=y$. But if $x=y$, $p(x,y)$ may not be 0.

If p is a partial metric on U, then the function $p^s\colon U\times U\to[0,+\infty)$ given by

$$p^s(x,y)=2p(x,y)-p(x,x)-p(y,y), \tag{1}$$

is a metric on U.

Example 2.7. ([3]) Let $U=\mathbb{R}^+$, where $\mathbb{R}^+=[0,+\infty)$ and $p\colon U\times U\to\mathbb{R}^+$ be given by $p(x,y)=\max\{x,y\}$ for all $x,y\in U$. Then (U,p) is a partial metric space.

Example 2.8. ([3]) Let I denote the set of all intervals $[a_1,b_1]$ for any real numbers $a_1\le b_1$. Let $p\colon I\times I\to[0,\infty)$ be a function such that

$$p\big([a_1,b_1],[a_2,b_2]\big)=\max\{b_1,b_2\}-\min\{a_1,a_2\}.$$

Then (I,p) is a partial metric space.

Example 2.9. ([8]) Let $U=\mathbb{R}$ and $p\colon U\times U\to\mathbb{R}^+$ be given by $p(x,y)=e^{\max\{x,y\}}$ for all $x,y\in U$. Then (U,p) is a partial metric space.

Each partial metric p on U generates a T_0 topology τ_ρ on U with the family of open p-balls $\{B_p(x,r):x\in U, r>0\}$ where $B_p(x,r)=\{y\in U:p(x,y)<p(x,x)+r\}$ for all $x\in U$ and $r>0$. Similarly, closed p-ball is defined as $B_p[x,r]=\{y\in U:p(x,y)\le p(x,x)+r\}$ for all $x\in U$ and $r>0$.

Definition 2.10. ([17]) Let (U,p) be a partial metric space. Then:

$(A1)$ a sequence $\{y_n\}$ converges to a point $y\in U$ if and only if $\lim_{n\to\infty}p(y,y_n)=p(y,y)$.

$(A2)$ a sequence $\{y_n\}$ in U is called a Cauchy sequence if and only if $\lim_{m,n\to\infty}$ $p(y_m,y_n)$ exists (and finite).

$(A3)$ A partial metric space (U,p) is said to be complete if every Cauchy sequence $\{y_n\}$ in U converges, with respect to τ_ρ, to a point $y\in U$, such that, $\lim_{m,n\to\infty}p(y_m,y_n)=p(y,y)$.

$(A4)$ A mapping $H\colon U\to U$ is said to be continuous at $y_0\in U$ if for every $\varepsilon>0$, there exists $\delta>0$ such that $H\big(B_p(y_0,\delta)\big)\subset B_p\big(H(y_0),\varepsilon\big)$.

Definition 2.11. ([17]) A partial metric space (U,p) is said to be complete if every Cauchy sequence $\{u_n\}$ in U converges to a point $u\in U$ with respect to τ_ρ. Furthermore,

$$\lim_{m,n\to\infty}p(u_m,u_n)=\lim_{n\to\infty}p(u_n,u)=p(u,u).$$

Definition 2.12. ([21]) (Control function) Let Φ be the set of all functions ϕ: $[0,+\infty)\to[0,+\infty)$ with the properties

(Φ_1) ϕ is continuous and non-decreasing,

(Φ_2) $\phi(t)<t$ for each $t>0$.

Obviously, if $\phi\in\Phi$, then $\phi(0)=0$ and $\phi(t)\le t$ for all $t\ge0$.

Lemma 2.13. *([2, 17, 18]) Let (U,p) be a partial metric space. Then*

$(B1)$ a sequence $\{u_n\}$ in (U,p) is a Cauchy sequence $\Leftrightarrow$ $\{u_n\}$ is a Cauchy sequence in the metric space (U,p^s),

$(B2)$ (U,p) is complete $\Leftrightarrow$ the metric space (U,p^s) is complete. Moreover,

$$\lim_{n\to\infty}p^s(u_n,u)=0\ \Leftrightarrow\ p(u,u)=\lim_{n\to\infty}p(u_n,u)=\lim_{n,m\to\infty}p(u_n,u_m).$$

Lemma 2.14. *([13]) Let (U,p) be a partial metric space. Then*

$(C1)$ if $x,y\in U$, $p(x,y)=0$, then $x=y$,

$(C2)$ if $x\neq y$, then $p(x,y)>0$.

One of the characterization of continuity of mappings in partial metric spaces was given by *Samet et al.* [24] as follows.

Lemma 2.15. *(see [24]) Let (U, p) be a partial metric space. The function $F\colon U \to U$ is continuous if given a sequence $\{u_n\}_{n\in\mathbb{N}}$ and $u \in U$ such that $p(u,u) = \lim_{n\to\infty} p(u,u_n)$, then $p(Fu,Fu) = \lim_{n\to\infty} p(Fu,Fu_n)$.*

Example 2.16. (see [24]) Let $U = \mathbb{R}^+$, where $\mathbb{R}^+ = [0,+\infty)$ endowed with the partial metric $p\colon U\times U \to \mathbb{R}^+$ defined $p(x,y) = \max\{x,y\}$ for all $x,y \in U$. Let $F\colon U \to U$ be a non-decreasing function. If F is continuous with respect to the standard metric $d(x,y) = |x-y|$ for all $x,y \in U$, F is continuous with respect to the partial metric p.

3. Main Results

The first result is the following:

Theorem 3.1. *Let $(U, p, \leq)$ be a partially ordered complete partial metric space. Suppose that the mapping $F\colon U\times U \to U$ satisfies the following conditions:*

(1)

$$p(F(x,y),F(u,v)) \leq \psi\big(\Delta_F(x,y,u,v)\big) - \varphi\big(\Delta_F(x,y,u,v)\big), \tag{2}$$

for all $x,y,u,v \in U$, where $\psi,\varphi \in \Phi$ and

$$\Delta_F(x,y,u,v) \;=\; \max\Big\{p(x,u), p(y,v), p(x,F(x,y)), p(y,F(y,x))\Big\},$$

(2) *either F is continuous or*

(3) *U has the following properties*

(*i*) *if a non-decreasing sequence $\{x_n\}$ in U converges to some point $x \in U$, then $x_n \leq x$ for all n,*

(*ii*) *if a non-increasing sequence $\{y_n\}$ in U converges to some point $y \in U$, then $y \leq y_n$ for all n.*

If there exist two elements $x_0, y_0 \in U$ with $x_0 \leq F(x_0,y_0)$ and $y_0 \geq F(y_0,x_0)$, then F has a coupled fixed point in U.

