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On determining sinusoidal components of source terms in elliptic equations from terminal data

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Abstract

We study an inverse source problem for an elliptic *partial differential equation* (PDE) with a separable time-dependent source $\phi(t)$. The reconstruction is based on terminal data, a setting that is ill-posed in the sense of Hadamard. We first establish a Lipschitz continuity (stability) estimate for the recovered source in $L^p(\Omega), p \geq 2$, with respect to the frequency parameter κ. Our second contribution develops a regularization for the final-value problem with separable sources and noisy measurements in $L^2(\Omega)$. A truncation-based scheme is analyzed to regularize the problem, and an L^2 error estimate is derived.

Keywords: ill-posed problems, Poisson operator, truncation method, Sobolev embeddings
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1. Introduction

Inverse source problems for partial differential equations (PDEs) have attracted substantial attention over the past few decades from both theoretical and applied viewpoints. The aim is to reconstruct an unknown

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source term from indirect measurements of the PDE solution. Such problems are mathematically challenging and arise naturally in acoustics, electromagnetics, seismology, medical imaging, and materials science. The modern notion of well-posedness goes back to Hadamard: a PDE problem is *well-posed* if the solution exists, is unique, and depends continuously on the data; otherwise, it is *ill-posed*. Inverse problems−and in particular inverse source problems−typically fall into the ill-posed category, where small data perturbations may cause large instabilities in the reconstruction. This necessitates the design of appropriate regularization methods.

The historical roots of inverse problems can be traced back to Hadamard in the early 20th century, who introduced the concept of a *well-posed problem*. In his framework, a PDE problem is said to be *well-posed* if three conditions are satisfied: existence, uniqueness, and continuous dependence of the solution on the given data. Problems that fail to meet any of these requirements are deemed *ill-posed*. Inverse problems—and in particular inverse source problems—typically fall into the ill-posed category, where small data perturbations may cause large instabilities in the reconstruction. This necessitates the design of appropriate regularization methods.

Comprehensive accounts of elliptic equations, wavelet analysis, and fractional calculus can be found in [1, 2, 3, 4]. In theoretical physics, the Poisson equation plays a central role; several recent developments are reported in [5, 6, 7]. The same model also underpins a wide range of applications—including non-destructive testing, corrosion detection, tomographic imaging, and geophysics [8, 9]. Its associated inverse source problem is a prototypical Hadamard–unstable task, where tiny perturbations in the input data may lead to large deviations in the solution. Owing to this ill-posedness, practical computations typically rely on regularization techniques; see, e.g., [18, 22].

Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$, and let $T > 0$. We consider the following inverse source problem for an elliptic-in-space PDE with the Poisson operator:

$$\begin{cases} -u_{tt}(x,t) - \Delta u(x,t) = \varphi(t) f(x), & (x,t) \in \Omega \times (0,T) \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = 0, & x \in \Omega \end{cases} \tag{1}$$

where the function φ will be specified below. The terminal condition is

$$u(x,T) = g(x),$$

where $g \in L^2(\Omega)$ is called the input data.

Now, we present several results related to the problem (1) in the case $\phi(t) = 1$. In [10], the truncation method was utilized for regularization, providing an approximation of the problem, and an error estimate between the regularized and exact solutions was obtained. In [11], the authors proposed a modified approach and established a Hölder-type error estimate for the difference between the regularized and exact solutions. In [12], the Tikhonov regularization method was considered in the extended Hilbert scale, and error estimates were obtained using both an a priori approach and posterior choice rules.

Up to now, there have been only a limited number of works that have studied the inverse source problem for the Poisson equation with general φ. In [27], this paper provides the first result on determining the source function with the variable-separable form $\varphi(t)f(x)$, with the specific choice $\varphi(t) = e^{-kt}$ for $k \geq 0$. To the best of our knowledge, there has not been any work that has examined the determination of the source function with the variable-separable form $\varphi(t)f(x)$, with the specific choice $\varphi(t) = \sin(\kappa t)$ for $\kappa > 0$ as in Problem (2).

In this work, we consider the following model:

$$-u_{tt}(x,t) - \Delta u(x,t) = \sin(\kappa t) f(x), \tag{2}$$

subject to homogeneous Dirichlet boundary conditions $u|_{\partial\Omega} = 0$. Here, T, κ are given positive numbers; $\Omega \subset \mathbb{R}^N, N \geq 1$ is a bounded and connected domain with a smooth boundary $\partial\Omega$. The function f is the source term.

In this setting we impose

$$u(x,T) = g(x), \quad u_t(x,0) = 0, \quad x \in \Omega,$$

where $g \in L^2(\Omega)$ denotes the input (terminal) data.

The time-harmonic source $\varphi(t) = \sin(\kappa t)$ arises under periodic excitation, e.g., in lock-in thermography, where harmonic heating generates "thermal waves" and the steady spatial field is governed by an elliptic model with a harmonic source. In electrical impedance tomography, small *alternating-current (AC)* injections at a fixed angular frequency lead to an elliptic forward model and an inverse reconstruction from boundary data; under the quasi-static approximation for *electroencephalography (EEG)* and *magnetoencephalography (MEG)*, the forward model reduces to a Poisson-type equation while the inverse task recovers neural sources. Therefore, developing regularization methods and L^2 error estimates for model (2) is directly relevant to these measurement protocols.

Our main results and contributions in this paper are summarized as follows:

- First contribution: We prove an L^p-Lipschitz dependence of the reconstructed source on $\kappa (p \geq 2)$.
- Second contribution: We devise a truncation-based regularization for the terminal (final-value) problem with separable sources and noisy L^2 data, and an L^2 error estimate is derived.

The remainder of the paper is organized as follows: Section 2 presents the preliminaries and introduces the abstract framework. Section 3 investigates the Lipschitz continuity of the source in L^p with respect to $\kappa (p \geq 2)$. Theorem 3.1 provides an explicit formula for the source, and Theorem 3.4 establishes its Lipschitz continuity with respect to κ. Section 4 introduces the regularized scheme and Theorem 4.1 provides the L^2 error estimate.

2. Preliminaries

Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, and define the operator

$$\mathcal{A} = -\Delta$$

subject to homogeneous Dirichlet boundary conditions. Suppose that $\mathcal{A}$ is a self-adjoint, positive operator with compact resolvent, and that there exists an orthonormal basis $\{e_n\}_{n\geq 1}$ of $L^2(\Omega)$ composed of eigenfunctions of $\mathcal{A}$, with corresponding eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to \infty \quad \text{as } n \to \infty$$

We denote by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(\Omega)$, and by $u_n(t) := \langle u(\cdot, t), e_n \rangle$ the Fourier coefficients of u. The norm in a Banach space B is indicated by $\| \cdot \|_B$.

