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Some Applications of the Weak Contraction Principle

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Abstract

Recently we obtained several extensions of the Banach contraction principle. One of them is the weak contraction principle or the Rus-Hicks-Rhoades contraction principle or Theorem P. There are a large number of examples or applications of Theorem P in the literature. Recently, Romaguera stated two corollaries of Theorem P are false. Our main aim of this paper is to clarify his claim.

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2010 MSC: 06A75, 47H10, 54E35, 54H25, 58E30, 65K10.

1. Prologue

Let (X, q) be a quasi-metric space and let $T : X \to X$ be a weak contraction or a Rus-Hicks-Rhoades map; that is,

$$q(T(x), T^2(x)) \le \alpha\, q(x, T(x)) \text{ for every } x \in X,$$

where $0 \le \alpha < 1$.

In our previous works [3], [4], [5], [7], we obtained the weak contraction principle or the Rus-Hicks-Rhoades contraction principle or Theorem P and its applications to quasi-metric spaces.

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Especially, the title of [4] (and its revised version [5]) should be corrected to *"Some metric fixed point theorems hold for quasi-metric spaces.*

Recently, motivated by [4], Salvador Romaguera [8] recalled several different notions of quasi-metric completeness that appear in the literature and revise how they influenced on the fixed point theory in quasi-metric spaces. In particular, he pointed out that there are several classical fixed point theorems that cannot be directly transferred to the quasi-metric setting without extra conditions, when Park's approach is considered.

He also gave an example which shows two consequences of Theorem P (that is, Theorem 6.3 and Corollary 6.5 in [4]) are not true. Our aim is to give new proofs of them and to show his claim does not hold.

This paper is organized as follows: For Section 2 is to introduce Theorem P and two corollaries with new proofs. These two corollaries are claimed to be false by Romaguera [8] in Section 3, where we show his claim is incorrect. Finally, Section 4 is epilogue.

2. The weak contraction principle

For quasi-metric spaces, the convergence of a sequence, Cauchy sequences, completeness, orbits, and orbital continuity are routinely defined; see Jleli-Samet [1] or our previous work [4]. For example,

Definition. Let (X, q) be a quasi-metric space and $T : X \to X$ a selfmap. The *orbit* of T at $x \in X$ is the set

$$O_T(x) = \{x, T(x), \cdots, T^n(x), \cdots\}.$$

The space X is said to be *T-orbitally complete* if every Cauchy sequence in $O_T(x)$ is convergent in X. A selfmap T of X is said to be *orbitally continuous* at $x_0 \in X$ if

$$\lim_{n\to\infty} x_n = x_0 \Longrightarrow \lim_{n\to\infty} T(x_n) = T(x_0)$$

for any sequence $\{x_n\}$ of X. A selfmap T of X is said to be *orbitally continuous* if so is at any point in X.

The following in Park [3], [4], [6], [7] is called the weak contraction principle or the Rus-Hicks-Rhoades (RHR) contraction principle:

Theorem P. *Let (X, q) be a quasi-metric space and let $T : X \to X$ be an RHR map; that is,*

$$q(T(x), T^2(x)) \le \alpha\, q(x, T(x)) \text{ for every } x \in X,$$

where $0 \le \alpha < 1$.

(i) *If X is T-orbitally complete, then, for each $x \in X$, there exists a point $x_0 \in X$ such that*

$$\lim_{n\to\infty} T^n(x) = x_0$$

and

$$q(T^n(x), x_0) \le \frac{\alpha^n}{1-\alpha} q(x, T(x)), \quad n = 1, 2, \cdots,$$

$$q(T^n(x), x_0) \le \frac{\alpha}{1-\alpha} q(T^{n-1}(x), T^n(x)), \quad n = 1, 2, \cdots.$$

(ii) x_0 *is a fixed point of T, and, equivalently,*

(iii) $T : X \to X$ *is orbitally continuous at $x_0 \in X$.*

This was proved in [4] by analyzing a typical proof of the Banach contraction principle given by Art Kirk [2], Theorem 2.2.

The original Rus-Hicks-Rhoades theorem can be extended to the following consequence of the Caristi type fixed point theorem; see [4], Theorem 6.3.