Proof. Let $x_0, y_0 \in U$ be such that $x_0 \leq F(x_0,y_0)$ and $y_0 \geq F(y_0,x_0)$. Let $x_1 = F(x_0,y_0)$ and $y_1 = F(y_0,x_0)$. Then $x_0 \leq x_1$ and $y_0 \geq y_1$. Again, let $x_2 = F(x_1,y_1)$ and $y_2 = F(y_1,x_1)$. Since F has the mixed monotone property on U, then we have $x_1 \leq x_2$ and $y_1 \geq y_2$. Repeating this process, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in U such that $x_{n+1} = F(x_n,y_n)$ and $y_{n+1} = F(y_n,x_n)$ for all $n \geq 0$ and

$$x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \ldots, \quad y_0 \geq y_1 \geq y_2 \geq \cdots \geq y_n \geq y_{n+1} \geq \ldots. \tag{3}$$

Now, using equation (2) for $(x,y) = (x_n,y_n)$ and $(u,v) = (x_{n+1},y_{n+1})$, we have

$$\begin{aligned} p(x_{n+1},x_{n+2}) &= p\big(F(x_n,y_n),F(x_{n+1},y_{n+1})\big) \\ &\leq \psi\big(\Delta_F(x_n,y_n,x_{n+1},y_{n+1})\big) - \varphi\big(\Delta_F(x_n,y_n,x_{n+1},y_{n+1})\big), \end{aligned} \tag{4}$$

where

$$\begin{aligned} \Delta_F(x_n,y_n,x_{n+1},y_{n+1}) &= \max\Big\{p(x_n,x_{n+1}), p(y_n,y_{n+1}), p(x_n,F(x_n,y_n)), \\ &\qquad p(y_n,F(y_n,x_n))\Big\} \\ &= \max\Big\{p(x_n,x_{n+1}), p(y_n,y_{n+1}), p(x_n,x_{n+1}), \\ &\qquad p(y_n,y_{n+1})\Big\} \\ &= \max\Big\{p(x_n,x_{n+1}), p(y_n,y_{n+1})\Big\}. \end{aligned}$$

Putting this value in equation (4), we get

$$p(x_{n+1}, x_{n+2}) \le \psi\big(\max\{p(x_n, x_{n+1}), p(y_n, y_{n+1})\}\big) - \varphi\big(\max\{p(x_n, x_{n+1}), p(y_n, y_{n+1})\}\big). \tag{5}$$

Similarly, we have

$$p(y_{n+1}, y_{n+2}) \le \psi\big(\max\{p(x_n, x_{n+1}), p(y_n, y_{n+1})\}\big) - \varphi\big(\max\{p(x_n, x_{n+1}), p(y_n, y_{n+1})\}\big). \tag{6}$$

From equations (5) and (6), we have

$$\max\{p(x_{n+1}, x_{n+2}), p(y_{n+1}, y_{n+2})\} \le \psi\big(\max\{p(x_n, x_{n+1}), p(y_n, y_{n+1})\}\big) - \varphi\big(\max\{p(x_n, x_{n+1}), p(y_n, y_{n+1})\}\big), \tag{7}$$

which implies

$$\max\{p(x_{n+1}, x_{n+2}), p(y_{n+1}, y_{n+2})\} < \psi\big(\max\{p(x_n, x_{n+1}), p(y_n, y_{n+1})\}\big) < \max\{p(x_n, x_{n+1}), p(y_n, y_{n+1})\}, \tag{8}$$

by the property of ψ. This means that $\{u_n := \max\{p(x_n, x_{n+1}), p(y_n, y_{n+1})\}\}$ is a decreasing sequence of positive real numbers. So, there exists an $L \ge 0$ such that

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \max\{p(x_n, x_{n+1}), p(y_n, y_{n+1})\} = L. \tag{9}$$

We shall show that $L = 0$. Suppose, on the contrary that, $L > 0$. Taking the limit as $n \to \infty$ and using the properties of ψ, φ in equation (7), we obtain

$$\begin{aligned} L &\le \lim_{n\to\infty} \psi(u_n) - \lim_{n\to\infty} \varphi(u_n) = \psi(L) - \varphi(L) \\ &< \psi(L) < L, \end{aligned}$$

which is a contradiction. Thus $L = 0$. Hence,

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0 \quad \text{and} \quad \lim_{n\to\infty} p(y_n, y_{n+1}) = 0. \tag{10}$$

Now, we show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{x_n\}$ or $\{y_n\}$ is not a Cauchy sequence, then there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}$, $\{x_{m(k)}\}$ of $\{x_n\}$ and $\{y_{n(k)}\}$, $\{y_{m(k)}\}$ of $\{y_n\}$ with $n(k) > m(k) \ge k$ such that

$$\max\{p(x_{n(k)}, x_{m(k)}), p(y_{n(k)}, y_{m(k)})\} \ge \varepsilon. \tag{11}$$

Furthermore, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \ge k$ and satisfies equation (11). Then

$$\max\{p(x_{n(k)-1}, x_{m(k)}), p(y_{n(k)-1}, y_{m(k)})\} < \varepsilon. \tag{12}$$

Using the triangle inequality and equation (12), we have

$$\begin{aligned} p(x_{n(k)}, x_{m(k)}) &\le p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)}) \\ &\quad - p(x_{n(k)-1}, x_{n(k)-1}) \\ &\le p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)}) \\ &< p(x_{n(k)}, x_{n(k)-1}) + \varepsilon, \end{aligned} \tag{13}$$

and

$$\begin{aligned} p(y_{n(k)}, y_{m(k)}) &\leq p(y_{n(k)}, y_{n(k)-1}) + p(y_{n(k)-1}, y_{m(k)}) \\ &\quad - p(y_{n(k)-1}, y_{n(k)-1}) \\ &\leq p(y_{n(k)}, y_{n(k)-1}) + p(y_{n(k)-1}, y_{m(k)}) \\ &< p(y_{n(k)}, y_{n(k)-1}) + \varepsilon. \end{aligned} \tag{14}$$

From equations (11), (13) and (14), we have

$$\begin{aligned} \varepsilon &\leq \max\left\{p(x_{n(k)}, x_{m(k)}), p(y_{n(k)}, y_{m(k)})\right\} \\ &\leq \max\left\{p(x_{n(k)}, x_{n(k)-1}), p(y_{n(k)}, y_{n(k)-1})\right\} + \varepsilon. \end{aligned} \tag{15}$$

Letting $k \to \infty$ in equation (15) and using equation (10), we get

$$\lim_{k\to\infty} \max\left\{p(x_{n(k)}, x_{m(k)}), p(y_{n(k)}, y_{m(k)})\right\} = \varepsilon. \tag{16}$$

By the triangle inequality, we have

$$\begin{aligned} p(x_{m(k)}, x_{n(k)}) &\leq p(x_{m(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{n(k)}) \\ &\quad - p(x_{n(k)-1}, x_{n(k)-1}) \\ &\leq p(x_{m(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{n(k)}), \end{aligned} \tag{17}$$

and

$$\begin{aligned} p(y_{m(k)}, y_{n(k)}) &\leq p(y_{m(k)}, y_{n(k)-1}) + p(y_{n(k)-1}, y_{n(k)}) \\ &\quad - p(y_{n(k)-1}, y_{n(k)-1}) \\ &\leq p(y_{m(k)}, y_{n(k)-1}) + p(y_{n(k)-1}, y_{n(k)}). \end{aligned} \tag{18}$$

From equations (11), (17) and (18), we have

$$\begin{aligned} \varepsilon &\leq \max\left\{p(x_{m(k)}, x_{n(k)}), p(y_{m(k)}, y_{n(k)})\right\} \\ &\leq \max\left\{p(x_{m(k)}, x_{n(k)-1}), p(y_{m(k)}, y_{n(k)-1})\right\} \\ &\quad + \max\left\{p(x_{n(k)-1}, x_{n(k)}), p(y_{n(k)-1}, y_{n(k)})\right\}. \end{aligned} \tag{19}$$

Again by the triangle inequality, we have

$$\begin{aligned} p(x_{m(k)}, x_{n(k)-1}) &\leq p(x_{m(k)}, x_{n(k)}) + p(x_{n(k)}, x_{n(k)-1}) \\ &\quad - p(x_{n(k)}, x_{n(k)}) \\ &\leq p(x_{m(k)}, x_{n(k)}) + p(x_{n(k)}, x_{n(k)-1}), \end{aligned} \tag{20}$$

and

$$\begin{aligned} p(y_{m(k)}, y_{n(k)-1}) &\leq p(y_{m(k)}, y_{n(k)}) + p(y_{n(k)}, y_{n(k)-1}) \\ &\quad - p(y_{n(k)}, y_{n(k)}) \\ &\leq p(y_{m(k)}, y_{n(k)}) + p(y_{n(k)}, y_{n(k)-1}), \end{aligned} \tag{21}$$