For $s \geq 0$, we define the Hilbert scale space

$$\mathbb{H}^s(\Omega) = \left\{ f = \sum_{n=1}^{\infty} f_n e_n \in L^2(\Omega) : \|f\|^2_{\mathbb{H}^s(\Omega)} = \sum_{n=1}^{\infty} f_n^2 \lambda_n^s < \infty \right\}.$$

Lemma 2.1. *(see [22]) The following inclusions hold true*

$$\left.\begin{array}{lll} L^p(\Omega) \hookrightarrow \mathbb{H}^\sigma(\Omega), & \text{if} & -\frac{N}{2} < \sigma \leq 0, \quad p \geq \frac{2N}{N-2\sigma}, \\ \mathbb{H}^\sigma(\Omega) \hookrightarrow L^p(\Omega), & \text{if} & 0 \leq \sigma < \frac{N}{2}, \quad p \leq \frac{2N}{N-2\sigma}. \end{array}\right\}$$

3. Inverse Source Problem with the Terminal Condition

We analyze the following PDE via spectral decomposition

$$\begin{cases} -u_{tt}(x,t) - \Delta u(x,t) = \sin(\kappa t) f(x), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = 0, \quad u_t(x,0) = 0, & x \in \Omega. \end{cases} \tag{3}$$

Here, T, κ are given positive numbers; $\Omega \subset \mathbb{R}^N, N \geq 1$ is a bounded and connected domain with a smooth boundary $\partial\Omega$. The function f is the (*unknown*) source term.

3.1. The explicit formula of the source function

Theorem 3.1. *Let $g \in L^2(\Omega)$. Assume that $-\kappa \sinh\left(\sqrt{\lambda_n}\, T\right) + \sqrt{\lambda_n} \sin(\kappa T) \neq 0$. Then the source term f of Problem (3) is given by*

$$f(x) = \sum_{n=1}^{\infty} \frac{\sqrt{\lambda_n}(\kappa^2 + \lambda_n)}{-\kappa \sinh(\sqrt{\lambda_n}\, T) + \sqrt{\lambda_n} \sin(\kappa T)} g_n e_n(x).$$

Proof. We express the solution $u(x,t)$ in the form of Fourier series as $u(x,t) = \sum_{n=1}^{\infty} u_n(t) e_n(x)$, with $u_n(t) = \langle u(\cdot,t), e_n \rangle$. Substituting into the main equation, we obtain:

$$-\sum_{n\geq 1} u_n''(t) e_n(x) + \sum_{n\geq 1} \lambda_n u_n(t) e_n(x) = \sum_{n\geq 1} f_n \sin(\kappa t) e_n(x),$$

where $f_n = \langle f, e_n \rangle$. From which we arrive at the following ordinary differential equations

$$u_n''(t) - \lambda_n u_n(t) = -f_n \sin(\kappa t). \tag{4}$$

The general solution of (4) equation is $u_n(t) = u_n^h(t) + u_n^p(t)$. The homogeneous solution is:

$$u_n^h(t) = A_n \cosh\left(\sqrt{\lambda_n}\, t\right) + B_n \sinh\left(\sqrt{\lambda_n}\, t\right).$$

For the particular solution, assume $u_n^p(t) = C_n \sin(\kappa t)$:

$$-C_n \kappa^2 \sin(\kappa t) - \lambda_n C_n \sin(\kappa t) = -f_n \sin(\kappa t),$$

which implies that $C_n = \frac{f_n}{\kappa^2 + \lambda_n}$ (since $\kappa^2 + \lambda_n > 0$). Thus, one gets

$$u_n^p(t) = \frac{f_n}{\kappa^2 + \lambda_n} \sin(\kappa t).$$

The general solution of (4) is defined by

$$u_n(t) = A_n \cosh\left(\sqrt{\lambda_n}\, t\right) + B_n \sinh\left(\sqrt{\lambda_n}\, t\right) + \frac{f_n}{\kappa^2 + \lambda_n} \sin(\kappa\, t).$$

Applying $u_n(0) = u_n'(0) = 0$, we find

$$u_n(0) = A_n = 0, \;\; u_n'(t) = B_n \sqrt{\lambda_n} \cosh\left(\sqrt{\lambda_n}\, t\right) + \frac{f_n \kappa}{\kappa^2 + \lambda_n} \cos(\kappa t),$$

and

$$u_n'(0) = B_n \sqrt{\lambda_n} + \frac{f_n \kappa}{\kappa^2 + \lambda_n} = 0.$$

Thus $B_n = -\frac{f_n \kappa}{\sqrt{\lambda_n}(\kappa^2+\lambda_n)}$. Hence,

$$u_n(t) = f_n \left(\frac{-\kappa \sinh\left(\sqrt{\lambda_n}\, t\right) + \sqrt{\lambda_n} \sin(\kappa t)}{\sqrt{\lambda_n}\left(\kappa^2 + \lambda_n\right)} \right).$$

The terminal condition gives:

$$g_n = u_n(T) = f_n \left(\frac{-\kappa \sinh\left(\sqrt{\lambda_n}\, T\right) + \sqrt{\lambda_n} \sin(\kappa T)}{\sqrt{\lambda_n}\left(\kappa^2 + \lambda_n\right)} \right).$$

This yields immediately that

$$f_n = \frac{\sqrt{\lambda_n}\left(\kappa^2 + \lambda_n\right)}{-\kappa \sinh\left(\sqrt{\lambda_n}\, T\right) + \sqrt{\lambda_n} \sin(\kappa T)} g_n.$$

Hence, the source is represented by the series

$$f(x) = \sum_{n=1}^{\infty} \frac{\sqrt{\lambda_n}\left(\kappa^2 + \lambda_n\right)}{-\kappa \sinh\left(\sqrt{\lambda_n}\, T\right) + \sqrt{\lambda_n} \sin(\kappa T)} g_n e_n(x).$$

□

3.2. The Lipschitz continuity of the source function with respect to the parameter κ in L^p

Lemma 3.2. *Let $T, \kappa > 0$ and $\alpha_n = \sqrt{\lambda_n}$ for $n \geq 1$. Then*

$$\kappa \sinh(\alpha_n T) - \alpha_n \sin(\kappa T) \;\geq\; \alpha_n\big(\kappa T - \sin(\kappa T)\big) \;=\; \sqrt{\lambda_n}\,\big(\kappa T - \sin(\kappa T)\big) \;\geq 0.$$

