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Delta-convergence and sequential delta-compactness on Banach spheres

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Abstract

In this paper, we consider notions of convergence weaker than one with a norm. We call them delta-convergence and dual-delta-convergence, and we investigate the sequential compactness. As an application, we prove a fixed point approximation theorem with the Krasnosel'skii type iterative scheme.

Keywords: Banach sphere, delta-convergence, fixed point approximation
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1. Introduction

Fixed point theory is one of the fundamental theories for applied mathematics. It has been investigated by many researchers on functional spaces such as Hilbert spaces or Banach spaces. Recently, some fixed point problems are discussed on a geodesic metric space which has curvature bounded above. Such a space is called a CAT(κ) space. When the parameter κ is positive, particularly, when it equals 1, the space has many effective properties that the unit sphere on the three dimensional Euclidean space has. In CAT(1) spaces, for instance, there is the following result:

Theorem 1.1 (Espínola–Fernández-León [2]). *Let X be a complete* CAT(1) *space such that*

$$\operatorname{diam} X < \frac{\pi}{2}.$$

Then, a nonexpansive mapping on X has a fixed point.

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In general geodesic spaces, there is not a concept of the usual weak convergence. Hence, for example, we cannot consider weak compactness of a subset. However, Lim [9] introduced delta-convergence as a notion of weak convergence on a metric space. Further, Kirk and Panyanak [5] applied it to investigate a fixed point problem on CAT(0) spaces in the manner of Hilbert spaces. In a CAT(1) space, we know the following:

Theorem 1.2 (Espínola–Fernández-León [2]). *Let X be a complete* CAT(1) *space and $\{x_n\}$ a sequence of X such that*

$$\inf_{y\in X} \limsup_{n\to\infty} d(y, x_n) < \frac{\pi}{2}.$$

Then, $\{x_n\}$ has a Δ-convergent subsequence.

Delta-convergence works well in many situations such as infinite dimensional cases, and it can be defined on a general metric space.

In this work, we deal with a notion of convergence on the unit sphere of a Banach space which is similar to the delta-convergence on CAT(κ) spaces. On Banach spheres, we cannot define a distance in the usual sense. Namely, in the case of a Hilbert space H, we define a spherical distance d on the unit sphere S_H by

$$d(x, y) = \arccos \langle x, y\rangle$$

for $x, y \in S_H$. Here, $\langle x, y\rangle$ means the inner product of x and y. From this reason, we need to devise the definition of a spherical metric. The authors [4] have defined a spherical metric on a Banach spheres using a bounded linear functional and the dual sphere, and they proved a fixed point theorem and a fixed point approximation theorem. Using this spherical metric, we investigate delta-convergence on Banach sphere in this work.

2. Preliminaries

Let X be a nonempty set and T a mapping on X. We denote the set of all fixed point of T by $\operatorname{Fix} T$, namely

$$\operatorname{Fix} T = \{x \in X \mid x = Tx\}.$$

In this paper, we always consider real linear spaces. Let E be a Banach space and E^* a dual space of E. We say that a sequence $\{x_n\}$ of E is convergent strongly if it is convergent with its norm. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. Let

$$S_E = \{x \in E \mid \|x\| = 1\}$$

be its unit sphere. The duality mapping J on E is defined by

$$Jx = \left\{x^* \in E^* \;\middle|\; \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\right\}$$

for $x \in E$. We know that Jx is a nonempty bounded closed convex subset of E^* for $x \in E$ and $J0_E = \{0_{E^*}\}$. Further, $JS_E = S_{E^*}$, where S_{E^*} is unit sphere of E^*.

Let E be a Banach space. E is said to be strictly convex if $x = y$ whenever

$$\|x + y\| = 2$$

for $x, y \in S_E$. Further, we say that E is uniformly convex if

$$\lim_{n\to\infty} \|x_n - y_n\| = 0$$

whenever

$$\lim_{n\to\infty} \|x_n + y_n\| = 2$$

for two sequences $\{x_n\}$ and $\{y_n\}$ of S_E. E is said to be smooth if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$$

exists for $x, y \in S_E$. The norm of E is said to be Fréchet differentiable if the limit is attained uniformly for $y \in S_E$ for fixed $x \in S_E$. It is said to be uniformly smooth if the limit is attained uniformly for $x, y \in S_E$.

We know the following properties of E and J:

- If E is uniformly convex, then it is reflexive and strictly convex;
- if E is uniformly smooth, then its norm is Fréchet differentiable;
- if E has the Fréchet differentiable norm, then it is smooth;
- E is smooth if and only if J is single-valued, and then

  $$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}=\langle y, Jx\rangle$$

  for $x, y \in S_E$;
- if E is smooth, then E is strictly convex if and only if J is injective;
- if E is smooth, then E is reflexive if and only if J is surjective;
- if the norm of E is Fréchet differentiable, then J is norm-to-norm continuous;
- if E is uniformly smooth, then J is uniformly norm-to-norm continuous on any bounded set;
- if E is reflexive, then E is strictly convex if and only if E^* is smooth;
- if E is reflexive, then E is smooth if and only if E^* is strictly convex;
- E is uniformly convex if and only if E^* is uniformly smooth;
- E is uniformly smooth if and only if E^* is uniformly convex.

For more details about Banach spaces, refer to [10] for instance.

In what follows, we introduce some notions about Banach spheres. For more details, see [4]. We know that

$$|\langle x, y^*\rangle| \le \|x\|\|y^*\| = 1$$

for $(x, y^*) \in S_E \times S_{E^*}$. We define a function ρ from $S_E \times S_{E^*}$ to $[0, \pi]$ by

$$\rho(x, y^*) = \arccos\langle x, y^*\rangle$$

for $(x, y^*) \in S_E \times S_{E^*}$. Then, the following hold:

- For $(x, y^*) \in S_E \times S_{E^*}$, $\rho(x, y^*) \ge 0$;
- if E is smooth and strictly convex, then $x = y$ if and only if $\rho(x, Jy) = 0$ for $x, y \in S_E$.

Here, J is the duality mapping on E. Notice that the function ρ is similar to a usual spherical metric defined on the unit sphere of Hilbert spaces, although its domain is not $S_E \times S_E$ but $S_E \times S_{E^*}$; obviously, it satisfies no symmetry unless E is a Hilbert space.

Let E be a Banach space. Then, $tx + (1-t)y \ne 0_E$ for $x, y \in S_E$ with $x \ne -y$ and $t \in [0, 1]$. Now, we can define a notion of convex combination on Banach spheres. For $x, y \in S_E$ with $x \ne -y$ and $t \in [0, 1]$, set

$$tx \oplus (1-t)y = \frac{tx + (1-t)y}{\|tx + (1-t)y\|} \in S_E.$$

Theorem 2.1 (Kimura–Sudo [4])**.** *Let E be a Banach space. Then,*

$$\cos \rho(tx \oplus (1-t)y, z^*) = \frac{t\cos\rho(x,z^*) + (1-t)\cos\rho(y,z^*)}{\|tx+(1-t)y\|}$$

for $x, y \in S_E$ with $x \neq -y$, $z^* \in S_{E^*}$ and $t \in [0,1]$.

Let E be a smooth Banach space and J the duality mapping on E. Let X be a nonempty subset of S_E. We say that X is admissible [4] if

$$\langle x, Jy \rangle > 0$$

for $x, y \in X$. Notice that X is admissible if and only if

$$\rho(x, Jy) < \frac{\pi}{2}$$

for $x, y \in X$. Moreover, we say that X has the nonnegative functional property [4] if

$$\inf_{x,y\in X} \langle x, Jy \rangle \geq 0,$$

or equivalently,

$$\sup_{x,y\in X} \rho(x, Jy) \leq \frac{\pi}{2}.$$

If X is admissible, then it has the nonnegative functional property.

