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Iterative scheme generating method for demiclosed and 2-demiclosed mappings

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Abstract

This study extends iterative scheme generating methods (ISGMs) to apply to general classes of mappings. In contrast to previous studies, this study examines quasi-nonexpansive and 2-demiclosed mappings, in addition to quasi-nonexpansive and demiclosed mappings. By doing so, we can consider a class of mappings called normally 2-generalized hybrid mappings. For these extended classes of mappings, we develop the ISGMs that generate various iterative schemes to locate common fixed points.

Keywords: Iterative scheme generating method, common fixed point, demiclosed mapping, 2-demiclosed mapping
2010 MSC: 47H05, 47H09

1. Introduction

Throughout this paper, we denote by H a real Hilbert space with an inner product $\langle \cdot, \ \cdot \rangle$ and the induced norm $\|\cdot\|$. Let T be a mapping from C into H, where C is a nonempty subset of H. We use a notation

$$F(T) = \{x \in C : Tx = x\}$$

to denote the set of fixed points of T. A mapping $T : C \to H$ is said to be (i) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C. \tag{1}$$

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For nonexpansive mappings, various approximation methods for finding fixed points have been studied because of their great application value. The following iteration is called the Mann type [22]:

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n) T x_n \quad \text{for all } n \in \mathbb{N} = \{1, 2, \cdots\}, \tag{2}$$

where $x_1 \in C$ is given arbitrarily and $\{\lambda_n\} \subset [0,1]$ satisfies appropriate conditions. It is known that Mann type iterative scheme yields weak convergence; see Reich [25].

Kocourek *et al.* [8] introduced a class of mappings that unifies nonexpansive mappings and other important classes of mappings. A mapping $T : C \to H$ is called (ii) *generalized hybrid* [8] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + (1-\alpha) \|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1-\beta) \|x - y\|^2 \tag{3}$$

for all $x, y \in C$. Letting $\alpha = 1$ and $\beta = 0$ in (3), we have the condition (1). This means that nonexpansive mappings are contained by the class of generalized hybrid mappings as particular cases. The class of generalized hybrid mappings includes many other types of mappings; for instance, *nonspreading mappings* [9], *hybrid mappings* [26], and λ-*hybrid mappings* [1].

The class of generalized hybrid mappings has been further extended. A mapping $T : C \to H$ is called (iii) *normally generalized hybrid* [28] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + \beta \|x - Ty\|^2 + \gamma \|Tx - y\|^2 + \delta \|x - y\|^2 \leq 0 \quad \text{for all } x, y \in C \tag{4}$$
$$\text{where (1) } \alpha + \beta + \gamma + \delta \geq 0, \text{ and (2) } \alpha + \beta > 0 \text{ or } \alpha + \gamma > 0.$$

It is easy to verify that the class of normally generalized hybrid mappings covers generalized hybrid mappings. Unlike the case of generalized hybrid mappings, a normally generalized hybrid mapping T does not have multiple distinct fixed points when $\alpha + \beta + \gamma + \delta > 0$. This was shown in Takahashi *et al.* [28]. A normally generalized hybrid mapping is quasi-nonexpansive (13) and demiclosed (14); see Takahashi *et al.* [28] or Proposition 2.1 in this paper. Consequently, generalized hybrid mappings, including nonexpansive mappings etc., are also quasi-nonexpansive and demiclosed.

Recently, for quasi-nonexpansive and demiclosed mappings, Kondo [19] proved the following theorem:

Theorem 1.1 ([19])**.** *Let C be a nonempty, closed, and convex subset of a real Hilbert space H and let $S, T : C \to C$ be quasi-nonexpansive and demiclosed mappings such that $F(S) \cap F(T) \neq \emptyset$. Denote by $P_{F(S)\cap F(T)}$ the metric projection from H onto $F(S) \cap F(T)$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Define a sequence $\{x_n\}$ in C as follows:*

$$x_1 \in C \text{ is given,}$$
$$x_{n+1} = a_n y_n + b_n S z_n + c_n T w_n$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy the following conditions:

$$\|y_n - q\| \leq \|x_n - q\|, \ \|z_n - q\| \leq \|x_n - q\|, \ \|w_n - q\| \leq \|x_n - q\| \tag{5}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$,

$$x_n - y_n \to 0, \ x_n - z_n \to 0, \text{ and } x_n - w_n \to 0. \tag{6}$$

Then, $\{x_n\}$ converges weakly to a point $\widehat{x} \in F(S) \cap F(T)$, where $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

In this theorem, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are given as "free sequences" that are only required to satisfy the conditions (5) and (6). This method generates infinitely many iterative schemes. For instance, consider the following two-step iterative scheme:

$$y_n = \lambda_n x_n + (1-\lambda_n) S x_n, \tag{7}$$
$$x_{n+1} = a_n y_n + b_n S y_n + c_n T y_n,$$

where an initial point $x_1 \in C$ is provided arbitrarily and $\lambda_n \to 1$ is required. For two-step iterative methods, see Ishikawa's seminal work [6]. It can be verified that $\{y_n\}$ in (7) satisfies the conditions $\|y_n - q\| \leq \|x_n - q\|$ and $x_n - y_n \to 0$, where q is an arbitrarily selected element in $F(S) \cap F(T)$. Consequently, we can conclude from Theorem 1.1 that the sequence $\{x_n\}$, defined recursively by (7), converges weakly to a common fixed point of S and T. Similarly, Theorem 1.1 can generate infinitely many other iterative schemes such as three-step and more general iterative schemes and we call it an *iterative scheme generating method (ISGM)*. This method was initiated by Kondo [14, 16], employing mean-valued sequences; see also [17, 18]. For three-step iterative procedure, see Noor [24]. For coincidence point theorem for two commutative nonlinear mappings, see Asadi and Karapinar [2].