Proof. Since $\sinh x \geq x$ for $x \geq 0$, we have

$$\kappa \sinh\left(\alpha_n T\right) \geq \kappa \alpha_n T.$$

Subtracting $\alpha_n \sin(\kappa T)$ from both sides gives

$$D_n := \kappa \sinh\left(\alpha_n T\right) - \alpha_n \sin(\kappa T) \geq \alpha_n \kappa T - \alpha_n \sin(\kappa T) = \alpha_n(\kappa T - \sin(\kappa T)).$$

Since $\sin x \leq x$ for all $x \geq 0$, we have $\kappa T - \sin(\kappa T) \geq 0$, hence the last term is nonnegative. □

Lemma 3.3. *(hyperbolic dominates polynomial). Fix $\kappa = \bar{\kappa} > 0$ and $T > 0$. Set*

$$A := \frac{|\sin(\bar{\kappa} T)|}{T} > 0, \quad B := \frac{\bar{\kappa}}{2} > 0.$$

Then there exists $\lambda_0 = \lambda_0(\bar{\kappa}, T) > 0$ such that for all $\lambda \geq \lambda_0$,

$$\sqrt{\lambda}\,|\sin(\bar{\kappa} T)| \leq \frac{\bar{\kappa}}{2} \sinh(\sqrt{\lambda}\, T).$$

Proof. Let $x = \sqrt{\lambda} T$. We must show $Ax \leq B \sinh x$ for all $x \geq x_0$ with some $x_0 > 0$. Consider $\phi(x) := \frac{B}{A}\frac{\sinh x}{x}$. Since

$$\phi'(x) = \frac{B}{A} \frac{x \cosh x - \sinh x}{x^2} > 0 \quad (x > 0),$$

ϕ is increasing on $(0, \infty)$ and $\phi(x) \to \infty$ as $x \to \infty$. Hence there exists $x_0 = x_0(\bar{\kappa}, T) > 0$ with $\phi\left(x_0\right) = 1$, and then $\phi(x) \geq 1$ for all $x \geq x_0$, i.e. $Ax \leq B \sinh x$. Returning to λ with $\lambda_0 := \left(x_0/T\right)^2$ yields the claim. □

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary and let $\{(\lambda_n, e_n)\}_{n\geq 1}$ be the orthonormal eigen-system of the Dirichlet Laplacian $-\Delta$ on Ω, i.e., $-\Delta e_n = \lambda_n e_n$ in Ω, $e_n|_{\partial\Omega} = 0$, and $\{e_n\}$ is orthonormal in $L^2(\Omega)$. For $T > 0$, $\kappa > 0$, and $g \in L^2(\Omega)$, define*

$$f(x,\kappa) = \sum_{n=1}^{\infty} \frac{\sqrt{\lambda_n}\,(\kappa^2 + \lambda_n)}{-\kappa \sinh\big(\sqrt{\lambda_n}\,T\big) + \sqrt{\lambda_n}\sin(\kappa T)} \langle g, e_n\rangle\, e_n(x), \tag{5}$$

whenever the denominator is nonzero for all n. The series (5) converges in $L^2(\Omega)$.

Let $\kappa_1, \kappa_2 > 0$, and let $\bar\kappa$ be any number between κ_1 and κ_2 such that the non-resonance condition

$$-\bar\kappa \sinh\big(\sqrt{\lambda_n}\,T\big) + \sqrt{\lambda_n}\sin(\bar\kappa T) \neq 0 \quad \text{for all } n \geq 1$$

holds. Then the following estimates are valid:

(1) *Lipschitz stability in L^2:*
There exists a constant $C_2(\bar\kappa, T) > 0$ such that

$$\|f(\cdot,\kappa_1) - f(\cdot,\kappa_2)\|_{L^2(\Omega)} \leq C_2(\bar\kappa, T)\,|\kappa_1 - \kappa_2|\,\|g\|_{L^2(\Omega)}.$$

(2) *Lipschitz stability in L^p, $p \geq 2$:*
Let $2 \leq p < \infty$ and $\sigma > \frac{N}{2} - \frac{N}{p}$. Set $s = \sigma - \frac{N}{2} + \frac{N}{p} > 0$. Assume $g(x) \in \mathbb{H}^\sigma(\Omega)$. There exists a constant $C_{p,\Omega} > 0$ and $M > 0$, such that,

$$\begin{aligned}\|f(\cdot,\kappa_1) - f(\cdot,\kappa_2)\|_{L^p(\Omega)} &\leq C_2(\bar\kappa, T)\,C_{p,\Omega}\,|\kappa_1 - \kappa_2|\,M^{\frac{N}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}\,\|g\|_{L^2(\Omega)} \\ &\quad + C_{p,\Omega}\,C(\bar\kappa, T)\,|\kappa_1 - \kappa_2|\,\big(M^{-1-\frac{s}{2}} + M^{-\frac{s}{2}}\big)\,\|g\|_{\mathbb{H}^\sigma(\Omega)}.\end{aligned}$$

(3) *Lipschitz stability in L^∞:*
Let $\sigma > \frac{N}{2}$. Assume $g(x) \in \mathbb{H}^\sigma(\Omega)$. Set $l = \sigma - \frac{N}{2} > 0$. There exists a constant $C_{\infty,\Omega} > 0$ and $M > 0$ such that,

$$\begin{aligned}\|f(\cdot,\kappa_1) - f(\cdot,\kappa_2)\|_{L^\infty(\Omega)} &\leq C_2(\bar\kappa, T) C_{\infty,\Omega}\,|\kappa_1 - \kappa_2|\,M^{\frac{N}{4}}\,\|g\|_{L^2(\Omega)} \\ &\quad + C_{\infty,\Omega} C(\bar\kappa, T)\,|\kappa_1 - \kappa_2|\,\Big(M^{-1-\frac{l}{2}} + M^{-\frac{l}{2}}\Big)\,\|g\|_{\mathbb{H}^\sigma(\Omega)}.\end{aligned}$$

Here $C(\bar\kappa, T)$, $C_2(\bar\kappa, T)$, $C_{p,\Omega}$, and $C_{\infty,\Omega}$ denote positive constants independent of g, κ_1, κ_2, and M (except for the displayed dependence), and $\mathbb{H}^\sigma$ is the Hilbert scale associated with the Dirichlet Laplacian.