The following is an example of a closed subset of a Banach sphere having the nonnegative functional property:

Example 2.2 (Kimura–Sudo [4])**.** Let $p > 1$ and let $q = p/(p-1)$. Let ℓ^p and ℓ^q be the Lebesgue real sequence spaces. Then, $(\ell^p)^* = \ell^q$ and they are smooth Banach spaces. Then, we know that

$$Jx = \left(\frac{x_k}{|x_k|} |x_k|^{p/q} \right)$$

for any $x = (x_k) \in S_{\ell^p}$, where J is the duality mapping on ℓ^p. Let

$$X = \{(x_k) \in S_{\ell^p} \mid \forall k \in \mathbb{N},\ x_k \geq 0\}.$$

Then, we know that X is closed, and it has the nonnegative functional property. It is obvious that $\langle x, Jy \rangle \geq 0$ for any $x, y \in X$. In addition, letting $x = (1, 0, 0, \dots)$ and $y = (0, 1, 0, \dots)$, we have $\langle x, Jy \rangle = 0$.

Lemma 2.3 (Kimura–Sudo [4])**.** *Let E be a smooth Banach space. Let X be a nonempty subset of S_E. Then, the following hold:*

(i) *If X has the nonnegative functional property, then $x \neq -y$ for $x, y \in X$;*

(ii) *if E is reflexive and strictly convex, and X is admissible, then JX is admissible, where J is the duality mapping on E;*

(iii) *if E is reflexive and strictly convex, and X has the nonnegative functional property, then JX has the nonnegative functional property.*

Let E be a Banach space and C a subset of S_E. We say that C is spherically convex if

$$tx \oplus (1-t)y \in C$$

for $x, y \in C$ with $x \neq -y$ and $t \in [0,1]$. The intersection of spherically convex subsets is spherically convex.

Let E be a smooth and uniformly convex Banach space. Let C be a nonempty, closed and spherically convex subset of S_E. For $x \in S_E$, we denote the value

$$\inf_{y\in C} \rho(y, Jx)$$

by $\rho(C, Jx)$, where J is the duality mapping on E. Let

$$D = \left\{ x \in S_E \;\middle|\; \rho(C, Jx) < \frac{\pi}{2} \right\}.$$

Then, for $x \in D$, there is a unique point $u_x \in C$ such that

$$\rho(u_x, Jx) = \inf_{y\in C} \rho(y, Jx) = \arccos\left(\sup_{y\in C} \langle y, Jx\rangle \right).$$

We call such a mapping $\Pi_C : x \mapsto u_x$ a spherical projection onto C. Note that $\operatorname{Fix} \Pi_C = C$. In what follows, instead of D, we denote $\operatorname{Dom} \Pi_C$.

To describe the geometrical properties of S_E and related mappings, we often need assumptions concerning the size of subsets of S_E. The nonnegative functional property is a more adequate assumption than admissibility for some cases.

Theorem 2.4 (Kimura–Sudo [4])**.** *Let E be a smooth and uniformly convex Banach space. Let C be a nonempty, closed and spherically convex subset of S_E having the nonnegative functional property, and Π_C a spherical projection onto C. Then,*

$$\cos\rho(u, J\Pi_C x)\cos\rho(\Pi_C x, Jx) \geq \cos\rho(u, Jx)$$

for $x \in \operatorname{Dom}\Pi_C$ and $u \in C$. Particularly,

$$\rho(u, J\Pi_C x) \leq \rho(u, Jx)$$

for $x \in \operatorname{Dom}\Pi_C$ and $u \in C$.

3. Cauchy sequences on Banach spheres

In this section, we consider convergence of a sequence on Banach spheres. The following fact is well known, and plays an important role for the main results in this section:

Theorem 3.1 (Ibaraki–Kimura [3])**.** *Let E be a uniformly smooth and uniformly convex Banach space. Then, there exist continuous, strictly increasing and convex functions $\underline{g}_1$ and $\overline{g}_1$ such that $\underline{g}_1(0) = \overline{g}_1(0) = 0$ and that*

$$\underline{g}_1(\|x - y\|) \leq 2 - 2\langle x, Jy\rangle \leq \overline{g}_1(\|x - y\|)$$

for $x, y \in S_E$, where J is the duality mapping on E.

Now, we obtain the following:

Theorem 3.2. *Let E be a uniformly smooth and uniformly convex Banach space, and J the duality mapping on E. Let $\{x_n\}$ be a sequence of S_E. Then, the following are equivalent:*

(i) *The sequence $\{x_n\}$ is a Cauchy sequence of E;*

(ii) *there exists a nonnegative real sequence $\{\beta_n\}$ converging to 0 such that*

$$\rho(x_m, Jx_n) \leq \beta_n$$

for $m, n \in \mathbb{N}$ with $m \geq n$;

(iii) *there exists a nonnegative real sequence* $\{\gamma_n\}$ *converging to* 0 *such that*

$$\rho(x_n, Jx_m) \leq \gamma_n$$

for $m, n \in \mathbb{N}$ *with* $m \geq n$.

Proof. We first suppose (i) and show (ii). Since $\{x_n\}$ is a Cauchy sequence of E, there exists a nonnegative real sequence $\{\alpha_n\}$ converging to 0 such that

$$\|x_m - x_n\| \leq \alpha_n$$

for $m, n \in \mathbb{N}$ with $m \geq n$. Fix $m, n \in \mathbb{N}$ with $m \geq n$. For details about the equivalent condition to the Cauchy sequences, see [11]. By Theorem 3.1, we can find a continuous, strictly increasing and convex function $\overline{g}_1$ such that $\overline{g}_1(0) = 0$, and that

$$2 - 2\langle x_m, Jx_n\rangle \leq \overline{g}_1(\|x_m - x_n\|) \leq \overline{g}_1(\alpha_n),$$

and hence

$$\cos\rho(x_m, Jx_n) \geq 1 - \frac{\overline{g}_1(\alpha_n)}{2}.$$

Therefore, we obtain

$$\rho(x_m, Jx_n) \leq \arccos\left(1 - \frac{\overline{g}_1(\alpha_n)}{2}\right).$$

It means that (ii) holds.

We next assume (ii) and deduce (iii). Then, there exists a nonnegative real sequence $\{\beta_n\}$ converging to 0 such that

$$\rho(x_m, Jx_n) \leq \beta_n$$

for $m, n \in \mathbb{N}$ with $m \geq n$. Fix $m, n \in \mathbb{N}$ with $m \geq n$. By Theorem 3.1, we can find a continuous, strictly increasing and convex function $\underline{g}_1$ such that $\underline{g}_1(0) = 0$, and that

$$\underline{g}_1(\|x_m - x_n\|) \leq 2 - 2\langle x_m, Jx_n\rangle = 2(1 - \cos\rho(x_m, Jx_n)) \leq 2(1 - \cos\beta_n),$$

and therefore

$$\|x_m - x_n\| \leq \underline{g}_1^{-1}(2(1 - \cos\beta_n)).$$

Now, again By Theorem 3.1, we can find a continuous, strictly increasing and convex function $\overline{g}_1$ such that $\overline{g}_1(0) = 0$, and that

$$2 - 2\langle x_n, Jx_m\rangle \leq \overline{g}_1(\|x_n - x_m\|) \leq \overline{g}_1(\underline{g}_1^{-1}(2(1 - \cos\beta_n))),$$

and hence

$$\cos\rho(x_n, Jx_m) \geq 1 - \frac{\overline{g}_1(\underline{g}_1^{-1}(2(1 - \cos\beta_n)))}{2}.$$

It implies that

$$\rho(x_n, Jx_m) \leq \arccos\left(1 - \frac{\overline{g}_1(\underline{g}_1^{-1}(2(1 - \cos\beta_n)))}{2}\right),$$

which means that (iii) holds.