As mappings (1), (3), and (4) with fixed points are quasi-nonexpansive and demiclosed, Theorem 1.1 can apply to those classes of mappings. In particular, if T is nonexpansive, then T^2 is also nonexpansive. Furthermore, the relationship $F(T) \cap F(T^2) = F(T)$ holds. Thus, replacing S in Theorem 1.1 by T^2, we have the following, where notations are adjusted:

Corollary 1.2. *Let C be a nonempty, closed, and convex subset of H and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Denote by $P_{F(T)}$ the metric projection from H onto $F(T)$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Define a sequence $\{x_n\}$ in C as follows:*

$$x_1 \in C \text{ is given,} \qquad x_{n+1} = a_n y_n + b_n T z_n + c_n T^2 w_n \tag{8}$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy the following conditions:

$$\|y_n - q\| \leq \|x_n - q\|, \ \|z_n - q\| \leq \|x_n - q\|, \ \|w_n - q\| \leq \|x_n - q\| \tag{9}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$,

$$x_n - y_n \to 0, \ x_n - z_n \to 0, \text{ and } x_n - w_n \to 0. \tag{10}$$

Then, $\{x_n\}$ converges weakly to a point $\widehat{x} \in F(T)$, where $\widehat{x} \equiv \lim_{n\to\infty} P_{F(T)} x_n$.

Although Corollary 1.2 is deduced from Theorem 1.1 as a special case, it can also generate infinitely many iterative schemes to find a fixed point of a nonexpansive mapping.

The class of mappings studied for fixed point approximation has been further extended. A mapping $T : C \to C$ is called (iv) 2-*generalized hybrid* [23] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_1 \left\|T^2x - Ty\right\|^2 + \alpha_2 \|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2) \|x - Ty\|^2 \\ &\leq \beta_1 \left\|T^2x - y\right\|^2 + \beta_2 \|Tx - y\|^2 + (1 - \beta_1 - \beta_2) \|x - y\|^2 \end{aligned} \tag{11}$$

for all $x, y \in C$. Generalized hybrid mappings (3) are contained by the class of 2-generalized hybrid mappings as special cases of $\alpha_1 = \beta_1 = 0$. A mapping $T : C \to C$ is termed (v) *normally* 2-*generalized hybrid* [20] if there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_2 \left\|T^2x - Ty\right\|^2 + \alpha_1 \|Tx - Ty\|^2 + \alpha_0 \|x - Ty\|^2 \\ &+\beta_2 \left\|T^2x - y\right\|^2 + \beta_1 \|Tx - y\|^2 + \beta_0 \|x - y\|^2 \leq 0 \\ &\text{for all } x, y \in C, \end{aligned} \tag{12}$$

$$\text{where (1) } \sum_{n=0}^{2} (\alpha_n + \beta_n) \geq 0 \text{ and (2) } \alpha_2 + \alpha_1 + \alpha_0 > 0.$$

The class of normally 2-generalized hybrid mappings contains all classes of mappings (i)–(iv); see Kondo and Takahashi [20]. Unlike the cases of generalized hybrid mappings and 2-generalized hybrid mappings, a normally 2-generalized hybrid mapping T does not have multiple distinct fixed points when $\sum_{n=0}^{2}(\alpha_n+\beta_n)>0$. For examples of these mapping classes, see [5, 10, 15] and papers cited therein.

It is known that mappings (iv) and (v) with fixed points are quasi-nonexpansive (13) and 2-dmiclosed (15); see Kondo and Takahashi [20] or Proposition 2.2 in this paper. As the 2-demiclosedness (15) is logically weaker than the demiclosedness (14), Theorem 1.1 cannot apply to the types of mappings (iv) and (v). For 2-generalized hybrid mappings and normally 2-generalized hybrid mappings characterized respectively by the conditions (11) and (12), the ISGMs have not yet been established.

In this work, we extend ISGMs to apply to quasi-nonexpansive and 2-demiclosed mappings, in addition to quasi-nonexpansive and demiclosed mappings, which allows us to take into account the class of normally 2-generalized hybrid mappings. For these broad classes of mappings, we develop the ISGMs that generate various iterative schemes to locate common fixed points. This work complements Theorem 1.1, which focuses only on quasi-nonexpansive and demiclosed mappings. In what follows, we prepare preliminary results in Section 2. Section 3 establishes the main theorems. Section 4 introduces some variations of iterative schemes derived from our main theorems. In Section 5, as an appendix, we prove Propositions 2.1 and 2.2 for reader's convenience.

2. Preliminaries

In this section, we concisely provide preliminary information. Let $\{x_n\}$ be a sequence in a real Hilbert space H and let $x \in H$. Notations $x_n \to x$ and $x_n \rightharpoonup x$ represent strong and weak convergence of $\{x_n\}$ to x, respectively. The following are known in the literature:

(a) $x_n \rightharpoonup x$ if and only if for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_{ij}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{ij}} \rightharpoonup x$;

(b) if $x_n \to x$ and $y_n \rightharpoonup y$, then $\langle x_n,\ y_n\rangle \to \langle x,\ y\rangle$;

(c) if C is a closed and convex subset of H, it is weakly closed, that is, $\{x_n\} \subset C$ and $x_n \rightharpoonup x$ imply $x \in C$.

Let C be a nonempty, closed, and convex subset of H. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if

$$\|Tx-q\| \le \|x-q\| \quad \text{for all } x \in C \text{ and } q \in F(T). \tag{13}$$

According to Itoh and Takahashi [7], a set of fixed points of a quasi-nonexpansive mapping is closed and convex. Let $\{x_n\}$ be a sequence in C. A mapping $T : C \to H$ is called *demiclosed* if

$$x_n - Tx_n \to 0 \text{ and } x_n \rightharpoonup p \Longrightarrow p \in F(T). \tag{14}$$

It is often said that $I - T$ is demiclosed if (14) holds, where I represents the identity mapping. Kondo [12] called a mapping $T : C \to C$ 2-*demiclosed* if

$$x_n - Tx_n \to 0,\ x_n - T^2x_n \to 0, \text{ and } x_n \rightharpoonup p \Longrightarrow p \in F(T). \tag{15}$$