Proof. To establish the Lipschitz continuity of the solution $f(x,\kappa)$ with respect to the frequency parameter κ, we denote the spectral multiplier by

$$m_n(\kappa) := \frac{\sqrt{\lambda_n}\,(\kappa^2 + \lambda_n)}{-\kappa \sinh\left(\sqrt{\lambda_n}\,T\right) + \sqrt{\lambda_n}\sin(\kappa T)}.$$

For any frequencies $\kappa_1, \kappa_2 > 0$, the mean value theorem implies there exists $\bar\kappa$ between κ_1 and κ_2 such that

$$m_n(\kappa_1) - m_n(\kappa_2) = \frac{\partial}{\partial\kappa} m(\kappa, \lambda_n)\Big|_{\kappa=\bar\kappa} (\kappa_1 - \kappa_2).$$

Thus, the difference in the solution is

$$f(x,\kappa_1) - f(x,\kappa_2) = \sum_{n=1}^{\infty} \left(\frac{\partial}{\partial\kappa} m(\kappa,\lambda_n)\Big|_{\kappa=\bar\kappa}\right)(\kappa_1 - \kappa_2)\,\langle g, e_n\rangle\, e_n(x).$$

(1) Lipschitz stability in L^2:

We first prove Lipschitz continuity in the $L^2(\Omega)$ norm. Since $\{e_n\}$ is an orthonormal basis in $L^2(\Omega)$, the L^2 norm of the difference is

$$\begin{aligned}\|f(\cdot,\kappa_1)-f(\cdot,\kappa_2)\|^2_{L^2(\Omega)} &= |\kappa_1-\kappa_2|^2\sum_{n=1}^{\infty}\left|\frac{\partial}{\partial\kappa}m(\kappa,\lambda_n)\Big|_{\kappa=\bar\kappa}\right|^2|\langle g,e_n\rangle|^2\\ &\le |\kappa_1-\kappa_2|^2\left(\sup_{n\ge1}\left|\frac{\partial}{\partial\kappa}m(\kappa,\lambda_n)\Big|_{\kappa=\bar\kappa}\right|^2\right)\sum_{n=1}^{\infty}|\langle g,e_n\rangle|^2\\ &= |\kappa_1-\kappa_2|^2\left(\sup_{n\ge1}\left|\frac{\partial}{\partial\kappa}m(\kappa,\lambda_n)\Big|_{\kappa=\bar\kappa}\right|^2\right)\|g\|^2_{L^2(\Omega)}.\end{aligned}\tag{6}$$

Taking the square root, we need to bound

$$\sup_{n\ge1}\left|\frac{\partial}{\partial\kappa}m(\kappa,\lambda_n)\Big|_{\kappa=\bar\kappa}\right|<\infty,$$

with a constant depending only on $\bar\kappa$ and T, under the non-resonance condition $\bar\kappa T-\sin(\bar\kappa T)>0$. Define the numerator and denominator of the multiplier:

$$\begin{aligned}N(\kappa,\lambda)&:=\sqrt{\lambda}\left(\kappa^2+\lambda\right),\\ D(\kappa,\lambda)&:=-\kappa\sinh(\sqrt{\lambda}T)+\sqrt{\lambda}\sin(\kappa T).\end{aligned}$$

Then, $m(\kappa,\lambda)=\frac{N(\kappa,\lambda)}{D(\kappa,\lambda)}$, and the derivative is

$$\frac{\partial}{\partial\kappa}m(\kappa,\lambda)=\frac{\frac{\partial}{\partial\kappa}N(\kappa,\lambda)D(\kappa,\lambda)-N(\kappa,\lambda)\frac{\partial}{\partial\kappa}D(\kappa,\lambda)}{D(\kappa,\lambda)^2}.\tag{7}$$

Compute the partial derivatives:

$$\begin{aligned}\frac{\partial}{\partial\kappa}N(\kappa,\lambda)&=2\sqrt{\lambda}\kappa,\\ \frac{\partial}{\partial\kappa}D(\kappa,\lambda)&=-\sinh(\sqrt{\lambda}T)+\sqrt{\lambda}T\cos(\kappa T).\end{aligned}$$

Substituting into (7), we obtain

$$\frac{\partial}{\partial\kappa}m(\kappa,\lambda)=\frac{2\sqrt{\lambda}\kappa(-\kappa\sinh(\sqrt{\lambda}T)+\sqrt{\lambda}\sin(\kappa T))-\sqrt{\lambda}\left(\kappa^2+\lambda\right)\left(-\sinh(\sqrt{\lambda}T)+\sqrt{\lambda}T\cos(\kappa T)\right)}{(-\kappa\sinh(\sqrt{\lambda}T)+\sqrt{\lambda}\sin(\kappa T))^2}.\tag{8}$$

To bound this, we estimate the denominator using standard inequalities: $\sinh z\ge z+\frac{z^3}{6}$ for all $z>0$ and Lemma 3.2 . Thus,

$$|D(\kappa,\lambda)|=\kappa\sinh(\sqrt{\lambda}T)-\sqrt{\lambda}\sin(\kappa T)\ge\sqrt{\lambda}(\kappa T-\sin(\kappa T))>0$$

and more sharply,

$$|D(\kappa,\lambda)|\ge\kappa(\sinh(\sqrt{\lambda}T)-\sqrt{\lambda}T)\ge\frac{\kappa T^3}{6}\lambda^{3/2}.\tag{9}$$

For the numerator of (8), we bound each term:

$$\begin{aligned}|2\sqrt{\lambda}\kappa(-\kappa\sinh(\sqrt{\lambda}T)+\sqrt{\lambda}\sin(\kappa T))|&\le 2\sqrt{\lambda}\,|\kappa|\left|-\kappa\sinh(\sqrt{\lambda}T)+\sqrt{\lambda}\sin(\kappa T)\right|,\\ \left|\sqrt{\lambda}\left(\kappa^2+\lambda\right)\left(-\sinh(\sqrt{\lambda}T)+\sqrt{\lambda}T\cos(\kappa T)\right)\right|&\le\sqrt{\lambda}\left(\kappa^2+\lambda\right)(\sinh(\sqrt{\lambda}T)+\sqrt{\lambda}T).\end{aligned}$$

Thus, the absolute value of the derivative is

$$\left|\frac{\partial}{\partial\kappa}m(\kappa,\lambda)\right| \leq \frac{2\sqrt{\lambda}|\kappa|\left|-\kappa\sinh(\sqrt{\lambda}T)+\sqrt{\lambda}\sin(\kappa T)\right|+\sqrt{\lambda}\left(\kappa^2+\lambda\right)(\sinh(\sqrt{\lambda}T)+\sqrt{\lambda}T)}{\left|-\kappa\sinh(\sqrt{\lambda}T)+\sqrt{\lambda}\sin(\kappa T)\right|^2}. \tag{10}$$