We finally suppose (iii) and show that (i) holds. Then, there exists a nonnegative real sequence $\{\gamma_n\}$ converging to 0 such that

$$\rho(x_n, Jx_m) \leq \gamma_n$$

for $m, n \in \mathbb{N}$ with $m \geq n$. Fix $m, n \in \mathbb{N}$ with $m \geq n$. By Theorem 3.1, we can find a continuous, strictly increasing and convex function $\underline{g}_1$ such that $\underline{g}_1(0) = 0$, and that

$$\underline{g}_1(\|x_n - x_m\|) \leq 2 - 2\langle x_n, Jx_m\rangle = 2 - 2\cos\rho(x_n, Jx_m) \leq 2 - 2\cos\gamma_n$$

and therefore

$$\|x_n - x_m\| \leq \underline{g}_1^{-1}(2 - 2\cos\gamma_n).$$

Hence, (iii) implies (i). Consequently, (i), (ii) and (iii) are equivalent to each other. □

In the same fashions of the previous theorem, we also obtain the following:

Theorem 3.3. *Let E be a uniformly smooth and uniformly convex Banach space, and J the duality mapping on E. Let $\{x_n\}$ be a sequence of S_E and $x_0 \in S_E$. Then, the following are equivalent:*

(i) *The sequence $\{x_n\}$ converges strongly to x_0;*

(ii) *a real sequence $\{\rho(x_n, Jx_0)\}$ converges to 0;*

(iii) *a real sequence $\{\rho(x_0, Jx_n)\}$ converges to 0.*

Theorem 3.4. *Let E be a uniformly smooth and uniformly convex Banach space, and J the duality mapping on E. Let $\{x_n\}$ and $\{y_n\}$ be sequences of S_E. Then, the following are equivalent:*

(i) *A real sequence $\{\|x_n - y_n\|\}$ converges to 0;*

(ii) *a real sequence $\{\|Jx_n - Jy_n\|\}$ converges to 0;*

(iii) *a real sequence $\{\rho(x_n, Jy_n)\}$ converges to 0;*

(iv) *a real sequence $\{\rho(y_n, Jx_n)\}$ converges to 0.*

4. Asymptotic centres of a sequence on a Banach sphere

Let E be a smooth, reflexive and strictly convex Banach space. Let $\{x_n\}$ be a sequence of S_E. We call $z \in S_E$ an asymptotic centre of $\{x_n\}$ if

$$\liminf_{n\to\infty} \langle z, Jx_n\rangle = \sup_{y\in S_E} \liminf_{n\to\infty} \langle y, Jx_n\rangle,$$

and call $z^* \in S_E$ a dual-asymptotic centre of $\{x_n\}$ if

$$\liminf_{n\to\infty} \langle x_n, z^*\rangle = \sup_{y^*\in S_{E^*}} \liminf_{n\to\infty} \langle x_n, y^*\rangle.$$

We denote the set of all asymptotic centres of $\{x_n\}$ by

$$\mathrm{AC}(\{x_n\}) = \left\{ x \in S_E \;\middle|\; \liminf_{n\to\infty} \langle x, Jx_n\rangle = \sup_{y\in S_E} \liminf_{n\to\infty} \langle y, Jx_n\rangle \right\},$$

and denote the set of all dual-asymptotic centres of $\{x_n\}$ by

$$\mathrm{AC}^*(\{x_n\}) = \left\{ x^* \in S_{E^*} \;\middle|\; \liminf_{n\to\infty} \langle x_n, x^*\rangle = \sup_{y^*\in S_{E^*}} \liminf_{n\to\infty} \langle x_n, y^*\rangle \right\}.$$

Example 4.1. Let ℓ^p and ℓ^q be the Lebesgue real sequence spaces such as Example 2.2. Let $\{e_n\}$ be a sequence of S_{ℓ^p} such as

$$e_n = (0, 0, \dots, 0, 1, 0, \dots),$$

where the component 1 appears at the nth coordinate. It is obvious that $\{e_n\}$ is not convergent strongly. Then, for $n \in \mathbb{N}$,

$$Je_n = e_n \in \ell^q.$$

Let $y \in S_{\ell^p}$ be an arbitrary point. Then, there exists $\{a_n\}$ such that

$$y = \sum_{i=1}^{\infty} a_i e_i$$

and that $\sum_{i=1}^{\infty} |a_i|^p = 1$. Therefore, we have

$$\liminf_{n\to\infty} \langle y, Je_n\rangle = \liminf_{n\to\infty} a_n \langle e_n, Je_n\rangle = \liminf_{n\to\infty} a_n = 0,$$

and hence $\mathrm{AC}(\{e_n\}) = S_{\ell^p}$. We next define a sequence $\{x_n\}$ of S_{ℓ^p} by

$$x_n = \frac{1}{2}e_1 \oplus \frac{1}{2}e_{n+1} = \frac{e_1 + e_{n+1}}{\|e_1 + e_{n+1}\|} = \frac{e_1 + e_{n+1}}{\sqrt[p]{2}}$$

for $n \in \mathbb{N}$. It is obvious that $\{x_n\}$ is not convergent strongly. Let $y^* \in S_{\ell^q}$ be an arbitrary point. Then, there exists $\{a_n^*\}$ such that

$$y^* = \sum_{i=1}^{\infty} a_i^* e_i$$

and that $\sum_{i=1}^{\infty} |a_i^*|^q = 1$. Therefore,

$$\liminf_{n\to\infty} \langle x_n, y^*\rangle = \liminf_{n\to\infty} \frac{\langle e_1 + e_{n+1}, y^*\rangle}{\sqrt[p]{2}} = \frac{a_1}{\sqrt[p]{2}} + \liminf_{n\to\infty} \frac{a_{n+1}^*}{\sqrt[p]{2}} = \frac{a_1}{\sqrt[p]{2}},$$

and thus

$$\liminf_{n\to\infty} \langle x_n, e_1\rangle = \frac{1}{\sqrt[p]{2}} = \sup_{y^*\in S_{\ell^q}} \liminf_{n\to\infty} \langle x_n, y^*\rangle.$$

It means that $\mathrm{AC}^*(\{x_n\}) = \{e_1\}$.

Now, we get the following lemma:

Lemma 4.2. *Let E be a smooth, reflexive and strictly convex Banach space. Let $\{x_n\}$ be a sequence of S_E. Then,*

$$\mathrm{AC}(\{x_n\}) = \mathrm{AC}^*(\{Jx_n\})$$

and

$$\mathrm{AC}^*(\{x_n\}) = \mathrm{AC}(\{Jx_n\}).$$

Proof. Take $x \in \mathrm{AC}(\{x_n\})$. Then,

$$\liminf_{n\to\infty} \langle Jx_n, x\rangle = \liminf_{n\to\infty} \langle x, Jx_n\rangle = \sup_{y\in S_E} \liminf_{n\to\infty} \langle y, Jx_n\rangle = \sup_{y\in S_E} \liminf_{n\to\infty} \langle Jx_n, y\rangle.$$