When considering 2-demiclosed mappings, we restrict the range of T to C since T^2x_n must be defined properly. If $T : C \to C$, a demiclosed mapping is 2-demiclosed. As mentioned in Introduction, the classes of mappings (i)–(iii) are quasi-nonexpansive and demiclosed, while the classes of mappings (iv) and (v) are quasi-nonexpansive and 2-demiclosed, if they have fixed points. More precisely, the following propositions hold:

Proposition 2.1 ([28]; see also Kocourek *et al.* [8]). *Let $T : C \to H$ be a normally generalized hybrid mapping (4) with a fixed point, where C is a nonempty, closed, and convex subset of H. Then, T is quasi-nonexpansive and demiclosed.*

Proposition 2.2 ([20]; see also Maruyama *et al.* [23])**.** *Let $T : C \to C$ be a normally 2-generalized hybrid mapping* (12) *with a fixed point, where C is a nonempty, closed, and convex subset of H. Then, T is quasi-nonexpansive and 2-demiclosed.*

Proofs of Propositions 2.1 and 2.2 are also provided in Section 5 in this paper for self-completeness. For examples of mappings that are quasi-nonexpansive and 2-demiclosed but are not demiclosed, see [5, 10, 15].

Let F be a nonempty, closed, and convex subset of H. Following the convention, we use the notation P_F to represent a *metric projection* from H onto F. A metric projection P_F is nonexpansive and satisfies the inequality

$$\langle x - P_F x,\ P_F x - q\rangle \geq 0 \text{ for all } x \in H \text{ and } q \in F. \tag{16}$$

The following lemmas are used in the next section:

Lemma 2.3 ([27])**.** *Let F be a nonempty, closed, and convex subset of H, let P_F be the metric projection from H onto F, and let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - q\| \leq \|x_n - q\|$ for all $q \in F$ and $n \in \mathbb{N}$, then $\{P_F x_n\}$ converges in F.*

Lemma 2.4 ([23, 29])**.** *Let $x, y, z, w, v \in H$ and $a, b, c, d, e \in \mathbb{R}$. Then, the following hold:*

(a) *If $a + b = 1$, then $\|ax + by\|^2 = a\|x\|^2 + b\|y\|^2 - ab\|x - y\|^2$;*

(b) *if $a + b + c = 1$, then*

$$\begin{aligned}&\|ax + by + cz\|^2 \\ &= a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x - y\|^2 - bc\|y - z\|^2 - ca\|z - x\|^2;\end{aligned}$$

(c) *if $a + b + c + d = 1$, then*

$$\begin{aligned}\|ax + by + cz + dw\|^2 &= a\|x\|^2 + b\|y\|^2 + c\|z\|^2 + d\|w\|^2 \\ &\quad -ab\|x - y\|^2 - ac\|x - z\|^2 - ad\|x - w\|^2 \\ &\quad -bc\|y - z\|^2 - bd\|y - w\|^2 - cd\|z - w\|^2;\end{aligned}$$

(d) *if $a + b + c + d + e = 1$, then*

$$\begin{aligned}&\|ax + by + cz + dw + ev\|^2 \\ = \ & a\|x\|^2 + b\|y\|^2 + c\|z\|^2 + d\|w\|^2 + e\|v\|^2 \\ & -ab\|x - y\|^2 - ac\|x - z\|^2 - ad\|x - w\|^2 - ae\|x - v\|^2 \\ & -bc\|y - z\|^2 - bd\|y - w\|^2 - be\|y - v\|^2 \\ & -cd\|z - w\|^2 - ce\|z - v\|^2 - de\|w - v\|^2.\end{aligned}$$

A proof of (c) is found in Kondo and Takahashi [21]. For Lemma 2.4, conditions $a, b, c, d, e \in [0, 1]$ are not necessary. If $a, b, c, d \in [0, 1]$, from (c), it follows that

$$\begin{aligned}&\|ax + by + cz + dw\|^2 \\ &\leq a\|x\|^2 + b\|y\|^2 + c\|z\|^2 + d\|w\|^2 - ab\|x - y\|^2 - ac\|x - z\|^2 - ad\|x - w\|^2.\end{aligned} \tag{17}$$

In the proof of Theorem 3.1, we use (17).

In Sections 3 and 4, we assume that there exists a common fixed point of nonlinear mappings. A set of sufficient conditions for the existence of a common fixed point is stated explicitly in the following theorem:

Theorem 2.5 ([3])**.** *Let C be a nonempty, closed, convex, and bounded subset of H. Let $S, T : C \to C$ be normally 2-generalized hybrid mappings such that $ST = TS$. Then, $F(S) \cap F(T)$ is nonempty.*

For common fixed point theorems, see also [4, 11, 13] and papers cited therein.

3. Main Results

In this section, we develop the ISGMs to deal with more general types of mappings than the earlier study [19]. We will prove two theorems in this section. Each of the two theorems generates an infinite number of iterative schemes that weakly approximate a common fixed point. The fundamental elements of the proof have been developed and refined in numerous prior studies; see papers cited in Kondo [10].

Theorem 3.1. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H, let $S : C \to C$ be a quasi-nonexpansive and demiclosed mapping, and let $T : C \to C$ be a quasi-nonexpansive and 2-demiclosed mapping. Suppose that $F(S) \cap F(T) \neq \emptyset$. Denote by $P_{F(S)\cap F(T)}$ the metric projection from H onto $F(S) \cap F(T)$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + c_n + d_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, $\underline{\lim}_{n\to\infty} a_n c_n > 0$, and $\underline{\lim}_{n\to\infty} a_n d_n > 0$. Define a sequence $\{x_n\}$ in C as follows:*

$$x_1 \in C : \text{ given,}$$
$$x_{n+1} = a_n y_n + b_n S z_n + c_n T w_n + d_n T^2 w_n$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy

$$\|y_n - q\| \leq \|x_n - q\|, \ \|z_n - q\| \leq \|x_n - q\|, \ \|w_n - q\| \leq \|x_n - q\| \tag{18}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$,

$$x_n - y_n \to 0, \ x_n - z_n \to 0, \text{ and } x_n - w_n \to 0. \tag{19}$$

Then, $\{x_n\}$ converges weakly to a point $\widehat{x} \in F(S) \cap F(T)$, where $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

Proof. As S and T are quasi-nonexpansive (13), $F(S)$ and $F(T)$ are closed and convex. Therefore, $F(S) \cap F(T)$ is also closed and convex. As $F(S) \cap F(T) \neq \emptyset$ is assumed, the metric projection $P_{F(S)\cap F(T)}$ from H onto $F(S) \cap F(T)$ is properly defined.