- For $\lambda \geq 1$, use (9), then

$$\begin{aligned} &\frac{2\sqrt{\lambda}|\kappa|\left|-\kappa\sinh(\sqrt{\lambda}T)+\sqrt{\lambda}\sin(\kappa T)\right|}{\left|-\kappa\sinh(\sqrt{\lambda}T)+\sqrt{\lambda}\sin(\kappa T)\right|^2} \leq \frac{2\sqrt{\lambda}|\kappa|}{\frac{\kappa T^3}{6}\lambda^{3/2}} = \frac{12|\kappa|}{\kappa T^3\lambda} \leq \frac{12}{T^3\lambda}, \\ &\frac{\sqrt{\lambda}\left(\kappa^2+\lambda\right)(\sinh(\sqrt{\lambda}T)+\sqrt{\lambda}T)}{\left(\frac{\kappa T^3}{6}\lambda^{3/2}\right)^2} = \frac{36\left(\kappa^2+\lambda\right)(\sinh(\sqrt{\lambda}T)+\sqrt{\lambda}T)}{\kappa^2T^6\lambda^{5/2}}. \end{aligned} \tag{11}$$

For large λ. Fix $\kappa=\bar{\kappa}>0$. Since $\sinh(\sqrt{\lambda}T)$ dominates $\sqrt{\lambda}$ as $\lambda\to\infty$, there exists $\lambda_0=\lambda_0(\bar{\kappa},T)$ such that, for all $\lambda\geq\lambda_0$,

$$|D(\bar{\kappa},\lambda)| = \bar{\kappa}\sinh(\sqrt{\lambda}T)-\sqrt{\lambda}\sin(\bar{\kappa}T) \geq \frac{\bar{\kappa}}{2}\sinh(\sqrt{\lambda}T),$$

(since Lemma 3.3). From (10) we then have

$$|\partial_\kappa m(\bar{\kappa},\lambda)| \leq \frac{2\sqrt{\lambda}|\bar{\kappa}|(\bar{\kappa}\sinh(\sqrt{\lambda}T)+\sqrt{\lambda}|\sin(\bar{\kappa}T)|)}{|D(\bar{\kappa},\lambda)|^2} + \frac{\sqrt{\lambda}\left(\bar{\kappa}^2+\lambda\right)(\sinh(\sqrt{\lambda}T)+\sqrt{\lambda}T)}{|D(\bar{\kappa},\lambda)|^2}.$$

Using (11), the two terms are bounded as follows:

$$\frac{2\sqrt{\lambda}|\bar{\kappa}|(\bar{\kappa}\sinh+\sqrt{\lambda}|\sin|)}{|D|^2} \leq \frac{C_1}{\lambda}; \quad \frac{\sqrt{\lambda}\left(\bar{\kappa}^2+\lambda\right)(\sinh+\sqrt{\lambda}T)}{|D|^2} \leq C_2(\bar{\kappa},T)\frac{\lambda^{3/2}(\sinh+\sqrt{\lambda}T)}{\sinh^2} \leq C_2e^{-\sqrt{\lambda}T}.$$

Hence,

$$|\partial_\kappa m(\bar{\kappa},\lambda)| \leq \frac{C_1}{\lambda}+C_2(\bar{\kappa},T)e^{-\sqrt{\lambda}T}, \quad \lambda\geq\lambda_0.$$

Therefore, there exists $C=C(\bar{\kappa},T)$ such that

$$\sup_{\lambda\geq\lambda_0}|\partial_\kappa m(\bar{\kappa},\lambda)| \leq C.$$

For $\lambda\in[1,\lambda_0]$, by Lemma 3.2 , we have $|D(\bar{\kappa},\lambda)|\geq c_*=\sqrt{\lambda_1}(\bar{\kappa}T-\sin(\bar{\kappa}T))>0$. Hence, using the quotient rule and the bounds $|N|\leq\sqrt{\lambda_0}\left(\bar{\kappa}^2+\lambda_0\right)$ and $|\partial_\kappa D|\leq\sinh T+T$, we obtain

$$\sup_{\lambda_n\in[1,\lambda_0]}|\partial_\kappa m\left(\bar{\kappa},\lambda_n\right)| \leq \frac{2\sqrt{\lambda_0}|\bar{\kappa}|}{c_*}+\frac{\sqrt{\lambda_0}\left(\bar{\kappa}^2+\lambda_0\right)(\sinh T+T)}{c_*^2} =: C_{\text{large}}\left(\bar{\kappa},T,\lambda_0\right)<\infty. \tag{12}$$

- For $\lambda\leq 1$. Fix $\kappa=\bar{\kappa}>0$. We have rough bounds

$$|N|=\sqrt{\lambda}\left(\kappa^2+\lambda\right)\leq\kappa^2+1, \quad |\partial_\kappa D|\leq\sinh(\sqrt{\lambda}T)+\sqrt{\lambda}T\leq\sinh T+T.$$

Inferred

$$\sup_{\lambda\in[\lambda_1,1]}|\partial_\kappa m(\kappa,\lambda)| \leq \frac{2|\kappa|}{c_*}+\frac{\left(\kappa^2+1\right)(\sinh T+T)}{c_*^2} =: C_{\text{small}}\left(\bar{\kappa},T\right)<\infty.$$

Combining the two ranges of λ, there exists a constant $C_2(\bar{\kappa},T)<\infty$ such that

$$\sup_{n\geq 1}\left|\frac{\partial}{\partial\kappa}m\left(\kappa,\lambda_n\right)\bigg|_{\kappa=\bar{\kappa}}\right| \leq \max\left\{C_{\text{small}}\left(\bar{\kappa},T\right),C_{\text{large}}\left(\bar{\kappa},T\right)\right\} =: C_2(\bar{\kappa},T)<\infty. \tag{13}$$

Combining (6) and (13), we obtain

$$\|f(\cdot,\kappa_1) - f(\cdot,\kappa_2)\|_{L^2(\Omega)} \le C_2(\bar{\kappa},T)\,|\kappa_1-\kappa_2|\,\|g\|_{L^2(\Omega)}.$$

proving the L^2 Lipschitz continuity under the non-resonance condition.