Therefor,

$$\max\left\{p(x_{m(k)}, x_{n(k)-1}), p(y_{m(k)}, y_{n(k)-1})\right\}$$

$$\begin{aligned}&\leq \max\left\{p(x_{m(k)},x_{n(k)}),p(y_{m(k)},y_{n(k)})\right\}\\&+\max\left\{p(x_{n(k)},x_{n(k)-1}),p(y_{n(k)},y_{n(k)-1})\right\}\\&\leq \max\left\{p(x_{n(k)},x_{n(k)-1}),p(y_{n(k)},y_{n(k)-1})\right\}+\varepsilon.\end{aligned}\tag{22}$$

From equations (19)-(22), we have

$$\begin{aligned}&\varepsilon-\max\left\{p(x_{n(k)},x_{n(k)-1}),p(y_{n(k)},y_{n(k)-1})\right\}\\&\leq \max\left\{p(x_{m(k)},x_{n(k)-1}),p(y_{m(k)},y_{n(k)-1})\right\}\\&\leq \max\left\{p(x_{n(k)},x_{n(k)-1}),p(y_{n(k)},y_{n(k)-1})\right\}+\varepsilon.\end{aligned}\tag{23}$$

Taking the limit as $k\to\infty$ in equation (23) and using equation (10), we get

$$\lim_{k\to\infty}\max\left\{p(x_{m(k)},x_{n(k)-1}),p(y_{m(k)},y_{n(k)-1})\right\}=\varepsilon.\tag{24}$$

Since $x_{m(k)}\leq x_{n(k)-1}$ and $y_{m(k)}\geq y_{n(k)-1}$, so from equation (2), we have

$$\begin{aligned}p(x_{m(k)+1},x_{n(k)})&=p(F(x_{m(k)},y_{m(k)}),F(x_{n(k)-1},y_{n(k)-1})\\&\leq \psi\big(\Delta_F(x_{m(k)},y_{m(k)},x_{n(k)-1},y_{n(k)-1})\big)\\&-\varphi\big(\Delta_F(x_{m(k)},y_{m(k)},x_{n(k)-1},y_{n(k)-1})\big),\end{aligned}\tag{25}$$

where

$$\begin{aligned}&\Delta_F(x_{m(k)},y_{m(k)},x_{n(k)-1},y_{n(k)-1})\\&=\max\Big\{p(x_{m(k)},x_{n(k)-1}),p(y_{m(k)},y_{n(k)-1}),\\&p(x_{m(k)},F(x_{m(k)},y_{m(k)})),p(y_{m(k)},F(y_{m(k)},x_{m(k)}))\Big\}\\&=\max\Big\{p(x_{m(k)},x_{n(k)-1}),p(y_{m(k)},y_{n(k)-1}),\\&p(x_{m(k)},x_{m(k)+1}),p(y_{m(k)},y_{m(k)+1})\Big\}.\end{aligned}$$

Taking the limit as $k\to\infty$ in the above and using equations (10) and (24), we get

$$\lim_{k\to\infty}\Delta_F(x_{m(k)},y_{m(k)},x_{n(k)-1},y_{n(k)-1})=\max\{\varepsilon,\varepsilon,0,0\}=\varepsilon.\tag{26}$$

From equations (25) and (26), we obtain

$$p(x_{m(k)+1},x_{n(k)})\leq\psi(\varepsilon)-\varphi(\varepsilon).\tag{27}$$

Similarly, we have

$$p(y_{m(k)+1},y_{n(k)})\leq\psi(\varepsilon)-\varphi(\varepsilon).\tag{28}$$

From equations (27) and (28), we obtain

$$\max\left\{p(x_{m(k)+1},x_{n(k)}),p(y_{m(k)+1},y_{n(k)})\right\}\leq\psi(\varepsilon)-\varphi(\varepsilon).\tag{29}$$

Using equation (24) in equation (29), we obtain

$$\varepsilon\leq\psi(\varepsilon)-\varphi(\varepsilon)<\psi(\varepsilon)<\varepsilon,\tag{30}$$

by the property of ψ, which is a contradiction. Hence, we conclude that

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0 \quad \text{and} \quad \lim_{n,m\to\infty} p(y_n, y_m) = 0. \tag{31}$$

Due to equation (1), we have

$$p^s(x_n, x_m) \le 2p(x_n, x_m) \quad \text{and} \quad p^s(y_n, y_m) \le 2p(y_n, y_m). \tag{32}$$

Letting $n, m \to \infty$ in equation (32) and using equation (31), we obtain

$$\lim_{n,m\to\infty} p^s(x_n, x_m) = 0 \quad \text{and} \quad \lim_{n,m\to\infty} p^s(y_n, y_m) = 0. \tag{33}$$

Then $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in the metric space (U, p^s). Since (U, p) is complete, it is also the case for (U, p^s). Then, there exist $f, g \in U$ such that

$$\lim_{n\to\infty} p^s(x_n, f) = 0 \quad \text{and} \quad \lim_{n\to\infty} p^s(y_n, g) = 0. \tag{34}$$

On the other hand we have

$$p^s(x_n, f) = 2p(x_n, f) - p(x_n, x_n) - p(f, f).$$

Letting $n \to \infty$ in the above equation and using equations (31) and (34), we obtain

$$\lim_{n\to\infty} p(x_n, f) = \frac{1}{2} p(f, f). \tag{35}$$

Again we have $p(f, f) \le p(f, x_n)$ for all $n \in \mathbb{N}$. On letting $n \to \infty$, we get that

$$p(f, f) \le \lim_{n\to\infty} p(f, x_n). \tag{36}$$

Using equation (35) in (36), we get that

$$\lim_{n\to\infty} p(f, x_n) = p(f, f) = 0. \tag{37}$$

By similar fashion, one can show that

$$\lim_{n\to\infty} p(g, y_n) = p(g, g) = 0. \tag{38}$$

Thus, we have

$$\lim_{n\to\infty} p(f, x_n) = p(f, f) = 0 \quad \text{and} \quad \lim_{n\to\infty} p(g, y_n) = p(g, g) = 0. \tag{39}$$

Now, we prove that $f = F(f, g)$ and $g = F(g, f)$. We shall distinguish the following cases.

Since (U, p) is a complete partial metric space, then there exist $f, g \in U$ such that $\lim_{n\to\infty} x_n = f$ and $\lim_{n\to\infty} y_n = g$.

Case I: We now show that if the assumption (2) holds, then (f, g) is a coupled fixed point of F.

As, we have

$$f = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = F(f, g),$$

and

$$g = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = F(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) = F(g, f).$$

Thus, (f, g) is a coupled fixed point of F.