Corollary P.1. *Let T be a selfmap of a complete quasi-metric space (M, q) satisfying*

$$q(T(x), T^2(x)) \leq \alpha\, q(x, T(x)) \quad \textit{for every } x \in M,$$

where $0 < \alpha < 1$. Then T has a fixed point and the statement (i) *of Theorem P holds.*

Proof. Completeness implies T-orbitally completeness. Then Corollary P.1 follows from Theorem P. □

The following consequence of Theorem P was given by Suzuki ([9], Corollary 1) for metric spaces. We obtained the following for quasi-metric spaces as [4], Corollary 6.5:

Corollary P.2. *Let (M, q) be a complete quasi-metric space and let T be a selfmap on M.*

(1) *For any function θ from $[0, 1)$ onto $[0, 1]$, there exists $r \in [0, 1)$ such that*

$$\theta(r) q(x, Tx) \leq q(x, y) \quad \textit{implies} \quad q(Tx, Ty) \leq r\, q(x, y)$$

for all $x, y \in M$. Then there exists a unique fixed point z of T. Moreover $\lim_n T^n x = z$ for all $x \in M$.

(2) *There exists $r \in (0, 1)$ such that every selfmap T on M satisfying the following has a fixed point:*

$$\frac{1}{10,000} q(x, Tx) \leq q(x, y) \quad \textit{implies} \quad q(Tx, Ty) \leq r\, q(x, y)$$

for all $x, y \in M$.

Proof. For $y = Tx$, we have $q(Tx, T^2x) \leq r\, q(x, Tx)$ for all $x \in M$. Then we can apply Corollary P.1. For the uniqueness, if we have two different fixed points x, y, then we have the contradiction:

$$q(x, y) = q(Tx, Ty) \leq r\, q(x, y). \ \square$$

3. Romaguera's Counterexample in 2024

Motivated by our [4], Romaguera [8] discussed various types of completeness of quasi-metric spaces. Moreover, he claimed the following counter-example of our Corollaries P1 and P2.

Example. Let $X = \mathbb{N} \cup \{\infty\}$ and let q be the quasi-metric on X given by $q(x, x) = 0$ for all $x \in X$, $q(n, \infty) = 0$ for all $n \in \mathbb{N}$, $q(\infty, n) = 1/n$ for all $n \in \mathbb{N}$, and $q(n, m) = 1/m$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Finally, let T be the self mapping of X defined as $T\infty = 1$, and $Tn = 2n$ for all $n \in \mathbb{N}$.

This example shows that Theorem 6.3 and Corollary 6.5 in [4] are not true.

Our Claim. *The above counterexample T does not work for Corollaries P.1 and P.2.*

Suppose $q(Tn, T^2n) \leq \alpha\, q(n, Tn)$. Then

$$q(2n, 4n) \leq \alpha\, q(n, 2n) \implies \frac{1}{4n} \leq \alpha \frac{1}{2n} \implies \frac{1}{2} \leq \alpha.$$

When $q(T\infty, T^2\infty) \leq \alpha\, q(\infty, T\infty)$. Then

$$q(1, 2) \leq \alpha\, q(\infty, 1) \implies \frac{1}{2} \leq \alpha \frac{1}{1} \implies \frac{1}{2} \leq \alpha.$$

In any case, it violates $0 < \alpha < 1$ in Corollary P.1.

Since Corollary P.2 follows from Corollary P.1, the above counter-example also works for P.2.

Consequently, the example is not a counter-example of Theorem 6.3 and Corollary 6.5 in [4].

4. Epilogue

The title of our previous works [4] and its revised version [5] should be corrected to "Some metric fixed point theorems hold for quasi-metric spaces." The original title was based on a naive view that every metric space is a quasi-metric space.

Romaguera's paper [8] is mainly concerned with various types of topologies on quasi-metric spaces. His opinion on Suzuki's works is definitely inadequate.

Some of our works aim to improve metric fixed point theory. We found that many traditional metric fixed theorems hold for quasi-metric spaces. Such examples are theorems due to Banach, Rus, Hicks-Rhoades, Suzuki, and many others.

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