(2) Lipschitz stability in L^p, $p \ge 2$:

To extend the result to $L^p(\Omega)$ for $p \ge 2$, we use spectral truncation at level $M \ge 1$ to split the difference at a spectral level M to be chosen later:

$$\begin{aligned} f(x,\kappa_1) - f(x,\kappa_2) = &\sum_{\lambda_n \le M} \left(\frac{\partial}{\partial \kappa} m(\kappa,\lambda_n)\Big|_{\kappa=\bar{\kappa}} \right)(\kappa_1-\kappa_2)\langle g,e_n\rangle e_n(x) \\ &+ \sum_{\lambda_n > M} \left(m_n(\kappa_1) - m_n(\kappa_2)\right)\langle g,e_n\rangle e_n(x). \end{aligned}$$

By the triangle inequality,

$$\begin{aligned} \|f(\cdot,\kappa_1) - f(\cdot,\kappa_2)\|_{L^p(\Omega)} \le &\left\| \sum_{\lambda_n \le M} \left(\frac{\partial}{\partial \kappa} m(\kappa,\lambda_n)\Big|_{\kappa=\bar{\kappa}} \right)(\kappa_1-\kappa_2)\langle g,e_n\rangle e_n \right\|_{L^p(\Omega)} \\ &+ \left\| \sum_{\lambda_n > M} \left(m_n(\kappa_1) - m_n(\kappa_2)\right)\langle g,e_n\rangle e_n \right\|_{L^p(\Omega)}. \end{aligned} \tag{14}$$

For the first term, apply the Sobolev embedding $H^s(\Omega) \hookrightarrow L^p(\Omega)$ with $s = \frac{N}{2} - \frac{N}{p} > 0$. There exists a constant $C_{p,\Omega} > 0$ such that for $u = \sum_{\lambda_n \le M} \langle u,e_n\rangle e_n$,

$$\|u\|_{L^p(\Omega)} \le C_{p,\Omega} \left\| (I-\Delta)^{s/2} u \right\|_{L^2(\Omega)} \le C_{p,\Omega}(1+M)^{s/2}\|u\|_{L^2(\Omega)} \le C_{p,\Omega} M^{\frac{N}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}\|u\|_{L^2(\Omega)}. \tag{15}$$

Set $u = \sum_{\lambda_n \le M} \left(\frac{\partial}{\partial\kappa} m(\kappa,\lambda_n)\big|_{\kappa=\bar{\kappa}} \right)(\kappa_1-\kappa_2)\langle g,e_n\rangle e_n$. Compute its L^2 norm:

$$\begin{aligned} \|u\|^2_{L^2(\Omega)} &= |\kappa_1-\kappa_2|^2 \sum_{\lambda_n \le M} \left| \frac{\partial}{\partial\kappa} m(\kappa,\lambda_n)\Big|_{\kappa=\bar{\kappa}} \right|^2 |\langle g,e_n\rangle|^2 \\ &\le |\kappa_1-\kappa_2|^2\, C_2(\bar{\kappa},T)^2 \|g\|^2_{L^2(\Omega)}. \end{aligned} \tag{16}$$

Applying (15), then

$$\left\| \sum_{\lambda_n \le M} \left(\frac{\partial}{\partial \kappa} m(\kappa,\lambda_n)\Big|_{\kappa=\bar{\kappa}} \right)(\kappa_1-\kappa_2)\langle g,e_n\rangle e_n \right\|_{L^p(\Omega)} \le C_2(\bar{\kappa},T)C_{p,\Omega}\,|\kappa_1-\kappa_2|\, M^{\frac{N}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}\|g\|_{L^2(\Omega)}. \tag{17}$$

For the high–frequency part, assume $g(x) \in \mathbb{H}^\sigma(\Omega)$ with $\sigma > \frac{N}{2} - \frac{N}{p}$. Set $s = \sigma - \frac{N}{2} + \frac{N}{p} > 0$. Using the inverse Sobolev embedding,

$$\left\| \sum_{\lambda_n > M} \left(m_n(\kappa_1) - m_n(\kappa_2)\right)\langle g,e_n\rangle e_n \right\|_{L^p(\Omega)} \le C_{p,\Omega} \left\| (-\Delta)^{\frac{N/2-N/p}{2}} \sum_{\lambda_n > M} \left(m_n(\kappa_1) - m_n(\kappa_2)\right)\langle g,e_n\rangle e_n \right\|_{L^2(\Omega)}.$$

Since $m_n(\kappa_1) - m_n(\kappa_2) = \frac{\partial}{\partial\kappa} m(\kappa,\lambda_n)\big|_{\kappa=\bar{\kappa}}(\kappa_1-\kappa_2)$, and

$$|\partial_\kappa m(\kappa,\lambda)|_{\kappa=\bar{\kappa}}| \le \frac{C_1}{\lambda_n} + C_2(\bar{\kappa},T)e^{-\sqrt{\lambda_n}T}, \lambda \ge \lambda_0(\bar{\kappa},T).$$

By choosing $M \geq \lambda_0$ to apply to all $\lambda_n > M$. Let $\alpha = \frac{N}{2} - \frac{N}{p}, \sigma > \alpha$ and $s = \sigma - \alpha > 0$. We have

$$\begin{aligned}\left\| \sum_{\lambda_n > M} (m_n(\kappa_1) - m_n(\kappa_2)) \langle g, e_n \rangle e_n \right\|_{L^p} &\leq C_{p,\Omega} \left\| (-\Delta)^{\alpha/2} \sum_{\lambda_n > M} (m_n(\kappa_1) - m_n(\kappa_2)) \langle g, e_n \rangle e_n \right\|_{L^2} \\ &= C_{p,\Omega} \left(\sum_{\lambda_n > M} \lambda_n^\alpha |m_n(\kappa_1) - m_n(\kappa_2)|^2 |\langle g, e_n \rangle|^2 \right)^{1/2} \\ &\leq C_{p,\Omega} |\kappa_1 - \kappa_2| \left(\sum_{\lambda_n > M} \lambda_n^\alpha \left(\frac{C_1}{\lambda_n} + C_2 e^{-\sqrt{\lambda_n} T} \right)^2 |\langle g, e_n \rangle|^2 \right)^{1/2} \\ &\leq C_{p,\Omega} |\kappa_1 - \kappa_2| (I_1 + I_2)^{1/2}.\end{aligned}$$

with

$$I_1 = C_1^2 \sum_{\lambda_n > M} \lambda_n^{\alpha-2} |\langle g, e_n \rangle|^2, \quad I_2 = C_2^2 \sum_{\lambda_n > M} \lambda_n^\alpha e^{-2\sqrt{\lambda_n} T} |\langle g, e_n \rangle|^2.$$