Case II: Suppose now that the conditions (3)(i) and (3)(ii) of the theorem hold.
Since $x_n \to f$ and $y_n \to g$ as $n \to \infty$, then we have

$$\begin{aligned} p(F(f,g),f) &\le p(F(f,g),x_{n+1}) + p(x_{n+1},f) - p(x_{n+1},x_{n+1}) \\ &\le p(F(f,g),x_{n+1}) + p(x_{n+1},f) \\ &= p(F(f,g),F(x_n,y_n) + p(x_{n+1},f) \\ &\le \psi\big(\Delta_F(f,g,x_n,y_n)\big) - \varphi\big(\Delta_F(f,g,x_n,y_n)\big) + p(x_{n+1},f), \end{aligned} \tag{40}$$

where

$$\begin{aligned} \Delta_F(f,g,x_n,y_n) &= \max\big\{p(f,x_n),p(g,y_n),p(f,F(f,g)),p(g,F(g,f))\big\} \\ &= \max\big\{p(f,x_n),p(g,y_n),p(f,f),p(g,g)\big\}. \end{aligned} \tag{41}$$

Letting $n \to \infty$ in equation (41) and using equation (39), we obtain

$$\Delta_F(f,g,x_n,y_n) = 0. \tag{42}$$

Letting $n \to \infty$ in equation (40) and using equation (39) and property of ψ, φ, we obtain

$$p(F(f,g),f) = 0. \tag{43}$$

This implies that $F(f,g) = f$. Similarly, one can show that $F(g,f) = g$. This completes the proof. □

If we take $\psi(t) = t$ and $\varphi(t) = (1-k)t$ for all $t > 0$ where $k \in (0,1)$ in Theorem 3.1, then we have the following result.

Corollary 3.2. *Let $(U,p,\le)$ be a partially ordered complete partial metric space. Suppose that the mapping $F\colon U \times U \to U$ satisfies the following conditions:*

(1)

$$p(F(x,y),F(u,v)) \le k\,\max\Big\{p(x,u),p(y,v),p(x,F(x,y)),p(y,F(y,x))\Big\}, \tag{44}$$

for all $x,y,u,v \in U$, where $k \in (0,1)$ is a constant,

(2) *either F is continuous or*

(3) *U has the following properties*

(i) *if a non-decreasing sequence $\{x_n\}$ in U converges to some point $x \in U$, then $x_n \le x$ for all n,*

(ii) *if a non-increasing sequence $\{y_n\}$ in U converges to some point $y \in U$, then $y \le y_n$ for all n.*

If there exist two elements $x_0, y_0 \in U$ with $x_0 \le F(x_0,y_0)$ and $y_0 \ge F(y_0,x_0)$, then F has a coupled fixed point in U.

Proof. As in the proof of Theorem 3.1, suppose that $x_0, y_0 \in U$ be such that $x_0 \le F(x_0,y_0)$ and $y_0 \ge F(y_0,x_0)$. Let $x_1 = F(x_0,y_0)$ and $y_1 = F(y_0,x_0)$. Then $x_0 \le x_1$ and $y_0 \ge y_1$. Again, let $x_2 = F(x_1,y_1)$ and $y_2 = F(y_1,x_1)$. Since F has the mixed monotone property on U, then we have $x_1 \le x_2$ and $y_1 \ge y_2$. Repeating this process, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in U such that $x_{n+1} = F(x_n,y_n)$ and $y_{n+1} = F(y_n,x_n)$ for all $n \ge 0$ and

$$x_0 \le x_1 \le x_2 \le \cdots \le x_n \le x_{n+1} \le \dots, \quad y_0 \ge y_1 \ge y_2 \ge \cdots \ge y_n \ge y_{n+1} \ge \dots.$$

Now, using equation (44) for $(x,y) = (x_{n-1},y_{n-1})$ and $(u,v) = (x_n,y_n)$, we have

$$\begin{aligned} p(x_n,x_{n+1}) &= p\big(F(x_{n-1},y_{n-1}),F(x_n,y_n)\big) \\ &\le k\,\max\Big\{p(x_{n-1},x_n),p(y_{n-1},y_n),p(x_{n-1},F(x_{n-1},y_{n-1})),p(y_{n-1},F(y_{n-1},x_{n-1}))\Big\} \\ &= k\,\max\Big\{p(x_{n-1},x_n),p(y_{n-1},y_n),p(x_{n-1},x_n),p(y_{n-1},y_n)\Big\} \\ &= k\,\max\Big\{p(x_{n-1},x_n),p(y_{n-1},y_n)\Big\}. \end{aligned} \tag{45}$$

Likewise, one can show that

$$\begin{aligned} p(y_n, y_{n+1}) &= p\big(F(y_{n-1}, x_{n-1}), F(y_n, x_n)\big) \\ &\le k \max\Big\{p(x_{n-1}, x_n), p(y_{n-1}, y_n)\Big\}. \end{aligned} \tag{46}$$

Now, we have the following cases:

Case (1): If $\max\Big\{p(x_{n-1}, x_n), p(y_{n-1}, y_n)\Big\} = p(x_{n-1}, x_n)$, then from equation (45), we obtain

$$p(x_n, x_{n+1}) \le k\, p(x_{n-1}, x_n). \tag{47}$$

Case (2): If $\max\Big\{p(x_{n-1}, x_n), p(y_{n-1}, y_n)\Big\} = p(y_{n-1}, y_n)$, then from equation (46), we obtain

$$p(y_n, y_{n+1}) \le k\, p(y_{n-1}, y_n). \tag{48}$$

Case (3): If $\max\Big\{p(x_{n-1}, x_n), p(y_{n-1}, y_n)\Big\} = p(y_{n-1}, y_n)$, then from equation (45), we obtain

$$p(x_n, x_{n+1}) \le k\, p(y_{n-1}, y_n). \tag{49}$$

Case (4): If $\max\Big\{p(x_{n-1}, x_n), p(y_{n-1}, y_n)\Big\} = p(x_{n-1}, x_n)$, then from equation (46), we obtain

$$p(y_n, y_{n+1}) \le k\, p(x_{n-1}, x_n). \tag{50}$$

Adding equations (47) and (48) or (49) and (50), we obtain

$$p(x_n, x_{n+1}) + p(y_n, y_{n+1}) \le k\,[p(x_{n-1}, x_n) + p(y_{n-1}, y_n)]. \tag{51}$$

Let $\mathcal{R}_n = p(x_n, x_{n+1}) + p(y_n, y_{n+1})$, then from equation (51), we obtain

$$\mathcal{R}_n \le k\,\mathcal{R}_{n-1}. \tag{52}$$

Continuing in the same manner, we obtain

$$\mathcal{R}_n \le k\,\mathcal{R}_{n-1} \le k^2\,\mathcal{R}_{n-2} \le k^3\,\mathcal{R}_{n-3} \le \cdots \le k^n\,\mathcal{R}_0. \tag{53}$$

If $\mathcal{R}_0 = 0$, then $p(x_0, x_1) + P(y_0, y_1) = 0$. Hence $p(x_0, x_1) = 0$ and $p(y_0, y_1) = 0$. Therefore by Lemma 2.14 $(C1)$, we get $x_0 = x_1 = F(x_0, y_0)$ and $y_0 = y_1 = F(y_0, x_0)$. This means that (x_0, y_0) is a coupled fixed point F. Now, assume that $\mathcal{R}_0 > 0$. For each $n \ge m$, where $n, m \in \mathbb{N}$, by using the condition $(PM4)$, we have

$$\begin{aligned} p(x_n, x_m) &\le p(x_n, x_{n-1}) + p(x_{n-1}, x_{n-2}) + \cdots + p(x_{m+1}, x_m) \\ &\quad - p(x_{n-1}, x_{n-1}) - p(x_{n-2}, x_{n-2}) - \cdots - p(x_{m+1}, x_{m+1}) \\ &\le p(x_n, x_{n-1}) + p(x_{n-1}, x_{n-2}) + \cdots + p(x_{m+1}, x_m). \end{aligned} \tag{54}$$