Observe that

$$\|x_{n+1} - q\| \leq \|x_n - q\| \tag{20}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Choose $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ arbitrarily. As S and T are quasi-nonexpansive, from (18), it follows that

$$\begin{aligned}
\|x_{n+1} - q\| &= \left\|a_n y_n + b_n S z_n + c_n T w_n + d_n T^2 w_n - q\right\| \\
&= \left\|a_n (y_n - q) + b_n (S z_n - q) + c_n (T w_n - q) + d_n \left(T^2 w_n - q\right)\right\| \\
&\leq a_n \|y_n - q\| + b_n \|S z_n - q\| + c_n \|T w_n - q\| + d_n \left\|T^2 w_n - q\right\| \\
&\leq a_n \|y_n - q\| + b_n \|z_n - q\| + c_n \|w_n - q\| + d_n \|T w_n - q\| \\
&\leq a_n \|y_n - q\| + b_n \|z_n - q\| + c_n \|w_n - q\| + d_n \|w_n - q\| \\
&\leq a_n \|x_n - q\| + b_n \|x_n - q\| + c_n \|x_n - q\| + d_n \|x_n - q\| \\
&= \|x_n - q\|.
\end{aligned}$$

This indicates that (20) holds true, as claimed. Consequently, we have the following: (1) $\{\|x_n - q\|\}$ converges in $\mathbb{R}$ for all $q \in F(S) \cap F(T)$; (2) $\{x_n\}$ is bounded; (3) according to Lemma 2.3, $\left\{P_{F(S)\cap F(T)} x_n\right\}$ converges in $F(S) \cap F(T)$. Define $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

We now prove that

$$y_n - S z_n \to 0, \ y_n - T w_n \to 0, \text{ and } y_n - T^2 w_n \to 0. \tag{21}$$

Choose $q \in F(S) \cap F(T)$ arbitrarily. Exploiting (17) and (18) yields

$$\begin{aligned}
& \|x_{n+1} - q\|^2 \\
= \;& \left\|a_n (y_n - q) + b_n (Sz_n - q) + c_n (Tw_n - q) + d_n \left(T^2 w_n - q\right)\right\|^2 \\
\leq \;& a_n \|y_n - q\|^2 + b_n \|Sz_n - q\|^2 + c_n \|Tw_n - q\|^2 + d_n \left\|T^2 w_n - q\right\| \\
& - a_n b_n \|y_n - Sz_n\|^2 - a_n c_n \|y_n - Tw_n\|^2 - a_n d_n \left\|y_n - T^2 w_n\right\|^2 \\
\leq \;& a_n \|y_n - q\|^2 + b_n \|z_n - q\|^2 + c_n \|w_n - q\|^2 + d_n \|w_n - q\| \\
& - a_n b_n \|y_n - Sz_n\|^2 - a_n c_n \|y_n - Tw_n\|^2 - a_n d_n \left\|y_n - T^2 w_n\right\|^2 \\
\leq \;& a_n^2 \|x_n - q\| + b_n^2 \|x_n - q\| + c_n^2 \|x_n - q\| + d_n \|x_n - q\| \\
& - a_n b_n \|y_n - Sz_n\|^2 - a_n c_n \|y_n - Tw_n\|^2 - a_n d_n \left\|y_n - T^2 w_n\right\|^2 \\
= \;& \|x_n - q\|^2 - a_n b_n \|y_n - Sz_n\|^2 - a_n c_n \|y_n - Tw_n\|^2 - a_n d_n \left\|y_n - T^2 w_n\right\|^2 .
\end{aligned}$$

Thus, it holds that

$$\begin{aligned}
& a_n b_n \|y_n - Sz_n\|^2 + a_n c_n \|y_n - Tw_n\|^2 + a_n d_n \left\|y_n - T^2 w_n\right\|^2 \\
& \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 .
\end{aligned}$$

As $\{\|x_n - q\|\}$ is convergent, the right-hand side converges to 0. From the hypotheses $\underline{\lim}_{n\to\infty} a_n b_n > 0$, $\underline{\lim}_{n\to\infty} a_n c_n > 0$, and $\underline{\lim}_{n\to\infty} a_n d_n > 0$, we obtain (21), as claimed.

Next, we show that

$$z_n - Sz_n \to 0,\ w_n - Tw_n \to 0, \text{ and } w_n - T^2 w_n \to 0. \tag{22}$$

Using (19) and (21), we have

$$\|z_n - Sz_n\| \leq \|z_n - x_n\| + \|x_n - y_n\| + \|y_n - Sz_n\| \to 0.$$

The part $w_n - Tw_n \to 0$ can be confirmed similarly. It follows that

$$\left\|w_n - T^2 w_n\right\| \leq \|w_n - x_n\| + \|x_n - y_n\| + \left\|y_n - T^2 w_n\right\| \to 0.$$

Therefore, (22) holds, as asserted.