- For I_1 : because $\lambda_n > M, \lambda_n^{\alpha-2} \leq M^{-2} \lambda_n^\alpha \leq M^{-2-s} \lambda_n^\sigma$ (due to $\sigma = \alpha + s$). Hence

$$I_1 \leq C M^{-2-s} \|g\|_{\mathbb{H}^\sigma(\Omega)}^2.$$

- With I_2 : use $e^{-2\sqrt{\lambda} T} \leq 1$ and $\lambda^\alpha \leq M^{-s} \lambda^\sigma$ when $\lambda > M$, we get

$$I_2 \leq C M^{-s} \|g\|_{\mathbb{H}^\sigma(\Omega)}^2.$$

Thus,

$$\left\| \sum_{\lambda_n > M} (m_n(\kappa_1) - m_n(\kappa_2)) \langle g, e_n \rangle e_n \right\|_{L^p(\Omega)} \leq C_{p,\Omega} C(\bar{\kappa}, T) |\kappa_1 - \kappa_2| \left(M^{-1-\frac{s}{2}} + M^{-\frac{s}{2}} \right) \|g\|_{\mathbb{H}^\sigma(\Omega)}. \tag{18}$$

Combining (17) and (18) into (14), then

$$\begin{aligned}\|f(\cdot, \kappa_1) - f(\cdot, \kappa_2)\|_{L^p(\Omega)} \leq & C_2(\bar{\kappa}, T) C_{p,\Omega} |\kappa_1 - \kappa_2| M^{\frac{N}{2}\left(\frac{1}{2} - \frac{1}{p}\right)} \|g\|_{L^2} \\ & + C_{p,\Omega} C(\bar{\kappa}, T) |\kappa_1 - \kappa_2| \left(M^{-1-\frac{s}{2}} + M^{-\frac{s}{2}} \right) \|g\|_{\mathbb{H}^\sigma(\Omega)}.\end{aligned}$$

(3) Lipschitz stability in L^∞:

For the $L^\infty(\Omega)$ norm, set $p = \infty$. For $\sigma > \frac{N}{2}$, the Sobolev embedding gives

$$\left\| \sum_{\lambda_n \leq M} \langle u, e_n \rangle e_n \right\|_{L^\infty(\Omega)} \leq C_{\infty,\Omega} M^{\frac{N}{4}} \|u\|_{L^2(\Omega)}.$$

Applying this to the truncated term, and (16), it follows that

$$\left\| \sum_{\lambda_n \leq M} \left(\frac{\partial}{\partial \kappa} m(\kappa, \lambda_n) \Big|_{\kappa = \bar{\kappa}} \right) (\kappa_1 - \kappa_2) \langle g, e_n \rangle e_n \right\|_{L^\infty(\Omega)} \leq C_2(\bar{\kappa}, T) C_{\infty,\Omega} |\kappa_1 - \kappa_2| M^{\frac{N}{4}} \|g\|_{L^2(\Omega)}.$$

For the high–frequency part, for $\sigma > \frac{N}{2}, l = \sigma - \frac{N}{2} > 0$ and choose $M \geq \lambda_0$. Inferred

$$\left\| \sum_{\lambda_n > M} \left(m_n\left(\kappa_1\right) - m_n\left(\kappa_2\right)\right)\left\langle g, e_n\right\rangle e_n \right\|_{L^\infty} \leq C_{\infty,\Omega} C(\bar{\kappa}, T)\left|\kappa_1 - \kappa_2\right| \left(M^{-1-\frac{l}{2}} + M^{-\frac{l}{2}}\right)\|g\|_{\mathbb{H}^\sigma(\Omega)}.$$

Hence, we obtain

$$\begin{aligned}&\left\|f\left(\cdot, \kappa_1\right) - f\left(\cdot, \kappa_2\right)\right\|_{L^\infty(\Omega)} \leq C_2(\bar{\kappa}, T) C_{\infty,\Omega}\left|\kappa_1 - \kappa_2\right| M^{\frac{N}{4}}\|g\|_{L^2(\Omega)} \\ &+C_{\infty,\Omega} C(\bar{\kappa}, T)\left|\kappa_1 - \kappa_2\right|\left(M^{-1-\frac{l}{2}} + M^{-\frac{l}{2}}\right)\|g\|_{\mathbb{H}^\sigma(\Omega)}.\end{aligned}$$

□

4. Regularization in the L^2 Space

In the eigenbasis $\{(\lambda_n, e_n)\}$,

$$g_n = \frac{D\left(\kappa, \lambda_n\right)}{\sqrt{\lambda_n}\left(\kappa^2 + \lambda_n\right)} f_n, \quad D(\kappa, \lambda) = -\kappa \sinh(\sqrt{\lambda} T) + \sqrt{\lambda} \sin(\kappa T).$$

By Lemma 3.2, $\left|D\left(\kappa, \lambda_n\right)\right| \sim \kappa \sinh\left(\sqrt{\lambda_n} T\right) \sim \frac{\kappa}{2} e^{\sqrt{\lambda_n} T}$ as $n \to \infty$; hence the multipliers of T_κ blow up and T_κ is unbounded on $L^2(\Omega)$. Equivalently, the inverse coefficients behave like $\frac{\sqrt{\lambda_n}\left(\kappa^2+\lambda_n\right)}{D(\kappa,\lambda_n)} \sim \frac{2}{\kappa}\lambda_n^{3/2} e^{-\sqrt{\lambda_n} T}$, so high-frequency noise in g is exponentially amplified. Therefore the recovery of f from g is *ill-posed* (Hadamard), motivating the spectral cut-off regularization in Theorem 4.1.

Theorem 4.1. *Suppose the observed data g^ε satisfies*

$$\|g^\varepsilon - g\|_{L^2(\Omega)} \leq \varepsilon.$$

Let us assume that κ is noised by κ_ε such that

$$|\kappa_\varepsilon - \kappa| \leq \varepsilon.$$

We define the regularized solution f^ε as

$$f^\varepsilon(x) = \sum_{\lambda_n \leq M_\varepsilon} \frac{\sqrt{\lambda_n}\left(\kappa_\varepsilon^2 + \lambda_n\right)}{-\kappa_\varepsilon \sinh\left(\sqrt{\lambda_n} T\right) + \sqrt{\lambda_n} \sin\left(\kappa_\varepsilon T\right)} g_n^\varepsilon e_n(x).$$

and assume that $f \in \mathbb{H}^\sigma(\Omega)$ for $\sigma > 0$. Then, the following estimate holds.