It implies that

$$x \in \mathrm{AC}^*(\{Jx_n\}) = \left\{ x^{**} \in S_E \;\middle|\; \liminf_{n\to\infty} \langle Jx_n, x^{**}\rangle = \sup_{y^{**}\in S_E} \liminf_{n\to\infty} \langle Jx_n, y^{**}\rangle \right\},$$

and hence $\mathrm{AC}(\{x_n\}) \subset \mathrm{AC}^*(\{Jx_n\})$. Conversely, let $x^{**} \in \mathrm{AC}^*(\{Jx_n\})$. Then,

$$\liminf_{n\to\infty} \langle x^{**}, Jx_n\rangle = \liminf_{n\to\infty} \langle Jx_n, x^{**}\rangle = \sup_{y^{**}\in S_E} \liminf_{n\to\infty} \langle Jx_n, y^{**}\rangle = \sup_{y^{**}\in S_E} \liminf_{n\to\infty} \langle y^{**}, Jx_n\rangle,$$

which implies that

$$x^{**} \in \mathrm{AC}(\{x_n\}) = \left\{ x \in S_E \;\middle|\; \liminf_{n\to\infty} \langle x, Jx_n\rangle = \sup_{y\in S_E} \liminf_{n\to\infty} \langle y, Jx_n\rangle \right\}$$

and thus $\mathrm{AC}^*(\{Jx_n\}) \subset \mathrm{AC}(\{x_n\})$. Therefore,

$$\mathrm{AC}(\{x_n\}) = \mathrm{AC}^*(\{Jx_n\}).$$

We also obtain

$$\mathrm{AC}^*(\{x_n\}) = \mathrm{AC}(\{Jx_n\}).$$

It completes the proof. □

We next prove the following:

Lemma 4.3. *Let E be a smooth and uniformly convex Banach space. Let $\{x_n\}$ be a sequence of S_E such that*

$$\sup_{y\in S_E} \liminf_{n\to\infty} \langle y, Jx_n\rangle > 0.$$

Then, $\{x_n\}$ has a unique asymptotic centre.

Proof. We define a real valued function g on S_E by

$$g(z) = \liminf_{n\to\infty} \langle z, Jx_n\rangle$$

for $z \in S_E$. We remark that $\mathrm{AC}(\{x_n\})$ coincides with the set of maximisers of g. Set

$$M = \sup_{y\in S_E} \liminf_{n\to\infty} \langle y, Jx_n\rangle \in \,]0, 1].$$

Then, there exists a sequence $\{z_i\}$ of S_E such that

$$M \geq g(z_i) \geq M - \frac{1}{i}.$$

Then, $g(z_i) \to M$ as $i \to \infty$. We first show that $\{z_i\}$ is a Cauchy sequence of E. We remark that for large $j_0 \in \mathbb{N}$,

$$M - \frac{1}{j_0} > 0.$$

Fix $i, j \in \mathbb{N}$ with $i \geq j \geq j_0$. We show that $z_i \neq -z_j$ by contradiction. Assume that $z_i = -z_j$. Then, we have

$$0 < \liminf_{n\to\infty} \langle z_i, Jx_n\rangle = \liminf_{n\to\infty} \langle -z_j, Jx_n\rangle = -\limsup_{n\to\infty} \langle z_j, Jx_n\rangle < 0$$

and thus this is a contradiction. Hence, since $z_i \neq -z_j$, we obtain

$$\begin{aligned} M &\geq g\left(\frac{1}{2}z_i \oplus \frac{1}{2}z_j\right) = \liminf_{n\to\infty} \left\langle \frac{1}{2}z_i \oplus \frac{1}{2}z_j, Jx_n \right\rangle = \liminf_{n\to\infty} \frac{\langle z_i + z_j, Jx_n\rangle}{\|z_i + z_j\|} \\ &\geq \frac{\liminf_{n\to\infty} \langle z_i, Jx_n\rangle + \liminf_{n\to\infty} \langle z_j, Jx_n\rangle}{\|z_i + z_j\|} = \frac{g(z_i) + g(z_j)}{\|z_i + z_j\|} \\ &\geq \frac{M - i^{-1} + M - j^{-1}}{\|z_i + z_j\|} \geq \frac{2(M - j^{-1})}{\|z_i + z_j\|}. \end{aligned}$$

Therefore,

$$\|z_i + z_j\| \geq \frac{2(M - j^{-1})}{M}.$$

Now, we assume that $\{z_i\}$ is not a Cauchy sequence. Then, there is $\varepsilon > 0$ such that for $k \in \mathbb{N}$ with $k \geq j_0$, there exist $i_k, j_k \in \mathbb{N}$ with $i_k \geq j_k \geq k$ such that

$$\|z_{i_k} - z_{j_k}\| > \varepsilon.$$

In this way, we can take two subsequences $\{z_{i_k}\}$ and $\{z_{j_k}\}$ of $\{z_i\}$. However, since

$$\|z_{i_k} + z_{j_k}\| \geq \frac{2(M - j_k^{-1})}{M}$$

for $k \in \mathbb{N}$ with $k \geq j_0$, we have

$$\lim_{k\to\infty} \|z_{i_k} + z_{j_k}\| = 2.$$

From the uniformly convexity of E,

$$\lim_{k\to\infty} \|z_{i_k} - z_{j_k}\| = 0.$$

This is a contradiction. Hence, $\{z_i\}$ is a Cauchy sequence of E. Let $z_0 \in E$ be its strong limit. We notice that $z_0 \in S_E$ since S_E is closed. From the continuity of g, we obtain $g(z_0) = M$ and hence z_0 is a maximiser of g.

Let $z_0, z_0' \in X$ be maximisers of g. We know that $z_0 \neq -z_0'$. Then,

$$M \geq g\left(\frac{1}{2}z_0 \oplus \frac{1}{2}z_0'\right) = \liminf_{n\to\infty} \left\langle \frac{1}{2}z_0 \oplus \frac{1}{2}z_0', Jx_n \right\rangle = \liminf_{n\to\infty} \frac{\langle z_0 + z_0', Jx_n\rangle}{\|z_0 + z_0'\|} \geq \frac{g(z_0) + g(z_0')}{\|z_0 + z_0'\|} = \frac{2M}{\|z_0 + z_0'\|}$$

and therefore $\|z_0 + z_0'\| = 2$. Since E is strictly convex, we have $z_0 = z_0'$. It means that g has a unique maximiser. □

Lemma 4.4. *Let E be a uniformly smooth and strictly convex Banach space. Let $\{x_n\}$ be a sequence of S_E such that*

$$\sup_{y^* \in S_{E^*}} \liminf_{n\to\infty} \langle x_n, y^* \rangle > 0.$$

Then, $\{x_n\}$ has a unique dual-asymptotic centre.

Proof. We know that $\mathrm{AC}^*(\{x_n\}) = \mathrm{AC}(\{Jx_n\})$ and the dual E^* of E is smooth and uniformly convex. Let J^* is the duality mapping on E^*. Since

$$\sup_{y^* \in S_{E^*}} \liminf_{n\to\infty} \langle y^*, J^* Jx_n \rangle = \sup_{y^* \in S_{E^*}} \liminf_{n\to\infty} \langle x_n, y^* \rangle > 0,$$

we obtain the desired result from Lemma 4.3. □

Theorem 4.5. *Let E be a uniformly smooth and uniformly convex Banach space. Let C be a nonempty, closed and spherically convex subset of S_E having the nonnegative functional property. Let $\{x_n\}$ be a sequence of C such that*

$$\sup_{y^* \in S_{E^*}} \liminf_{n\to\infty} \langle x_n, y^* \rangle > 0.$$

Then, $J^{-1}\,\mathrm{AC}^*(\{x_n\})$ is included in C.