Our aim is to prove that $x_n \rightharpoonup \widehat{x} \left(\equiv \lim_{k\to\infty} P_{F(S)\cap F(T)} x_k\right)$. Choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily. As $\{x_{n_i}\}$ is bounded, there exists a subsequence $\left\{x_{n_{ij}}\right\}$ of $\{x_{n_i}\}$ such that $x_{n_{ij}} \rightharpoonup p$ for some $p \in H$. From (19), it holds that $z_{n_{ij}} \rightharpoonup p$ and $w_{n_{ij}} \rightharpoonup p$. As S is demiclosed (14) and T is 2-demiclosed (15), from (22), we obtain $p \in F(S) \cap F(T)$. Therefore, from (16), it follows that

$$\left\langle x_{n_{ij}} - P_{F(S)\cap F(T)} x_{n_{ij}},\ P_{F(S)\cap F(T)} x_{n_{ij}} - p\right\rangle \geq 0$$

for all $j \in \mathbb{N}$. As $x_{n_{ij}} \rightharpoonup p$ and $P_{F(S)\cap F(T)} x_n \to \widehat{x}$, we have $\langle p - \widehat{x}, \widehat{x} - p\rangle \geq 0$. This implies that $p = \widehat{x}$. We have shown that for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\left\{x_{n_{ij}}\right\}$ of $\{x_{n_i}\}$ such that $x_{n_{ij}} \rightharpoonup p = \widehat{x}$. This means that $x_n \rightharpoonup \widehat{x}$. This concludes the proof. □

Remark 3.2. Here, two remarks are presented.

(i) Remind that a normally generalized hybrid mapping (4) with a fixed point is quasi-nonexpansive and demiclosed (Proposition 2.1), and a normally 2-generalized hybrid mapping (12) with a fixed point is quasi-nonexpansive and 2-demiclosed (Proposition 2.2). Thus, Theorem 3.1 can apply to those classes of mappings.

(ii) As a demiclosed mapping is 2-demiclosed, the mapping T in Theorem 3.1 can also be viewed as a nonexpansive mapping (1), a generalized hybrid mapping (3), or a normally generalized hybrid mapping, rather than a normally 2-generalized hybrid mapping.

In the proof of Theorem 3.1, the equality (17), which is derived from Lemma 2.4-(c), is exploited. The following theorem can be established using Lemma 2.4-(d). The basic flow of the proof of the following theorem is the same as the above theorem, we will omit it here:

Theorem 3.3. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H and let $S, T : C \to C$ be quasi-nonexpansive and 2-demiclosed mappings. Suppose that $F(S) \cap F(T) \neq \emptyset$. Denote by $P_{F(S)\cap F(T)}$ the metric projection from H onto $F(S) \cap F(T)$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, and $\{e_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $a_n + b_n + c_n + d_n + e_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, $\underline{\lim}_{n\to\infty} a_n c_n > 0$, $\underline{\lim}_{n\to\infty} a_n d_n > 0$, and $\underline{\lim}_{n\to\infty} a_n e_n > 0$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} &x_1 \in C : \text{ given,} \\ &x_{n+1} = a_n y_n + b_n S z_n + c_n S^2 z_n + d_n T w_n + e_n T^2 w_n \end{aligned}$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy

$$\|y_n - q\| \leq \|x_n - q\|, \ \|z_n - q\| \leq \|x_n - q\|, \text{ and } \|w_n - q\| \leq \|x_n - q\| \tag{23}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ and

$$x_n - y_n \to 0, \ x_n - z_n \to 0, \text{ and } x_n - w_n \to 0. \tag{24}$$

Then, $\{x_n\}$ converges weakly to a point $\widehat{x} \in F(S) \cap F(T)$, where $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

4. Corollaries

Each Theorems 3.1 and 3.3 yields infinitely many iterative schemes that approximate common fixed points of nonlinear mappings. This section presents some variations derived from these theorems, as illustrations. We begin with simple cases. Set $y_n = z_n = w_n = x_n$ in Theorems 3.1 and 3.3. Then, the required conditions (18) and (19) (equivalently, (23) and (24)) are fulfilled. By this operation, we obtain the following two corollaries:

Corollary 4.1. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H, let $S : C \to C$ be a quasi-nonexpansive and demiclosed mapping, and let $T : C \to C$ be a quasi-nonexpansive and 2-demiclosed mapping. Suppose that $F(S) \cap F(T) \neq \emptyset$. Denote by $P_{F(S)\cap F(T)}$ the metric projection from H onto $F(S) \cap F(T)$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $a_n + b_n + c_n + d_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, $\underline{\lim}_{n\to\infty} a_n c_n > 0$, and $\underline{\lim}_{n\to\infty} a_n d_n > 0$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} &x_1 \in C : \text{ given,} \\ &x_{n+1} = a_n x_n + b_n S x_n + c_n T x_n + d_n T^2 x_n \end{aligned}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges weakly to a point $\widehat{x} \in F(S) \cap F(T)$, where $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

Corollary 4.2. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H and let $S, T : C \to C$ be quasi-nonexpansive and 2-demiclosed mappings. Suppose that $F(S) \cap F(T) \neq \emptyset$. Denote by $P_{F(S)\cap F(T)}$ the metric projection from H onto $F(S) \cap F(T)$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, and $\{e_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $a_n + b_n + c_n + d_n + e_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, $\underline{\lim}_{n\to\infty} a_n c_n > 0$, $\underline{\lim}_{n\to\infty} a_n d_n > 0$, and $\underline{\lim}_{n\to\infty} a_n e_n > 0$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} &x_1 \in C : \text{ given,} \\ &x_{n+1} = a_n x_n + b_n S x_n + c_n S^2 x_n + d_n T x_n + e_n T^2 x_n \end{aligned}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges weakly to a point $\widehat{x} \in F(S) \cap F(T)$, where $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

These results Corollaries 4.1 and 4.2 correspond Theorem 3.1 and 3.4 in Kondo and Takahashi [21], respectively.