Similarly, one can obtain

$$p(y_n, y_m) \le p(y_n, y_{n-1}) + p(y_{n-1}, y_{n-2}) + \cdots + p(y_{m+1}, y_m). \tag{55}$$

Thus,

$$\begin{aligned} \mathcal{R}_{nm} = p(x_n, x_m) + p(y_n, y_m) &\le \mathcal{R}_{n-1} + \mathcal{R}_{n-2} + \cdots + \mathcal{R}_m \\ &\le (k^{n-1} + k^{n-2} + \cdots + k^m)\mathcal{R}_0 \\ &\le \Big(\frac{k^m}{1-k}\Big)\mathcal{R}_0. \end{aligned} \tag{56}$$

By definition of metric p^s, we have $p^s(x,y) \leq 2p(x,y)$, therefore for any $n \geq m$

$$p^s(x_n,x_m)+p^s(y_n,y_m) \leq 2[p(x_n,x_m)+p(y_n,y_m)] = 2\mathcal{R}_{nm} \leq \Big(\frac{2k^m}{1-k}\Big)\mathcal{R}_0, \quad (57)$$

which implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (U,p^s) since $k<1$. Since the partial metric space (U,p) is complete, by Lemma 2.13 $(B2)$, the metric space (U,p^s) is also complete, so there exist $L_1, L_2 \in U$ such that

$$\lim_{n\to\infty} p^s(x_n,L_1) = \lim_{n\to\infty} p^s(y_n,L_2) = 0. \quad (58)$$

From Lemma 2.13 $(B2)$, we obtain

$$p(L_1,L_1) = \lim_{n\to\infty} p(x_n,L_1) = \lim_{n\to\infty} p(x_n,x_n), \quad (59)$$

and

$$p(L_2,L_2) = \lim_{n\to\infty} p(y_n,L_2) = \lim_{n\to\infty} p(y_n,y_n). \quad (60)$$

But, from condition $(PM2)$ and equation (53), we have

$$p(x_n,x_n) \leq p(x_n,x_{n+1}) \leq \mathcal{R}_n \leq k^n\mathcal{R}_0, \quad (61)$$

and since $k<1$, hence letting $n\to\infty$, we get $\lim_{n\to\infty} p(x_n,x_n)=0$. It follows that

$$p(L_1,L_1) = \lim_{n\to\infty} p(x_n,L_1) = \lim_{n\to\infty} p(x_n,x_n) = 0. \quad (62)$$

Similarly, we obtain

$$p(L_2,L_2) = \lim_{n\to\infty} p(y_n,L_2) = \lim_{n\to\infty} p(y_n,y_n) = 0. \quad (63)$$

Now, we show that $L_1 = F(L_1,L_2)$ and $L_2 = F(L_2,L_1)$. We shall distinguish the following cases.

Case (i): We now show that if the assumption (2) holds, then (L_1,L_2) is a coupled fixed point of F. As, we have

$$L_1 = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n,y_n) = F(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = F(L_1,L_2),$$

and

$$L_2 = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n,x_n) = F(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) = F(L_2,L_1).$$

Thus, (L_1,L_2) is a coupled fixed point of F.

Case (ii): Suppose now that the conditions $(3)(i)$ and $(3)(ii)$ of the theorem hold.

Since $x_n \to L_1$ and $y_n \to L_2$ as $n\to\infty$, then we have

$$\begin{aligned} p(F(L_1,L_2),L_1) &\leq p(F(L_1,L_2),x_{n+1}) + p(x_{n+1},L_1) - p(x_{n+1},x_{n+1}) \\ &\leq p(F(L_1,L_2),x_{n+1}) + p(x_{n+1},L_1) \\ &= p(F(L_1,L_2),F(x_n,y_n)) + p(x_{n+1},L_1) \\ &\leq k \max\Big\{p(L_1,x_n),p(L_2,y_n),p(L_1,F(L_1,L_2)),p(L_2,F(L_2,L_1))\Big\} \\ &\quad + p(x_{n+1},L_1) \\ &= k \max\Big\{p(L_1,x_n),p(L_2,y_n),p(L_1,L_1),p(L_2,L_2)\Big\} + p(x_{n+1},L_1). \end{aligned} \quad (64)$$

Letting $n\to\infty$ in equation (64) and using equations (62) and (63), we obtain

$$p(F(L_1,L_2),L_1) \leq 0 \;\Rightarrow\; p(F(L_1,L_2),L_1) = 0.$$

Hence by Lemma 2.14 $(C1)$, we get $F(L_1, L_2) = L_1$. Similarly, one can show that $F(L_2, L_1) = L_2$. This completes the proof. □

Corollary 3.3. *Let $(U, p, \leq)$ be a partially ordered complete partial metric space. Suppose that the mapping $F: U \times U \to U$ satisfies the following conditions:*

(1)

$$p(F(x,y), F(u,v)) \leq a_1\, p(x,u) + a_2\, p(y,v) + a_3\, p(x, F(x,y)) + a_4\, p(y, F(y,x)), \tag{65}$$

for all $x, y, u, v \in U$, where a_1, a_2, a_3, a_4 are nonnegative reals such that $a_1 + a_2 + a_3 + a_4 < 1$,

(2) *either F is continuous or*

(3) *U has the following properties*

(i) if a non-decreasing sequence $\{x_n\}$ in U converges to some point $x \in U$, then $x_n \leq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\}$ in U converges to some point $y \in U$, then $y \leq y_n$ for all n.

If there exist two elements $x_0, y_0 \in U$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point in U.

Proof. Follows from Corollary 3.2, by noting that

$$a_1\, p(x,u) + a_2\, p(y,v) + a_3\, p(x, F(x,y)) + a_4\, p(y, F(y,x))$$

$$\begin{aligned} &\leq (a_1 + a_2 + a_3 + a_4) \max\Big\{p(x,u), p(y,v), p(x, F(x,y)), p(y, F(y,x))\Big\} \\ &= k \max\Big\{p(x,u), p(y,v), p(x, F(x,y)), p(y, F(y,x))\Big\}, \end{aligned}$$

where $k = a_1 + a_2 + a_3 + a_4 < 1$. □

Remark 3.4. Corollary 3.2 and Corollary 3.3 extend and generalize Theorem 2.1 of [2] from complete partial metric spaces to partially ordered complete partial metric spaces.

If we take $a_1 = k, a_2 = l$ and $a_3 = a_4 = 0$ where $k, l \in (0,1)$ in Corollary 3.3, then we have the following result.

Corollary 3.5. *Let $(U, p, \leq)$ be a partially ordered complete partial metric space. Suppose that the mapping $F: U \times U \to U$ satisfies the following conditions:*

(1)

$$p(F(x,y), F(u,v)) \leq k\, p(x,u) + l\, p(y,v), \tag{66}$$

for all $x, y, u, v \in U$, where k, l are nonnegative reals such that $k + l < 1$,

(2) *either F is continuous or*

(3) *U has the following properties*

(i) if a non-decreasing sequence $\{x_n\}$ in U converges to some point $x \in U$, then $x_n \leq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\}$ in U converges to some point $y \in U$, then $y \leq y_n$ for all n.

If there exist two elements $x_0, y_0 \in U$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point in U.

Remark 3.6. Corollary 3.5 also generalizes Theorem 2.1 of [2] from complete partial metric spaces to partially ordered complete partial metric spaces.

If we take $a_1 = a_2 = k$ and $a_3 = a_4 = 0$ where $k \in (0,1)$ in Corollary 3.3, then we have the following result.