Proof. Let $\{x_0\} = J^{-1}\,\mathrm{AC}^*(\{x_n\})$. Let Π_C be a spherical projection onto C. Then, since

$$\sup_{y \in C} \langle y, Jx_0 \rangle \geq \liminf_{n\to\infty} \langle x_n, Jx_0 \rangle = \sup_{y^* \in S_{E^*}} \liminf_{n\to\infty} \langle x_n, y^* \rangle > 0,$$

we have

$$\rho(C, Jx_0) < \frac{\pi}{2}$$

and thus $x_0 \in \mathrm{Dom}\,\Pi_C$. From the property of the spherical projection, since

$$\rho(x_n, J\Pi_C x_0) \leq \rho(x_n, Jx_0)$$

for $n \in \mathbb{N}$, we have

$$\limsup_{n\to\infty} \rho(x_n, J\Pi_C x_0) \le \limsup_{n\to\infty} \rho(x_n, Jx_0)$$

and therefore

$$\liminf_{n\to\infty} \langle x_n, J\Pi_C x_0\rangle \ge \liminf_{n\to\infty} \langle x_n, Jx_0\rangle.$$

Since $\mathrm{AC}^*(\{x_n\})$ consists of one point, we get $x_0 = \Pi_C x_0 \in C$. □

We obtain the following in the same fashion of the previous theorem:

Theorem 4.6. *Let E be a uniformly smooth and uniformly convex Banach space. Let C be a nonempty, closed subset of S_E such that it has the nonnegative functional property and JC is spherically convex. Let $\{x_n\}$ be a sequence of C such that*

$$\sup_{y\in S_E} \liminf_{n\to\infty} \langle y, Jx_n\rangle > 0.$$

Then, $\mathrm{AC}(\{x_n\})$ *is included in C.*

5. Delta-convergence on a Banach sphere

Let E be a smooth, reflexive and strictly convex Banach space. Let $\{x_n\}$ be a sequence of S_E and $x_0 \in S_E$. We say that $\{x_n\}$ delta-converges to a delta-limit x_0 if $\{x_0\} = \mathrm{AC}(\{x_{n_i}\})$ for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$. Moreover, we say that $\{x_n\}$ dual-delta-converges to a dual-delta-limit x_0 if $\{x_0\} = J^* \mathrm{AC}^*(\{x_{n_i}\})$ for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

The original definition of delta-convergence was defined by Lim [9] in the setting of metric spaces. Our definition above differs slightly from the original definition since the asymptotic centre of a sequence is defined by the dual pairing instead of the metric.

Theorem 5.1. *Let E be a smooth and uniformly convex Banach space. Let $\{x_n\}$ be a sequence of S_E converging strongly to $x_0 \in S_E$. Then, $\{x_n\}$ delta-converges to x_0.*

Proof. Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily. Then,

$$\liminf_{i\to\infty} \langle x_0, Jx_{n_i}\rangle = \langle x_0, Jx_0\rangle = 1.$$

Thus, since

$$\liminf_{i\to\infty} \langle y, Jx_{n_i}\rangle \le 1 = \liminf_{i\to\infty} \langle x_0, Jx_{n_i}\rangle$$

for $y \in S_E$, we have

$$\sup_{y\in S_E} \liminf_{i\to\infty} \langle y, Jx_{n_i}\rangle = \liminf_{i\to\infty} \langle x_0, Jx_{n_i}\rangle = 1.$$

Thus, $\{x_0\} = \mathrm{AC}(\{x_{n_i}\})$, which implies that $\{x_n\}$ delta-converges to x_0. □

Theorem 5.2. *Let E be a uniformly smooth and strictly convex Banach space. Let $\{x_n\}$ be a sequence of S_E converging strongly to $x_0 \in S_E$. Then, $\{x_n\}$ dual-delta-converges to x_0.*

Proof. Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily. Then,

$$\liminf_{i\to\infty} \langle x_{n_i}, Jx_0\rangle = \langle x_0, Jx_0\rangle = 1.$$

Thus, since

$$\liminf_{i\to\infty} \langle x_{n_i}, y^*\rangle \le 1 = \liminf_{i\to\infty} \langle x_{n_i}, Jx_0\rangle$$

for $y^* \in S_{E^*}$, we have

$$\sup_{y^*\in S_{E^*}} \liminf_{i\to\infty} \langle x_{n_i}, y^*\rangle = \liminf_{i\to\infty} \langle x_{n_i}, Jx_0\rangle = 1.$$

Thus, $\{Jx_0\} = \mathrm{AC}^*(\{x_{n_i}\})$, which implies that $\{x_n\}$ dual-delta-converges to x_0. □

Consequently, both notions of delta-convergence are weaker than strong convergence. We next show that the following duality theorem:

Theorem 5.3. *Let E be a smooth, reflexive and strictly convex Banach space. Let $\{x_n\}$ be a sequence of S_E and $x_0 \in S_E$. Then, $\{x_n\}$ delta-converges to x_0 if and only if $\{Jx_n\}$ dual-delta-converges to Jx_0. Further, $\{x_n\}$ dual-delta-converges to x_0 if and only if $\{Jx_n\}$ delta-converges to Jx_0.*

Proof. Suppose that $\{x_n\}$ delta-converges to x_0. For any subsequence $\{Jx_{n_i}\}$ of $\{Jx_n\}$, since

$$\{x_0\} = \mathrm{AC}(\{x_{n_i}\}) = \mathrm{AC}^*(\{Jx_{n_i}\}),$$

we have $\{Jx_0\} = J\,\mathrm{AC}^*(\{Jx_{n_i}\})$. It means that $\{Jx_n\}$ dual-delta-converges to Jx_0. Suppose that $\{Jx_n\}$ dual-delta-converges to Jx_0. For any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, since

$$\{Jx_0\} = J\,\mathrm{AC}^*(\{Jx_{n_i}\}) = J\,\mathrm{AC}(\{x_{n_i}\}),$$

we have $\{x_0\} = \mathrm{AC}(\{x_{n_i}\})$. It means that $\{x_n\}$ delta-converges to x_0. Consequently, $\{x_n\}$ delta-converges to x_0 if and only if $\{Jx_n\}$ dual-delta-converges to Jx_0. We can also prove that $\{x_n\}$ dual-delta-converges to x_0 if and only if $\{Jx_n\}$ delta-converges to Jx_0. □

Let E be a smooth, reflexive and strictly convex Banach space. Let $\{x_n\}$ be a sequence of S_E. We say that $\{x_n\}$ is spherically bounded if

$$\sup_{x\in S_E} \liminf_{n\to\infty} \langle x, Jx_n\rangle > 0,$$

or equivalently

$$\inf_{x\in S_E} \limsup_{n\to\infty} \rho(x, Jx_n) < \frac{\pi}{2}.$$

We further say that $\{x_n\}$ is dual-spherically bounded if

$$\sup_{x^*\in S_{E^*}} \liminf_{n\to\infty} \langle x_n, x^*\rangle > 0,$$

or equivalently

$$\inf_{x^*\in S_{E^*}} \limsup_{n\to\infty} \rho(x_n, x^*) < \frac{\pi}{2}.$$

We next show that sequential delta-compactness of a spherically bounded sequence. To prove this, we use the similar fashion to [1, Proposition 3.1.2].

Theorem 5.4. *Let E be a smooth and uniformly convex Banach space. Let $\{x_n\}$ be a spherically bounded sequence of S_E. Then, $\{x_n\}$ has a delta-convergent subsequence.*

Proof. Let J be the duality mapping on E. Set

$$r_1 = \inf_{\{u_n\}\subset\{x_n\}} \left(\inf_{x\in S_E} \limsup_{n\to\infty} \rho(x, Ju_n) \right).$$

We take a subsequence $\{x_n^1\}$ of $\{x_n\}$ as

$$\inf_{x\in S_E} \limsup_{n\to\infty} \rho(x, Jx_n^1) \le r_1 + \frac{1}{1}.$$

Further, set

$$r_2 = \inf_{\{u_n^1\}\subset\{x_n^1\}} \left(\inf_{x\in S_E} \limsup_{n\to\infty} \rho(x, Ju_n^1) \right).$$

Then, we can take a subsequence $\{x_n^2\}$ of $\{x_n^1\}$ as

$$\inf_{x\in S_E}\limsup_{n\to\infty}\rho(x, Jx_n^2) \le r_2 + \frac{1}{2}.$$

In this way, for a sequence $\{x_n^k\}$, let

$$r_{k+1} = \inf_{\{u_n^k\}\subset\{x_n^k\}}\left(\inf_{x\in S_E}\limsup_{n\to\infty}\rho(x, Ju_n^k)\right)$$

and take a subsequence $\{x_n^{k+1}\}$ of $\{x_n^k\}$ as

$$\inf_{x\in S_E}\limsup_{n\to\infty}\rho(x, Jx_n^{k+1}) \le r_{k+1} + \frac{1}{k+1}.$$