Next, setting $y_n = z_n = w_n$ and $y_n = \lambda_n x_n + \mu_n S x_n + \nu_n S^2 x_n + \xi_n T x_n$ in Theorem 3.1, we obtain the following:

Corollary 4.3. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H, let $S : C \to C$ be a quasi-nonexpansive and demiclosed mapping, and let $T : C \to C$ be a quasi-nonexpansive and 2-demiclosed mapping. Suppose that $F(S) \cap F(T) \neq \emptyset$. Denote by $P_{F(S)\cap F(T)}$ the metric projection from H onto $F(S) \cap F(T)$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $a_n + b_n + c_n + d_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, $\underline{\lim}_{n\to\infty} a_n c_n > 0$, and $\underline{\lim}_{n\to\infty} a_n d_n > 0$. Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, and $\{\xi_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $\lambda_n + \mu_n + \nu_n + \xi_n = 1$ for all $n \in \mathbb{N}$ and $\lambda_n \to 1$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} x_1 &\in C : \text{ given,} \\ y_n &= \lambda_n x_n + \mu_n S x_n + \nu_n S^2 x_n + \xi_n T x_n, \\ x_{n+1} &= a_n y_n + b_n S y_n + c_n T y_n + d_n T^2 y_n \end{aligned}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges weakly to a point $\hat{x} \in F(S) \cap F(T)$, where $\hat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

Proof. According to Theorem 3.1, it is sufficient to prove that

$$\|y_n - q\| \leq \|x_n - q\| \quad \text{for all } q \in F(S) \cap F(T) \text{ and } n \in \mathbb{N} \text{ and} \tag{25}$$

$$x_n - y_n \to 0 \quad \text{as } n \to \infty. \tag{26}$$

Choose $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ arbitrarily. As S and T are quasi-nonexpansive, it holds that

$$\begin{aligned} \|y_n - q\| &= \left\|\lambda_n x_n + \mu_n S x_n + \nu_n S^2 x_n + \xi_n T x_n - q\right\| \\ &\leq \lambda_n \|x_n - q\| + \mu_n \|S x_n - q\| + \nu_n \left\|S^2 x_n - q\right\| + \xi_n \|T x_n - q\| \\ &\leq \lambda_n \|x_n - q\| + \mu_n \|x_n - q\| + \nu_n \|x_n - q\| + \xi_n \|x_n - q\| \\ &= \|x_n - q\|. \end{aligned}$$

This shows that (25) holds.

Using (25), we can verify that $\|x_{n+1} - q\| \leq \|x_n - q\|$ for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Indeed, as S and T are quasi-nonexpansive,

$$\begin{aligned} \|x_{n+1} - q\| &= \left\|a_n y_n + b_n S y_n + c_n T y_n + d_n T^2 y_n - q\right\| \\ &\leq a_n \|y_n - q\| + b_n \|S y_n - q\| + c_n \|T y_n - q\| + d_n \left\|T^2 y_n - q\right\| \\ &\leq a_n \|y_n - q\| + b_n \|y_n - q\| + c_n \|y_n - q\| + d_n \|y_n - q\| \\ &\leq \|x_n - q\|. \end{aligned}$$

From this, we can observe that $\{x_n\}$ is bounded.

As S and T are quasi-nonexpansive, $\{S x_n\}$, $\{S^2 x_n\}$, and $\{T x_n\}$ are also bounded. Indeed, for $q \in F(S)$. As S is quasi-nonexpansive, we have

$$\begin{aligned} \left\|S^2 x_n\right\| &\leq \left\|S^2 x_n - q\right\| + \|q\| \\ &\leq \|S x_n - q\| + \|q\| \\ &\leq \|x_n - q\| + \|q\|. \end{aligned}$$

As $\{x_n\}$ is bounded, $\{S x_n\}$ and $\{S^2 x_n\}$ are also bounded. Similarly, it can be shown that $\{T x_n\}$ is bounded, as asserted.

Now, we observe that (26) is true. As $\lambda_n \to 1$ is assumed, we have $\mu_n, \nu_n, \xi_n \to 0$. As $\{Sx_n\}$, $\{S^2x_n\}$, and $\{Tx_n\}$ are bounded, it follows that

$$\begin{aligned} \|x_n - y_n\| &= \left\|x_n - \left(\lambda_n x_n + \mu_n S x_n + \nu_n S^2 x_n + \xi_n T x_n\right)\right\| \\ &\leq (1-\lambda_n)\|x_n\| + \mu_n \|Sx_n\| + \nu_n \left\|S^2 x_n\right\| + \xi_n \|Tx_n\| \to 0. \end{aligned}$$

This indicates that (26) holds. Thus, the desired result follows from Theorem 3.1. □

Setting $S = I$ and $y_n = z_n$ in Theorem 3.1 and adjusting notations, we obtain the following:

Corollary 4.4. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H and let $T : C \to C$ be a quasi-nonexpansive and 2-demiclosed mapping. Suppose that $F(T) \neq \emptyset$. Denote by $P_{F(T)}$ the metric projection from H onto $F(T)$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$ and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} &x_1 \in C: \text{ given}, \\ &x_{n+1} = a_n y_n + b_n T w_n + c_n T^2 w_n \end{aligned} \tag{27}$$

for all $n \in \mathbb{N}$, where $\{y_n\}$ an $\{w_n\}$ are sequences in C that satisfy

$$\|y_n - q\| \leq \|x_n - q\|, \ \|w_n - q\| \leq \|x_n - q\|$$

for all $q \in F(T)$ and $n \in \mathbb{N}$,

$$x_n - y_n \to 0, \text{ and } x_n - w_n \to 0.$$

Then, $\{x_n\}$ converges weakly to a point $\widehat{x} \in F(T)$, where $\widehat{x} \equiv \lim_{n\to\infty} P_{F(T)} x_n$.

Remark 4.5. We present two remarks.

(i) Compare Corollary 4.4 with Corollary 1.2. Although (27) and (8) seem similar, there are differences: in (27), the sequence $\{w_n\}$ in terms of T and T^2 must be the same, unlike the case of (8).

(ii) Although Corollary 4.4 is derived from Theorem 3.1, it can also generate infinitely many iterative schemes to locate fixed points of quasi-nonexpansive and 2-demiclosed mappings.

Other variations derived from the main theorems can be considered with reference to Kondo [14, 16, 17, 18, 19].

5. Appendix

In this section, as an appendix, we provide proofs of Propositions 2.1 and 2.2. Although they have already been established in existing studies, we reproduce them for readers' convenience.