Corollary 3.7. *Let $(U, p, \leq)$ be a partially ordered complete partial metric space. Suppose that the mapping $F : U \times U \to U$ satisfies the following conditions:*

(1)

$$p(F(x, y), F(u, v)) \leq \frac{k}{2}\,[p(x, u) + p(y, v)], \tag{67}$$

for all $x, y, u, v \in U$, where $k \in (0, 1)$ is a constant,

(2) *either F is continuous or*

(3) *U has the following properties*

(*i*) *if a non-decreasing sequence $\{x_n\}$ in U converges to some point $x \in U$, then $x_n \leq x$ for all n,*

(*ii*) *if a non-increasing sequence $\{y_n\}$ in U converges to some point $y \in U$, then $y \leq y_n$ for all n.*

If there exist two elements $x_0, y_0 \in U$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then F has a coupled fixed point in U.

Remark 3.8. Corollary 3.7 extends and generalizes Theorem 2.1 and Theorem 2.2 of [5] from partially ordered complete metric spaces to partially ordered complete partial metric spaces.

Remark 3.9. Corollary 3.7 also generalizes Corollary 2.2 of [2] from complete partial metric spaces to partially ordered complete partial metric spaces.

Now, we consider some additional conditions to ensure the uniqueness of a coupled fixed point in the setting of partially ordered complete partial metric spaces. Moreover, we study appropriate conditions to ensure that for a coupled fixed point (x, y) we have $x = y$.

Notice that if $(U, \leq)$ is a partially ordered set, we endow the product space $U \times U$ with the partial order relation given by

$$(u, v) \leq (x, y) \quad \Leftrightarrow \quad x \geq u \text{ and } y \leq v.$$

We say that two pairs (f, g) and (u, v) are comparable, that is, every pair of elements has either a lower bound or an upper bound.

Theorem 3.10. *In addition to the hypotheses of Theorem 3.1, suppose that, for every $(a, b), (c, d) \in U \times U$, there exists a pair $(u, v) \in U \times U$ such that (u, v) is comparable to (a, b) and (c, d). Then F has a unique coupled fixed point. Moreover $p(a, a) = 0$.*

Proof. Suppose that (x, y) and (z, t) are coupled fixed points of F, that is, $x = F(x, y)$, $y = F(y, x)$, $z = F(z, t)$ and $t = F(t, z)$.

Let (u, v) be an element of $U \times U$ comparable to both (x, y) and (z, t). Suppose that $(x, y) \geq (u, v)$ (the proof is similar in other cases). We consider the following two cases.

Case I. If (x, y) and (z, t) are comparable, then we have

$$\begin{aligned} p(x, z) &= p\big(F(x, y), F(z, t)\big) \\ &\leq \psi\Big(\Delta_F(x, y, z, t)\Big) - \varphi\Big(\Delta_F(x, y, z, t)\Big), \end{aligned}$$

where

$$\begin{aligned} \Delta_F(x, y, z, t) &= \max\Big\{p(x, z), p(y, t), p(x, F(x, y)), p(y, F(y, x))\Big\} \\ &= \max\Big\{p(x, z), p(y, t), p(x, x), p(y, y)\Big\} \\ &= \max\Big\{p(x, z), p(y, t)\Big\}. \end{aligned}$$

Using in the above inequality and using the property of ψ, φ, we obtain

$$\begin{aligned} p(x,z) &\leq \psi\Big(\max\Big\{p(x,z),p(y,t)\Big\}\Big) \\ &\quad -\varphi\Big(\max\Big\{p(x,z),p(y,t)\Big\}\Big) \\ &< \psi\Big(\max\Big\{p(x,z),p(y,t)\Big\}\Big) \\ &< \max\Big\{p(x,z),p(y,t)\Big\}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} p(y,t) &= p\big(F(y,x),F(t,z)\big) \\ &\leq \psi\Big(\Delta_F(y,x,t,z)\Big)-\varphi\Big(\Delta_F(y,x,t,z)\Big), \end{aligned}$$

where

$$\begin{aligned} \Delta_F(y,x,t,z) &= \max\Big\{p(y,t),p(x,z),p(y,F(y,x)),p(x,F(x,y))\Big\} \\ &= \max\Big\{p(x,z),p(y,t),p(y,y),p(x,x)\Big\} \\ &= \max\Big\{p(x,z),p(y,t)\Big\}. \end{aligned}$$

Using in the above inequality and using the property of ψ, φ, we obtain

$$\begin{aligned} p(y,t) &\leq \psi\Big(\max\Big\{p(x,z),p(y,t)\Big\}\Big) \\ &\quad -\varphi\Big(\max\Big\{p(x,z),p(y,t)\Big\}\Big) \\ &< \psi\Big(\max\Big\{p(x,z),p(y,t)\Big\}\Big) \\ &< \max\Big\{p(x,z),p(y,t)\Big\}. \end{aligned}$$

It follows that

$$\max\Big\{p(x,z),p(y,t)\Big\} < \max\Big\{p(x,z),p(y,t)\Big\},$$

which is a contradiction. Hence, $\max\Big\{p(x,z),p(y,t)\Big\}=0$, that is, $p(x,z)=0$ and $p(y,t)=0$ and so $x=z$, $y=t$. This shows the uniqueness of coupled fixed point.

Case II. Suppose now that (x,y) and (z,t) are not comparable, then there exists an element $(u,v)\in U\times U$ is comparable to both (x,y) and (z,t). Now, since by iteration $F^n(x,y)=x$, $F^n(y,x)=y$, $F^n(z,t)=z$, $F^n(t,z)=t$, $F^n(u,v)=u$ and $F^n(v,u)=v$, we have

$$\begin{aligned} p\Big(\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}z\\t\end{pmatrix}\Big) &= p\Big(\begin{pmatrix}F^n(x,y)\\F^n(y,x)\end{pmatrix},\begin{pmatrix}F^n(z,t)\\F^n(t,z)\end{pmatrix}\Big) \\ &\leq p\Big(\begin{pmatrix}F^n(x,y)\\F^n(y,x)\end{pmatrix},\begin{pmatrix}F^n(u,v)\\F^n(v,u)\end{pmatrix}\Big) \\ &\quad +p\Big(\begin{pmatrix}F^n(u,v)\\F^n(v,u)\end{pmatrix},\begin{pmatrix}F^n(z,t)\\F^n(t,z)\end{pmatrix}\Big) \\ &\leq \psi\Big(\Delta_F(x,y,u,v)\Big)-\varphi\Big(\Delta_F(x,y,u,v)\Big) \\ &\quad +\psi\Big(\Delta_F(y,x,v,u)\Big)-\varphi\Big(\Delta_F(y,x,v,u)\Big) \\ &\quad +\psi\Big(\Delta_F(u,v,z,t)\Big)-\varphi\Big(\Delta_F(u,v,z,t)\Big) \\ &\quad +\psi\Big(\Delta_F(v,u,t,z)\Big)-\varphi\Big(\Delta_F(v,u,t,z)\Big). \end{aligned}$$

where

$$\begin{aligned}\Delta_F(x,y,u,v) &= \max\Big\{p(x,u), p(y,v), p(x,F(x,y)), p(y,F(y,x))\Big\}\\ &= \max\Big\{p(x,u), p(y,v), p(x,x), p(y,y)\Big\} = 0.\end{aligned}$$

Similarly,

$$\Delta_F(y,x,v,u) = 0,\ \Delta_F(u,v,z,t) = 0 \text{ and } \Delta_F(v,u,t,z) = 0.$$

Using this in the above inequality and the property of ψ, φ, we obtain

$$p\Big(\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}z\\t\end{pmatrix}\Big) = 0.$$