Then, $\{r_k\}$ is increasing and bounded above. Therefore, $\{r_k\}$ converges to some real number r. Now, we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ as $x_{n_k} = x_k^k$ for $k \in \mathbb{N}$. Fix $i \in \mathbb{N}$ arbitrarily. For sufficiently large $k \in \mathbb{N}$, $\{x_{n_k}\}$ is a subsequence of $\{x_n^i\}$. Thus,

$$r_{i+1} \le \inf_{x\in S_E}\limsup_{k\to\infty}\rho(x, Jx_{n_k}).$$

Moreover, since

$$\inf_{x\in S_E}\limsup_{k\to\infty}\rho(x, Jx_{n_k}) \le r_{i+1} + \frac{1}{i+1},$$

letting $i \to \infty$, we obtain

$$\inf_{x\in S_E}\limsup_{k\to\infty}\rho(x, Jx_{n_k}) = r$$

and thus

$$r = \inf_{x\in S_E}\limsup_{k\to\infty}\rho(x, Jx_{n_k}) \le \inf_{x\in S_E}\limsup_{n\to\infty}\rho(x, Jx_n) < \frac{\pi}{2}.$$

In the same way, for any subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$, we get

$$\inf_{x\in S_E}\limsup_{l\to\infty}\rho(x, Jx_{n_{k_l}}) = r < \frac{\pi}{2}.$$

Let $\{x_0\} = \mathrm{AC}(\{x_{n_k}\})$. Then, for any subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$, we have

$$\limsup_{l\to\infty}\rho(x_0, Jx_{n_{k_l}}) \le \limsup_{k\to\infty}\rho(x_0, Jx_{n_k}) = \inf_{x\in S_E}\limsup_{k\to\infty}\rho(x, Jx_{n_k}) = r = \inf_{x\in S_E}\limsup_{l\to\infty}\rho(x, Jx_{n_{k_l}})$$

and thus $\{x_0\} = \mathrm{AC}(\{x_{n_{k_l}}\})$. It means that $\{x_{n_k}\}$ delta-converges to $x_0 \in S_E$. □

Theorem 5.5. *Let E be a uniformly smooth and strictly convex Banach space. Let $\{x_n\}$ be a dual-spherically bounded sequence of S_E. Then, $\{x_n\}$ has a dual-delta-convergent subsequence.*

Proof. We show that $\{Jx_n\}$ has a delta-convergent subsequence. Since $\{x_n\}$ is dual-spherically bounded,

$$\sup_{x^*\in S_{E^*}}\liminf_{n\to\infty}\langle x^*, J^*Jx_n\rangle = \sup_{x^*\in S_{E^*}}\liminf_{n\to\infty}\langle x_n, x^*\rangle > 0$$

and thus $\{Jx_n\}$ is spherically bounded. Note that E^* is smooth and uniformly convex. From Theorem 5.4, $\{Jx_n\}$ has a delta-convergent subsequence $\{Jx_{n_i}\}$. Let $x_0^* \in S_{E^*}$ be its delta-limit. Then, $\{x_{n_i}\}$ dual-delta-converges to J^*x^*. It completes the proof. □

At the end of this section, we obtain the following:

Lemma 5.6. *Let E be a smooth and uniformly convex Banach space. Let $\{x_n\}$ be a spherically bounded sequence of S_E. Then, $\{x_n\}$ delta-converges to $x_0 \in S_E$ if and only if x_0 is a delta-limit of every delta-convergent subsequence of $\{x_n\}$.*

Proof. Since the 'only if' part is obvious, we prove the 'if' part. Suppose that x_0 is a delta-limit of every delta-converging subsequence of $\{x_n\}$. Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily and let $\{z\} = \mathrm{AC}(\{x_{n_i}\})$. Here, we take a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that

$$\lim_{j\to\infty} \left\langle x_0, x_{n_{i_j}} \right\rangle = \liminf_{i\to\infty} \langle x_0, x_{n_i} \rangle$$

and that $\{x_{n_{i_j}}\}$ is a delta-convergent sequence. From the assumption, x_0 is its delta-limit. Then, we obtain

$$\liminf_{i\to\infty} \langle x_0, x_{n_i} \rangle = \lim_{j\to\infty} \left\langle x_0, x_{n_{i_j}} \right\rangle \geq \liminf_{j\to\infty} \left\langle z, x_{n_{i_j}} \right\rangle \geq \liminf_{i\to\infty} \langle z, x_{n_i} \rangle.$$

Since z is a unique asymptotic centre of $\{x_{n_i}\}$, we obtain $z = x_0$ and thus $\{x_n\}$ delta-converges to x_0. □

Lemma 5.7. *Let E be a uniformly smooth and strictly convex Banach space. Let $\{x_n\}$ be a dual-spherically bounded sequence of S_E. Then, $\{x_n\}$ dual-delta-converges to $x_0 \in S_E$ if and only if x_0 is a dual-delta-limit of every dual-delta-convergent subsequence of $\{x_n\}$.*

6. Applications to fixed point approximation

In this section, we prove a delta-convergence theorem for a spherically nonspreading mapping on a Banach sphere.

Let E be a smooth Banach space and X nonempty subset of S_E having the nonnegative functional property. We call a mapping T from X into itself a spherically nonspreading mapping if

$$\cos\rho(Tx, JTy) + \cos\rho(Ty, JTx) \geq \cos\rho(Tx, Jy) + \cos\rho(Ty, Jx)$$

for $x, y \in X$, where J is the duality mapping on E. A notion of nonspreadingness is first introduced by Kohsaka and Takahashi [7] in Banach spaces, and generalised to geodesic space in [6]. For a nonempty closed spherically convex subset C of X, the spherical projection Π_C from $X \cap \mathrm{Dom}\, \Pi_C$ onto C is spherically nonspreading; see [4].

Lemma 6.1 (Kimura–Sudo [4])**.** *Let E be a smooth Banach space and X nonempty, closed and spherically convex subset of S_E having the nonnegative functional property. Let T be a spherically nonspreading mapping on X. Then, the following hold:*

- *Its fixed point set $\mathrm{Fix}\, T$ is closed and spherically convex;*
- *if T has a fixed point, then*
$$\rho(p, JTx) \leq \rho(p, Jx)$$
for $x \in X$ and $p \in \mathrm{Fix}\, T$.

We first prove the following result corresponding to demiclosedness of a mapping:

Lemma 6.2. *Let E be a uniformly smooth and strictly convex Banach space and X a nonempty, closed and spherically convex subset of S_E having the nonnegative functional property. Let T be a spherically nonspreading mapping on X. Then,*

$$J^*\, \mathrm{AC}^*(\{x_n\}) \subset \mathrm{Fix}\, T$$

for a dual-spherically bounded sequence $\{x_n\}$ of X such that

$$\lim_{n\to\infty} \|Jx_n - JTx_n\| = 0.$$

Here, J and J^* are the duality mappings on E and E^*, respectively.