Proof of Proposition 2.1.
Let $T : C \to H$ be a normally generalized hybrid mapping (4) that satisfies $F(T) \neq \emptyset$ with parameters $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. First, we show that T is quasi-nonexpansive (13). Let $x \in C$ and $q \in F(T)$. Assume that $\alpha + \beta > 0$. Then, from the condition (4),

$$\alpha \|Tq - Tx\|^2 + \beta \|q - Tx\|^2 + \gamma \|Tq - x\|^2 + \delta \|q - x\|^2 \leq 0. \tag{28}$$

As $q = Tq$, we have

$$(\alpha + \beta) \|Tx - q\|^2 + (\gamma + \delta) \|x - q\|^2 \leq 0.$$

As $\alpha + \beta + \gamma + \delta \geq 0$, it holds that

$$\begin{aligned}(\alpha + \beta) \|Tx - q\|^2 &\leq -(\gamma + \delta) \|x - q\|^2 \\ &\leq (\alpha + \beta) \|x - q\|^2 .\end{aligned}$$

Dividing by $\alpha + \beta\,(> 0)$, we obtain $\|Tx - q\|^2 \leq \|x - q\|^2$. In the case of $\alpha + \gamma > 0$, replacing the roles of x and q in (28), we can also obtain $\|Tx - q\| \leq \|x - q\|$. Therefore, T is quasi-nonexpansive, as claimed.

Next, we demonstrate that T is demiclosed (14). Let $\{x_n\}$ be a sequence in C such that $x_n - Tx_n \to 0$ and $x_n \rightharpoonup p$. Note that as C is closed and convex, it is weakly closed. Thus, from $x_n \rightharpoonup p$, we have $p \in C$. As T is a mapping from C into H, an element $Tp\,(\in H)$ exists. Our goal is to demonstrate that $Tp = p$. Assume that $\alpha + \beta > 0$. As T is a normally generalized hybrid mapping with parameters $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, it holds that

$$\alpha \|Tx_n - Tp\|^2 + \beta \|x_n - Tp\|^2 + \gamma \|Tx_n - p\|^2 + \delta \|x_n - p\|^2 \leq 0. \tag{29}$$

This implies that

$$\begin{aligned}&\alpha \left(\|Tx_n - x_n\|^2 + 2 \langle Tx_n - x_n,\ x_n - Tp\rangle + \|x_n - Tp\|^2\right) + \beta \|x_n - Tp\|^2 \\ &+\gamma \left(\|Tx_n - x_n\|^2 + 2 \langle Tx_n - x_n,\ x_n - p\rangle + \|x_n - p\|^2\right) + \delta \|x_n - p\|^2 \leq 0.\end{aligned}$$

Hence,

$$\begin{aligned}&\alpha \left(\|Tx_n - x_n\|^2 + 2 \langle Tx_n - x_n,\ x_n - Tp\rangle\right) + (\alpha + \beta) \|x_n - Tp\|^2 \\ &+\gamma \left(\|Tx_n - x_n\|^2 + 2 \langle Tx_n - x_n,\ x_n - p\rangle\right) + (\gamma + \delta) \|x_n - p\|^2 \leq 0.\end{aligned}$$

Therefore,

$$\begin{aligned}&\alpha \left(\|Tx_n - x_n\|^2 + 2 \langle Tx_n - x_n,\ x_n - Tp\rangle\right) \\ &+ (\alpha + \beta) \left(\|x_n - p\|^2 + 2 \langle x_n - p,\ p - Tp\rangle + \|p - Tp\|^2\right) \\ &+\gamma \left(\|Tx_n - x_n\|^2 + 2 \langle Tx_n - x_n,\ x_n - p\rangle\right) + (\gamma + \delta) \|x_n - p\|^2 \leq 0.\end{aligned}$$

As $\alpha + \beta + \gamma + \delta \geq 0$, subtracting $(\alpha + \beta + \gamma + \delta) \|x_n - p\|^2\,(\geq 0)$ from the LHS, we obtain

$$\begin{aligned}&\alpha \left(\|Tx_n - x_n\|^2 + 2 \langle Tx_n - x_n,\ x_n - Tp\rangle\right) \\ &+ (\alpha + \beta) \left(2 \langle x_n - p,\ p - Tp\rangle + \|p - Tp\|^2\right) \\ &+\gamma \left(\|Tx_n - x_n\|^2 + 2 \langle Tx_n - x_n,\ x_n - p\rangle\right) \leq 0.\end{aligned}$$

Note that as $\{x_n\}$ weakly converges, it is bounded. Using $x_n - Tx_n \to 0$ and $x_n \rightharpoonup p$, we have

$$(\alpha + \beta) \|p - Tp\|^2 \leq 0$$

as $n \to \infty$. As we temporarily assume that $\alpha + \beta > 0$, it follows that $\|p - Tp\|^2 \leq 0$. This implies that $p \in F(T)$. Replacing the roles of x_n and p in (29), we can obtain the proof for the case $\alpha + \gamma > 0$. This completes the proof. □

Proof of Proposition 2.2.

Let $T : C \to C$ be a normally 2-generalized hybrid mapping (12) with $F(T) \neq \emptyset$. We verify that T is quasi-nonexpansive (13). Select $x \in C$ and $q \in F(T)$ arbitrarily. According to the condition (12), there exist $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2 \in \mathbb{R}$ such that $\sum_{i=0}^{2} (\alpha_i + \beta_i) \geq 0$, $\alpha_2 + \alpha_1 + \alpha_0 > 0$, and

$$\begin{aligned} &\alpha_2 \left\|T^2 q - Tx\right\|^2 + \alpha_1 \|Tq - Tx\|^2 + \alpha_0 \|q - Tx\|^2 \\ &+\beta_2 \left\|T^2 q - x\right\|^2 + \beta_1 \|Tq - x\|^2 + \beta_0 \|q - x\|^2 \leq 0. \end{aligned}$$

As $q \in F(T)$, it holds that $q = Tq = T^2 q$. Hence,

$$(\alpha_2 + \alpha_1 + \alpha_0) \|Tx - q\|^2 + (\beta_2 + \beta_1 + \beta_0) \|x - q\|^2 \leq 0.$$

Using $\sum_{i=0}^{2} (\alpha_i + \beta_i) \geq 0$, we have

$$\begin{aligned} (\alpha_2 + \alpha_1 + \alpha_0) \|Tx - q\|^2 &\leq - (\beta_2 + \beta_1 + \beta_0) \|x - q\|^2 \\ &\leq (\alpha_2 + \alpha_1 + \alpha_0) \|x - q\|^2 . \end{aligned}$$

Dividing by $\alpha_2 + \alpha_1 + \alpha_0 \,(> 0)$ yields $\|Tx - q\|^2 \leq \|x - q\|^2$, which indicates that T is quasi-nonexpansive.