Thus, $x = z$ and $y = t$. Hence, the coupled fixed point of F is unique. This completes the proof. □

Theorem 3.11. *In addition to the hypotheses of Theorem 3.1, suppose that x_0, y_0 in U are comparable, then the coupled fixed point $(x,y) \in U \times U$ satisfies $x = y$. Moreover $p(a,a) = 0$.*

Proof. Recall that $x_0 \in U$ is such that $x_0 \leq F(x_0,y_0)$. Now, if $x_0 \leq y_0$, we claim that for all $n \in \mathbb{N}$, $x_n \leq y_n$. Indeed, by the mixed monotone property of F,

$$x_1 = F(x_0,y_0) \leq F(y_0,x_0) = y_1.$$

Assume that $x_n \leq y_n$ for some n. Now, consider

$$\begin{aligned}x_{n+1} &= F^{n+1}(x_0,y_0) = F\big(F^n(x_0,y_0), F^n(y_0,x_0)\big)\\ &= F(x_n,y_n) \leq F(y_n,x_n) = y_{n+1}.\end{aligned}$$

Hence, $x_n \leq y_n$ for all n. Taking the limit as $n \to \infty$, we get

$$x = \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n = y.$$

From the contractive condition (2), we get

$$\begin{aligned}p(x,y) &= p(F(x,y), F(y,x))\\ &\leq \psi\big(\Delta_F(x,y,y,x)\big) - \varphi\big(\Delta_F(x,y,y,x)\big),\end{aligned}$$

where

$$\begin{aligned}\Delta_F(x,y,y,x) &= \max\Big\{p(x,y), p(y,x), p(x,F(x,y)), p(y,F(y,x))\Big\}\\ &= \max\Big\{p(x,y), p(x,y), p(x,x), p(y,y)\Big\} = 0.\end{aligned}$$

Using this in the above inequality and using the property of ψ, φ, we get $p(x,y) = 0$ and so $x = y$.

Similarly, if $x_0 \geq y_0$, then it is possible to show $x_n \geq y_n$ for all n and that $p(x,y) = 0$. This completes the proof. □

Remark 3.12. Theorem 3.10 and Theorem 3.11 extend and generalize Theorem 2.4 and Theorem 2.6 of [5] from partially ordered complete metric spaces to partially ordered complete partial metric spaces.

Example 3.13. Let $U = [0,1]$. Then $(U,\leq)$ is a partially ordered set with a natural ordering of real numbers. Let $p\colon U \times U \to [0,1]$ be defined by $p(x,y) = |x-y|$ for all $x,y \in U$. Consider the mapping $F\colon U \times U \to [0,1]$ defined by

$$F(x,y) = \begin{cases} \frac{x^2-y^2+1}{3}, & \text{if } x \leq y,\\ \frac{1}{3}, & \text{if } x > y,\end{cases}$$

for all $x,y \in U$. Then

(1) (U,p) is a complete partial metric space since (U,p^s) is complete;

(2) F has the mixed monotone property;
(3) F is continuous;
(4) $0 \leq F(0,1)$ and $1 \geq F(1,0)$;
(5) there exist two control function ψ and φ such that

$$p\big(F(x,y), F(u,v)\big) \quad \leq \psi\Big(\Delta_F(x,y,u,v)\Big) - \varphi\Big(\Delta_F(x,y,u,v)\Big),$$

for all $x, y, u, v \in U$ with $x \leq u$ and $y \geq v$. Thus, by Theorem 3.1, F has a coupled fixed point. Moreover, $(\frac{1}{3}, \frac{1}{3})$ is the unique coupled fixed point of F.

Proof. The proofs of $(1)-(4)$ are obvious.

For any $x \leq u$ and $y \geq v$, we have

$$p(x,u) = u - x, \quad p(y,v) = y - v.$$

The proof of (5) is divided into the following cases.

Case 1. If $u \leq v$. In this case, $x \leq u \leq v \leq y$, and so

$$F(x,y) = \frac{x^2 - y^2 + 1}{3}, \quad F(u,v) = \frac{u^2 - v^2 + 1}{3}.$$

Hence, we get

$$\begin{aligned} p\big(F(x,y), F(u,v)\big) &= p\Big(\frac{x^2 - y^2 + 1}{3}, \frac{u^2 - v^2 + 1}{3}\Big) \\ &= \frac{1}{3}(u^2 - v^2 - x^2 + y^2) \leq \frac{1}{3}\max\big\{u^2 - x^2, y^2 - v^2\big\} \\ &\leq \frac{1}{3}\max\big\{u - x, y - v\big\} = \frac{1}{3}\max\big\{p(x,u), p(y,v)\big\} \\ &\leq \frac{1}{3}\max\big\{p(x,u), p(y,v), p(x, F(x,y)), p(y, F(y,x))\big\}. \end{aligned}$$

Case 2. If $u > v$. In this case, $x \leq u \leq y$, and so

$$F(x,y) = \frac{x^2 - y^2 + 1}{3}, \quad F(u,v) = \frac{1}{3}.$$

Hence, we get

$$\begin{aligned} p\big(F(x,y), F(u,v)\big) &= p\Big(\frac{x^2 - y^2 + 1}{3}, \frac{1}{3}\Big) = \frac{1}{3}(y^2 - x^2) \\ &\leq \frac{1}{3}(y^2 - x^2 + u^2 - v^2) \leq \frac{1}{3}\max\big\{u^2 - x^2, y^2 - v^2\big\} \\ &\leq \frac{1}{3}\max\big\{u - x, y - v\big\} = \frac{1}{3}\max\big\{p(x,u), p(y,v)\big\} \\ &\leq \frac{1}{3}\max\big\{p(x,u), p(y,v), p(x, F(x,y)), p(y, F(y,x))\big\}. \end{aligned}$$

Case 3. If $x > y$. In this case, $u \leq v \leq y$, and so

$$F(x,y) = \frac{1}{3}, \quad F(u,v) = \frac{u^2 - v^2 + 1}{3}.$$

Hence, we get

$$\begin{aligned} p\big(F(x,y), F(u,v)\big) &= p\Big(\frac{1}{3}, \frac{u^2 - v^2 + 1}{3}\Big) = \frac{1}{3}(u^2 - v^2) \\ &\leq \frac{1}{3}(u^2 - v^2 + y^2 - x^2) \leq \frac{1}{3}\max\big\{u^2 - x^2, y^2 - v^2\big\} \\ &\leq \frac{1}{3}\max\big\{u - x, y - v\big\} = \frac{1}{3}\max\big\{p(x,u), p(y,v)\big\} \\ &\leq \frac{1}{3}\max\big\{p(x,u), p(y,v), p(x, F(x,y)), p(y, F(y,x))\big\}. \end{aligned}$$

Thus, in all the above cases, the condition (5) is satisfied for the control functions $\psi(t) = t$ and $\varphi(t) = \frac{2}{3}t$ for all $t > 0$. Since $U = [0, 1]$ is a totally ordered set, by Theorem 3.11, $(\frac{1}{3}, \frac{1}{3})$ is the unique coupled fixed point of F.

□

4. An application to the Volterra type integral equations

The following system of Volterra type integral equations:

$$x(t) = h(t) + \int_0^T \lambda(t,s)[f(s,x(s)) + g(s,y(s))]ds,$$
$$y(t) = h(t) + \int_0^T \lambda(t,s)[f(s,y(s)) + g(s,x(s))]ds, \tag{68}$$

where the space $U = C([0,T],\mathbb{R})$ of continuous functions defined in $[0,T]$. Define $p\colon U \times U \to [0,+\infty)$ by

$$p(x,y) = \max_{t\in[0,T]} |x(t) - y(t)|, \tag{69}$$

for all $x, y \in U$. Then (U,p) is a complete partial metric space.