Proof. Since J^* is uniformly norm-to-norm continuous on S_{E^*}, we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Let $\{w^*\} = \mathrm{AC}^*(\{x_n\})$. Set $w = J^*w^*$. Since T is spherically nonspreading, for fixed $n \in \mathbb{N}$,

$$\cos\rho(Tx_n, JTw) + \cos\rho(Tw, JTx_n) \geq \cos\rho(Tx_n, Jw) + \cos\rho(Tw, Jx_n)$$

and hence

$$\langle Tx_n, JTw\rangle \geq \langle Tx_n, Jw\rangle + \langle Tw, Jx_n - JTx_n\rangle.$$

Note that

$$\langle Tx_n, JTw\rangle = \langle Tx_n - x_n, JTw\rangle + \langle x_n, JTw\rangle$$

and

$$\langle Tx_n, Jw\rangle = \langle Tx_n - x_n, Jw\rangle + \langle x_n, Jw\rangle.$$

Thus, we have

$$\liminf_{n\to\infty} \langle Tx_n, JTw\rangle = \liminf_{n\to\infty} \langle x_n, JTw\rangle$$

and

$$\liminf_{n\to\infty} \langle Tx_n, Jw\rangle = \liminf_{n\to\infty} \langle x_n, Jw\rangle.$$

Therefore,

$$\begin{aligned}
\liminf_{n\to\infty} \langle x_n, JTw\rangle &= \liminf_{n\to\infty} \langle Tx_n, JTw\rangle \\
&\geq \liminf_{n\to\infty} \left(\langle Tx_n, Jw\rangle + \langle Tw, Jx_n - JTx_n\rangle\right) \\
&= \liminf_{n\to\infty} \langle Tx_n, Jw\rangle \\
&= \liminf_{n\to\infty} \langle x_n, Jw\rangle = \liminf_{n\to\infty} \langle x_n, w^*\rangle.
\end{aligned}$$

Since $\mathrm{AC}^*(\{x_n\})$ is a singleton, we have $Jw = w^* = JTw$ and therefore $w \in \mathrm{Fix}\,T$, which implies that $J^*\,\mathrm{AC}^*(\{x_n\}) \subset \mathrm{Fix}\,T$. □

Lemma 6.3. *Let E be a uniformly smooth and uniformly convex Banach space and X a nonempty, closed and spherically convex subset of S_E having the nonnegative functional property such that JX is spherically convex. Let T be a spherically nonspreading mapping on X which has a fixed point. Let $\{\alpha_n\}$ be a real sequence of $[0,1]$. For an initial point $x_1 \in X \cap \mathrm{Dom}\,\Pi_{\mathrm{Fix}\,T}$, define a sequence $\{x_n\}$ of X as follows:*

$$x_{n+1} = \frac{J^*(\alpha_n Jx_n + (1-\alpha_n)JTx_n)}{\|\alpha_n Jx_n + (1-\alpha_n)JTx_n\|} = J^*(\alpha_n Jx_n \oplus (1-\alpha_n)JTx_n)$$

for $n \in \mathbb{N}$. Then, a sequence $\{\Pi_{\mathrm{Fix}\,T}x_n\}$ converges strongly to some $x_0 \in \mathrm{Fix}\,T$, where $\Pi_{\mathrm{Fix}\,T}$ is a spherical projection onto $\mathrm{Fix}\,T$.

Proof. Since $x_1 \in \mathrm{Dom}\,\Pi_{\mathrm{Fix}\,T}$,

$$\rho(\mathrm{Fix}\,T, Jx_1) = \rho(\Pi_{\mathrm{Fix}\,T}x_1, Jx_1) < \frac{\pi}{2}.$$

Further,

$$\begin{aligned}
\cos\rho(\Pi_{\mathrm{Fix}\,T}x_1, Jx_{n+1}) &= \cos\rho(\Pi_{\mathrm{Fix}\,T}x_1, \alpha_n Jx_n \oplus (1-\alpha_n)JTx_n) \\
&\geq \alpha_n \cos\rho(\Pi_{\mathrm{Fix}\,T}x_1, Jx_n) + (1-\alpha_n)\cos\rho(\Pi_{\mathrm{Fix}\,T}x_1, JTx_n) \\
&\geq \cos\rho(\Pi_{\mathrm{Fix}\,T}x_1, Jx_n).
\end{aligned}$$

Therefore,

$$\rho(\operatorname{Fix} T, Jx_n) \leq \rho(\mathit{\Pi}_{\operatorname{Fix} T} x_1, Jx_n) \leq \rho(\mathit{\Pi}_{\operatorname{Fix} T} x_1, Jx_1) < \frac{\pi}{2}.$$

It means that $\{x_n\}$ is included in $\operatorname{Dom} \mathit{\Pi}_{\operatorname{Fix} T}$ and hence a sequence $\{\mathit{\Pi}_{\operatorname{Fix} T} x_n\}$ is well defined. In what follows, we denote $\mathit{\Pi}_{\operatorname{Fix} T}$ by $\mathit{\Pi}$. We show that $\{\mathit{\Pi} x_n\}$ is a Cauchy sequence. Fix $n \in \mathbb{N}$. Then,

$$\begin{aligned}\cos \rho(\mathit{\Pi} x_{n+1}, Jx_{n+1}) &\geq \cos \rho(\mathit{\Pi} x_n, Jx_{n+1}) = \cos \rho(\mathit{\Pi} x_n, \alpha_n Jx_n \oplus (1-\alpha_n) JTx_n) \\ &\geq \alpha_n \cos \rho(\mathit{\Pi} x_n, Jx_n) + (1-\alpha_n) \cos \rho(\mathit{\Pi} x_n, JTx_n) \geq \cos \rho(\mathit{\Pi} x_n, Jx_n)\end{aligned}$$

and thus

$$0 < \langle p, Jx_1 \rangle \leq \langle \mathit{\Pi} x_n, Jx_n \rangle \leq \langle \mathit{\Pi} x_{n+1}, Jx_{n+1} \rangle \leq 1.$$

It implies that $\{\langle \mathit{\Pi} x_n, Jx_n \rangle\}$ is convergent and that there exists a nonnegative real sequence $\{\alpha_n\}$ converging to 0 such that

$$\langle \mathit{\Pi} x_m, Jx_m \rangle - \langle \mathit{\Pi} x_n, Jx_n \rangle = |\langle \mathit{\Pi} x_m, Jx_m \rangle - \langle \mathit{\Pi} x_n, Jx_n \rangle| \leq \alpha_n$$

for $m, n \in \mathbb{N}$ with $m \geq n$. Now, fix $m, n \in \mathbb{N}$ with $m \geq n$ arbitrarily. From the property of $\mathit{\Pi}$, we get

$$\cos \rho(\mathit{\Pi} x_n, J\mathit{\Pi} x_m) \cos \rho(\mathit{\Pi} x_m, Jx_m) \geq \cos \rho(\mathit{\Pi} x_n, Jx_m)$$

and thus

$$\cos \rho(\mathit{\Pi} x_n, J\mathit{\Pi} x_m) \geq 1 - \frac{\cos \rho(\mathit{\Pi} x_m, Jx_m) - \cos \rho(\mathit{\Pi} x_n, Jx_m)}{\cos \rho(\mathit{\Pi} x_m, Jx_m)} = 1 - \frac{\langle \mathit{\Pi} x_m, Jx_m \rangle - \langle \mathit{\Pi} x_n, Jx_m \rangle}{\langle \mathit{\Pi} x_m, Jx_m \rangle}.$$

Moreover, since

$$\langle \mathit{\Pi} x_n, Jx_m \rangle = \cos \rho(\mathit{\Pi} x_n, Jx_m) \geq \cos \rho(\mathit{\Pi} x_n, Jx_n) = \langle \mathit{\Pi} x_n, Jx_n \rangle,$$

we have

$$\begin{aligned}\langle \mathit{\Pi} x_n, J\mathit{\Pi} x_m \rangle = \cos \rho(\mathit{\Pi} x_n, J\mathit{\Pi} x_m) &\geq 1 - \frac{\langle \mathit{\Pi} x_m, Jx_m \rangle - \langle \mathit{\Pi} x_n, Jx_m \rangle}{\langle \mathit{\Pi} x_m, Jx_m \rangle} \\ &\geq 1 - \frac{\langle \mathit{\Pi} x_m, Jx_m \rangle - \langle \mathit{\Pi} x_n, Jx_n \rangle}{\langle \mathit{\Pi} x_m, Jx_m \rangle} \\ &= 1 - \frac{|\langle \mathit{\Pi} x_m, Jx_m \rangle - \langle \mathit{\Pi} x_n, Jx_n \rangle|}{\langle \mathit{\Pi} x_m, Jx_m \rangle} \geq 1 - \frac{\alpha_n}{\langle p, Jx_1 \rangle}.\end{aligned}$$