Next, we demonstrate that T is 2-demiclosed (15). Assume that $x_n - Tx_n \to 0$, $x_n - T^2 x_n \to 0$, and $x_n \rightharpoonup p$, where $\{x_n\}$ is a sequence in C. For the same reason as the proof of Proposition 2.1, it holds that $p \in C$ and consequently, $Tp \,(\in C)$ exists. We aim to demonstrate that $Tp = p$. As T is normally 2-generalized hybrid, it holds that

$$\begin{aligned} &\alpha_2 \left\|T^2 x_n - Tp\right\|^2 + \alpha_1 \|Tx_n - Tp\|^2 + \alpha_0 \|x_n - Tp\|^2 \\ &+\beta_2 \left\|T^2 x_n - p\right\|^2 + \beta_1 \|Tx_n - p\|^2 + \beta_0 \|x_n - p\|^2 \leq 0. \end{aligned}$$

From this,

$$\begin{aligned} &\alpha_2 \left(\left\|T^2 x_n - x_n\right\|^2 + 2 \left\langle T^2 x_n - x_n,\ x_n - Tp\right\rangle + \|x_n - Tp\|^2\right) \\ &+\alpha_1 \left(\|Tx_n - x_n\|^2 + 2 \langle Tx_n - x_n,\ x_n - Tp\rangle + \|x_n - Tp\|^2\right) + \alpha_0 \|x_n - Tp\|^2 \\ &+\beta_2 \left(\left\|T^2 x_n - x_n\right\|^2 + 2 \left\langle T^2 x_n - x_n,\ x_n - p\right\rangle + \|x_n - p\|^2\right) \\ &+\beta_1 \left(\|Tx_n - x_n\|^2 + 2 \langle Tx_n - x_n,\ x_n - p\rangle + \|x_n - p\|^2\right) + \beta_0 \|x_n - p\|^2 \leq 0, \end{aligned}$$

which implies

$$\begin{aligned} &\alpha_2 \left(\left\|T^2 x_n - x_n\right\|^2 + 2 \left\langle T^2 x_n - x_n,\ x_n - Tp\right\rangle\right) \\ &+\alpha_1 \left(\|Tx_n - x_n\|^2 + 2 \langle Tx_n - x_n,\ x_n - Tp\rangle\right) + (\alpha_2 + \alpha_1 + \alpha_0) \|x_n - Tp\|^2 \\ &+\beta_2 \left(\left\|T^2 x_n - x_n\right\|^2 + 2 \left\langle T^2 x_n - x_n,\ x_n - p\right\rangle\right) \\ &+\beta_1 \left(\|Tx_n - x_n\|^2 + 2 \langle Tx_n - x_n,\ x_n - p\rangle\right) + (\beta_2 + \beta_1 + \beta_0) \|x_n - p\|^2 \leq 0. \end{aligned}$$

It follows that

$$\begin{aligned} &\alpha_2 \left(\left\|T^2 x_n - x_n\right\|^2 + 2 \left\langle T^2 x_n - x_n,\ x_n - Tp\right\rangle\right) \\ &+\alpha_1 \left(\|Tx_n - x_n\|^2 + 2 \langle Tx_n - x_n,\ x_n - Tp\rangle\right) \\ &+ (\alpha_2 + \alpha_1 + \alpha_0) \left(\|x_n - p\|^2 + 2 \langle x_n - p,\ p - Tp\rangle + \|p - Tp\|^2\right) \\ &+\beta_2 \left(\left\|T^2 x_n - x_n\right\|^2 + 2 \left\langle T^2 x_n - x_n,\ x_n - p\right\rangle\right) \\ &+\beta_1 \left(\|Tx_n - x_n\|^2 + 2 \langle Tx_n - x_n,\ x_n - p\rangle\right) + (\beta_2 + \beta_1 + \beta_0) \|x_n - p\|^2 \leq 0. \end{aligned}$$

As $\sum_{i=0}^{2}(\alpha_i+\beta_i)\geq 0$, subtracting $\left(\sum_{i=0}^{2}(\alpha_i+\beta_i)\right)\|x_n-p\|^2\,(\geq 0)$ from the LHS, we have

$$\begin{gathered}\alpha_2\left(\left\|T^2x_n-x_n\right\|^2+2\left\langle T^2x_n-x_n,\ x_n-Tp\right\rangle\right)\\ +\alpha_1\left(\|Tx_n-x_n\|^2+2\left\langle Tx_n-x_n,\ x_n-Tp\right\rangle\right)\\ +(\alpha_2+\alpha_1+\alpha_0)\left(2\left\langle x_n-p,\ p-Tp\right\rangle+\|p-Tp\|^2\right)\\ +\beta_2\left(\left\|T^2x_n-x_n\right\|^2+2\left\langle T^2x_n-x_n,\ x_n-p\right\rangle\right)\\ +\beta_1\left(\|Tx_n-x_n\|^2+2\left\langle Tx_n-x_n,\ x_n-p\right\rangle\right)\leq 0.\end{gathered}$$

Note that as $\{x_n\}$ is weakly convergent, it is bounded. Using $x_n-Tx_n\to 0$, $x_n-T^2x_n\to 0$, and $x_n\rightharpoonup p$, we obtain

$$(\alpha_2+\alpha_1+\alpha_0)\|p-Tp\|^2\leq 0$$

in the limit as $n\to\infty$. Dividing by $\alpha_2+\alpha_1+\alpha_0\,(>0)$, we have $p\in F(T)$. This ends the proof. □

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