Let $U = C([0,T],\mathbb{R})$ with the natural partial order relation, that is, $x, y \in C([0,T],\mathbb{R})$,

$$x \le y \iff x(t) \le y(t),\ t \in [0,T].$$

Theorem 4.1. *Assume the following conditions are hold:*
(1) *the mappings $f, g\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous;*
(2) *$h\colon [0,T] \to \mathbb{R}$ is continuous;*
(3) *$\lambda\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous;*
(4) *there exists $b > 0$ and k is a nonnegative constant with $0 \le k < 1$, such that for all $x, y \in U$, $x \le y$,*

$$0 \le f(s,y) - f(s,x) \le b\frac{k}{2}(y-x),$$
$$0 \le g(s,y) - g(s,x) \le b\frac{k}{2}(y-x);$$

(5)

$$b \max_{t\in[0,T]} \int_0^T |\lambda(t,s)|ds \le 1,$$

(6) *there exist $u_0, v_0 \in U$ such that*

$$u_0(t) \ge h(t) + \int_0^T \lambda(t,s)[f(s,u_0(s)) + g(s,v_0(s))]ds,$$
$$v_0(t) \le h(t) + \int_0^T \lambda(t,s)[f(s,v_0(s)) + g(s,u_0(s))]ds.$$

Then, the system of Volterra integral equation (68) has a unique solution in $U \times U$ with $U = C([0,T],\mathbb{R})$.

Proof. Define the mapping $F\colon U \times U \to U$ by

$$F(x,y)(t) = h(t) + \int_0^T \lambda(t,s)[f(s,x(s)) + g(s,y(s))]ds, \tag{70}$$

for all $x, y \in U$ and $t \in [0,T]$.

From assumption (4), clearly F has mixed monotone property.

For $x, y, u, v \in U$ with $x \geq u$ and $y \leq v$, we have

$$\begin{aligned} p(F(x,y), F(u,v)) &= \max_{t\in[0,T]} |F(x,y)(t) - F(u,v)(t)| \\ &= \max_{t\in[0,T]} \Big| h(t) + \int_0^T \lambda(t,s)[f(s,x(s)) + g(s,y(s))]ds \\ &\quad - \Big(h(t) + \int_0^T \lambda(t,s)[f(s,u(s)) + g(s,v(s))]ds \Big) \Big| \\ &= \max_{t\in[0,T]} \Big| \int_0^T \lambda(t,s)[f(s,x(s)) - f(s,u(s)) \\ &\qquad + g(s,y(s)) - g(s,v(s))]ds \Big| \\ &\leq \max_{t\in[0,T]} \int_0^T \Big(\big| f(s,x(s)) - f(s,u(s)) \big| \\ &\qquad + \big| g(s,y(s)) - g(s,v(s)) \big| \Big) |\lambda(t,s)| ds \\ &\leq \max_{t\in[0,T]} \frac{k}{2} b \int_0^T \Big(\big| x(s) - u(s) \big| + \big| y(s) - v(s) \big| \Big) |\lambda(t,s)| ds \\ &\leq \frac{k}{2} \Big(\max_{t\in[0,T]} \big| x(t) - u(t) \big| + \max_{t\in[0,T]} \big| y(t) - v(t) \big| \Big) \times \\ &\qquad \Big(b \max_{t\in[0,T]} \int_0^T |\lambda(t,s)| ds \Big) \\ &\leq \frac{k}{2} \Big(\max_{t\in[0,T]} \big| x(t) - u(t) \big| + \max_{t\in[0,T]} \big| y(t) - v(t) \big| \Big) \\ &= \frac{k}{2} [p(x,u) + p(y,v)], \end{aligned}$$

where $0 \leq k < 1$.

So that

$$p(F(x,y), F(u,v)) \leq \frac{k}{2} [p(x,u) + p(y,v)],$$

a contractive condition in Corollary 3.7. Thus F has a coupled fixed point in U, that is, the system of Volterra type integral equation has a solution. Finally, let (x, y) be a coupled lower and upper solution of the integral equation (68), then by assumption (6) of the Theorem 4.1, we have $x \leq F(x,y) \leq F(y,x) \leq y$. Corollary 3.7 gives us that F has a coupled fixed point, say $(\alpha, \beta) \in U \times U$. Since $x \leq y$, Theorem 3.11 says us that $\alpha = \beta$ and this implies $\alpha = F(\alpha, \alpha)$ and α is the unique solution of the integral equation (68). □

The aforesaid application is illustrated by the following example.

Example 4.2. Let $U = C([0,1], \mathbb{R})$. Now consider the integral equation in U as

$$F(x,y)(t) = \frac{t^3 + 7}{4} + \int_0^1 \frac{s^2}{24(t+3)} \Big[x(s) + \frac{2}{y(s)+3} \Big] ds. \tag{71}$$

Then clearly the above equation is in the form of following equation:

$$F(x,y)(t) = h(t) + \int_0^T \lambda(t,s)[f(s,x(s)) + g(s,y(s))]ds,$$

for all $x, y \in U$ and $t \in [0,T]$, where

$$h(t) = \frac{t^3 + 7}{4},\ \lambda(t,s) = \frac{s^2}{24(t+3)},\ f(s,t) = s,\ g(s,t) = \frac{2}{s+3} \text{ and } T = 1.$$

That is, equation (71) is a special case of equation (68).

Here it is easy to verify that the functions $h(t)$, $\lambda(t,s)$, $f(s,t)$ and $g(s,t)$ are continuous. Moreover, there exist $b=9$ and $k=\frac{1}{2}$ with $0<k<1$ such that

$$0 \le f(s,y) - f(s,x) \le b\,\frac{k}{2}(y-x),$$
$$0 \le g(s,y) - g(s,x) \le b\,\frac{k}{2}(y-x);$$

for all $x, y \in \mathbb{R}$ with $y \ge x$ and $s \in [0,1]$ and

$$\begin{aligned} b \max_{t\in[0,T]} \int_0^T \lambda(t,s)ds &= 9 \max_{t\in[0,1]} \int_0^1 \frac{s^2}{24(t+3)} ds \\ &= 9 \max_{t\in[0,1]} \left\{ \frac{1}{72(t+3)} \right\} < 1. \end{aligned}$$

Thus the conditions (1)-(5) of Theorem 4.1 are satisfied.

Now consider $u_0(t)=1$ and $v_0(t)=1$. Then we have

$$h(t) + \int_0^T \lambda(t,s)[f(s,v_0(s)) + g(s,u_0(s))]ds$$

$$\begin{aligned} &= \frac{t^3+7}{4} + \int_0^1 \frac{s^2}{24(t+3)} \big[1 + \frac{2}{4}\big] ds \\ &= \frac{t^3+7}{4} + \frac{1}{48(t+3)} \ge 1. \end{aligned}$$

That is, $v_0 \le F(v_0,u_0)$. Similarly, it can be shown that $u_0 \ge F(u_0,v_0)$.

Thus all the conditions of Theorem 4.1 are satisfied. It follows that the integral equation (71) has a solution in $U \times U$ with $U = C([0,1],\mathbb{R})$.

5. Conclusion

In this work, we examine coupled fixed point solutions for contractive type conditions involving control functions in the context of partial metric spaces that are partially ordered. Furthermore, we show some implications of the established results, as well as an example to support the established results. An application of the Volterra integral equation is also provided. Our findings expand and generalize several previously published results from the literature.

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