Then,

$$\rho(\mathit{\Pi} x_n, J\mathit{\Pi} x_m) \leq \arccos\left(1 - \frac{\alpha_n}{\langle p, Jx_1 \rangle}\right),$$

which implies that $\{\mathit{\Pi} x_n\}$ is a Cauchy sequence from Theorem 3.2, and therefore it converges strongly to some $x_0 \in \operatorname{Fix} T$. □

Now we obtain a fixed point approximation theorem with the Krasnosel'skii type iterative scheme [8]. Before that, we give the following condition:

Let E be a smooth Banach space and J the duality mapping on E. We say that J is sequentially delta-continuous if a sequence $\{Jx_n\}$ delta-converges to Jx_0 whenever a spherically bounded sequence $\{x_n\}$ of S_E delta-converges to $x_0 \in S_E$.

Theorem 6.4. *Let E be a uniformly smooth and uniformly convex Banach space and X a nonempty, admissible, closed and spherically convex subset of S_E such that JX is spherically convex. Let T be a spherically nonspreading mapping on X which has a fixed point. Assume that J is sequentially delta-continuous. For an initial point $x_1 \in X$, define a sequence $\{x_n\}$ of X as follows:*

$$x_{n+1} = \frac{J^*(Jx_n + JTx_n)}{\|Jx_n + JTx_n\|} = J^*\left(\frac{1}{2} Jx_n \oplus \frac{1}{2} JTx_n\right) \in X$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ delta-converges to a fixed point

$$x_0 = \lim_{n\to\infty} \mathit{\Pi}_{\mathrm{Fix}\,T} x_n.$$

Proof. Since T is spherically nonspreading, for $p \in \mathrm{Fix}\,T$, we have

$$\cos\rho(p, Jx_{n+1}) = \cos\rho\left(p, \frac{1}{2}Jx_n \oplus \frac{1}{2}JTx_n\right) \geq \frac{1}{2}\cos\rho(p, Jx_n) + \frac{1}{2}\cos\rho(p, JTx_n) \geq \cos\rho(p, Jx_n)$$

for $n \in \mathbb{N}$ and hence a limit of $\{\cos\rho(p, Jx_n)\}$ exists for all $p \in \mathrm{Fix}\,T$, and the limit is positive. Moreover, since

$$\sup_{y\in S_E} \liminf_{n\to\infty} \langle y, Jx_n\rangle \geq \liminf_{n\to\infty} \langle p, Jx_n\rangle \geq \langle p, Jx_1\rangle > 0,$$

the generated sequence $\{x_n\}$ is spherically bounded. Fix $p \in \mathrm{Fix}\,T$ and $n \in \mathbb{N}$ arbitrarily. Then,

$$\cos\rho(p, Jx_{n+1}) = \cos\rho\left(p, \frac{1}{2}Jx_n \oplus \frac{1}{2}JTx_n\right) = \frac{\cos\rho(p, Jx_n) + \cos\rho(p, JTx_n)}{\|Jx_n + JTx_n\|} \geq \frac{2\cos\rho(p, Jx_n)}{\|Jx_n + JTx_n\|}$$

and hence

$$\|Jx_n + JTx_n\| \geq \frac{2\cos\rho(p, Jx_n)}{\cos\rho(p, Jx_{n+1})}.$$

Thus,

$$2 \geq \|Jx_n + JTx_n\| \geq \frac{2\cos\rho(p, Jx_n)}{\cos\rho(p, Jx_{n+1})} \to 2$$

as $n \to \infty$. From the uniformly convexity of E^*,

$$\lim_{n\to\infty} \|Jx_n - JTx_n\| = 0.$$

Take a delta-convergent sequence $\{x_{n_i}\}$ of $\{x_n\}$ and let $w \in X$ be its delta-limit. Then, from sequential delta-continuity of J, a sequence $\{Jx_{n_i}\}$ delta-converges to Jw. Since

$$\{w\} = J^{-1}\{Jw\} = J^{-1}\,\mathrm{AC}(\{Jx_{n_i}\}) = J^{-1}\,\mathrm{AC}^*(\{x_{n_i}\}),$$

from Lemma 6.2, we have $w \in \mathrm{Fix}\,T$. Note that from Lemma 6.3, a sequence $\{\mathit{\Pi}_{\mathrm{Fix}\,T} x_n\}$ converges strongly to $x_0 \in \mathrm{Fix}\,T$. Since

$$\begin{aligned}
\liminf_{n\to\infty} \langle x_0, Jx_n\rangle &= \liminf_{n\to\infty} \langle x_0 + \mathit{\Pi}_{\mathrm{Fix}\,T} x_n - \mathit{\Pi}_{\mathrm{Fix}\,T} x_n, Jx_n\rangle \\
&= \liminf_{n\to\infty} \left(-\langle \mathit{\Pi}_{\mathrm{Fix}\,T} x_n - x_0, Jx_n\rangle + \langle \mathit{\Pi}_{\mathrm{Fix}\,T} x_n, Jx_n\rangle\right) \\
&\geq \liminf_{n\to\infty} \left(-\|\mathit{\Pi}_{\mathrm{Fix}\,T} x_n - x_0\| + \langle \mathit{\Pi}_{\mathrm{Fix}\,T} x_n, Jx_n\rangle\right) \\
&= \liminf_{n\to\infty} \langle \mathit{\Pi}_{\mathrm{Fix}\,T} x_n, Jx_n\rangle \geq \liminf_{n\to\infty} \langle w, Jx_n\rangle,
\end{aligned}$$

we get $w = x_0$, which implies that $\{x_n\}$ delta-converges to x_0. This is the desired result. □

Let H be a Hilbert space. Then, since the duality mapping on H is the identity mapping and $\mathrm{AC}(\{x_n\}) = \mathrm{AC}^*(\{x_n\})$ for a sequence $\{x_n\}$ of S_H, it is an example having the sequentially delta-continuous duality mapping. However, we have not known that there is an infinite dimensional Banach space having the non-trivial sequentially delta-continuous duality mapping yet. At the end of this section, we consider the following example:

Example 6.5. Let $m \in \mathbb{N}$ and $p > 1$. We define a norm on $\mathbb{R}^m$ by

$$\|x\| = \left(\sum_{k=1}^{m} |x_k|^p \right)^{1/p}$$

for $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$. We denote it by $\ell^{p;m}$ and then it is a uniformly smooth and uniformly convex Banach space. Moreover, for $q = p/(p-1) > 1$, we have $(\ell^{p;m})^* = \ell^{q;m}$. Let J be the duality mapping on $\ell^{p;m}$ and S be the unit sphere of $\ell^{p;m}$. Take an arbitrary spherically bounded sequence $\{x_n\}$ of S delta-converging to $x_0 \in S$. Then, since S is bounded and closed, $\{x_n\}$ has a convergent subsequence. Let $\{x_{n_i}\}$ be such a sequence of $\{x_n\}$ and $y_0 \in S$ its limit. We remark that $\{x_0\} = \mathrm{AC}(\{x_{n_i}\})$. Thus, $\{x_{n_i}\}$ converges to x_0. Therefore, $\{Jx_{n_i}\}$ converges to Jx_0 and hence it delta-converges to Jx_0. It means that J is sequentially delta-continuous.

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