$$\begin{aligned}\|f^\varepsilon - f\|_{L^2(\Omega)} &\leq \|h^\varepsilon - \theta^\varepsilon\|_{L^2(\Omega)} + \|f^\varepsilon - h^\varepsilon\|_{L^2(\Omega)} + \|\theta^\varepsilon - f\|_{L^2(\Omega)} \\ &\leq \widetilde{C}(\kappa, T)\left(M_\varepsilon + 1\right)\varepsilon + \left(A(\kappa, T) + B(\kappa, T)\frac{M_\varepsilon + \kappa^2}{\kappa T - \sin(\kappa T)}\right)\varepsilon\|g\|_{L^2(\Omega)} + M_\varepsilon^{-\sigma/2}\|f\|_{\mathbb{H}^\sigma}.\end{aligned}$$

where $\widetilde{C}$, $A(\kappa, T), B(\kappa, T)$ is a positive constant depending on κ, T. The parameter M_ε is chosen to satisfy

$$\lim_{\varepsilon \to 0} M_\varepsilon = +\infty, \quad \lim_{\varepsilon \to 0} M_\varepsilon \varepsilon = 0.$$

Proof. Let us set the following functions

$$h^\varepsilon(x) = \sum_{\lambda_n \leq M_\varepsilon} \frac{\sqrt{\lambda_n}\left(\lambda_n + \kappa_\varepsilon^2\right)}{\sqrt{\lambda_n} \sin\left(\kappa_\varepsilon T\right) - \kappa_\varepsilon \sinh\left(\sqrt{\lambda_n} T\right)} g_n e_n(x),$$

$$\theta^\varepsilon(x) = \sum_{\lambda_n \leq M_\varepsilon} \frac{\sqrt{\lambda_n}\left(\lambda_n + \kappa^2\right)}{\sqrt{\lambda_n} \sin(\kappa T) - \kappa \sinh\left(\sqrt{\lambda_n} T\right)} g_n e_n(x).$$

Step 1. Estimate of $\|f^\varepsilon - h^\varepsilon\|_{L^2}$*:*

By Parseval's theorem, we find that

$$\begin{aligned}\|f^\varepsilon - h^\varepsilon\|_{L^2}^2 &= \sum_{\lambda_n \le M_\varepsilon} \left| \frac{\sqrt{\lambda_n}\left(\lambda_n + \kappa_\varepsilon^2\right)}{\sqrt{\lambda_n}\sin\left(\kappa_\varepsilon T\right) - \kappa_\varepsilon \sinh\left(\sqrt{\lambda_n}\,T\right)} \right|^2 |g_n^\varepsilon - g_n|^2 \\ &\le \left(\sup_{\lambda_n \le M_\varepsilon} \left| \frac{\sqrt{\lambda_n}\left(\lambda_n + \kappa_\varepsilon^2\right)}{\sqrt{\lambda_n}\sin\left(\kappa_\varepsilon T\right) - \kappa_\varepsilon \sinh\left(\sqrt{\lambda_n}\,T\right)} \right| \right)^2 \sum_{\lambda_n \le M_\varepsilon} |g_n^\varepsilon - g_n|^2 .\end{aligned}$$

Since the fact that

$$\sum_{\lambda_n \le M_\varepsilon} |g_n^\varepsilon - g_n|^2 \le \|g^\varepsilon - g\|_{L^2}^2 \le \varepsilon^2.$$

Set $\phi(x) = xT - \sin(xT)$. By the non–resonance condition we have $\phi(\kappa) = \kappa T - \sin(\kappa T) > 0$. Since ϕ is continuous at $x = \kappa$, there exist $\varepsilon_0 > 0$ and a constant $c_* = c_*(\kappa, T) > 0$ such that

$$\phi(\eta) \ge \tfrac{1}{2}\phi(\kappa) =: c_*(\kappa, T) \qquad \text{for all } |\eta - \kappa| \le \varepsilon_0.$$

In particular, for any $\varepsilon \le \varepsilon_0$ and any κ_ε with $|\kappa_\varepsilon - \kappa| \le \varepsilon$,

$$\kappa_\varepsilon T - \sin(\kappa_\varepsilon T) = \phi(\kappa_\varepsilon) \ge c_*(\kappa, T) \ge \tfrac{1}{2}(\kappa T - \sin(\kappa T)).$$

Hence,

$$\left| \frac{\sqrt{\lambda_n}(\lambda_n + \kappa_\varepsilon^2)}{\sqrt{\lambda_n}\sin(\kappa_\varepsilon T) - \kappa_\varepsilon \sinh(\sqrt{\lambda_n}T)} \right| \le \frac{\lambda_n + \kappa_\varepsilon^2}{c_*(\kappa, T)} \le \frac{M_\varepsilon + \kappa_\varepsilon^2}{c_*(\kappa, T)}.$$

Let $C_3(\kappa, T) := 1/c_*(\kappa, T)$ (note that C_3 is independent of ε). Using Parseval's identity and $\sum_{\lambda_n \le M_\varepsilon} |g_n^\varepsilon - g_n|^2 \le \varepsilon^2$, we obtain

$$\|f^\varepsilon - h^\varepsilon\|_{L^2(\Omega)} \le C_3(\kappa, T)\big(M_\varepsilon + \kappa_\varepsilon^2\big)\,\varepsilon. \tag{19}$$

Moreover, since $\kappa_\varepsilon \in [\kappa - \varepsilon_0, \kappa + \varepsilon_0]$, we may absorb κ_ε^2 into the constant to get

$$\|f^\varepsilon - h^\varepsilon\|_{L^2(\Omega)} \le \widetilde{C}(\kappa, T)\big(M_\varepsilon + 1\big)\,\varepsilon.$$

Step 2. Estimate of $\|h^\varepsilon - \theta^\varepsilon\|_{L^2}$*:*

By Parseval's theorem, we find that

$$\|h^\varepsilon - \theta^\varepsilon\|_{L^2}^2 = \sum_{\lambda_n \le M_\varepsilon} \left| \frac{\sqrt{\lambda_n}\left(\lambda_n + \kappa_\varepsilon^2\right)}{\sqrt{\lambda_n}\sin\left(\kappa_\varepsilon T\right) - \kappa_\varepsilon \sinh\left(\sqrt{\lambda_n}\,T\right)} - \frac{\sqrt{\lambda_n}\left(\lambda_n + \kappa^2\right)}{\sqrt{\lambda_n}\sin(\kappa T) - \kappa \sinh\left(\sqrt{\lambda_n}\,T\right)} \right|^2 |g_n|^2 . \tag{20}$$

Using the mean value theorem, there exists ξ between κ_ε and κ such that:

$$\begin{aligned}&\left| \frac{\sqrt{\lambda_n}\left(\lambda_n + \kappa_\varepsilon^2\right)}{\sqrt{\lambda_n}\sin\left(\kappa_\varepsilon T\right) - \kappa_\varepsilon \sinh\left(\sqrt{\lambda_n}\,T\right)} - \frac{\sqrt{\lambda_n}\left(\lambda_n + \kappa^2\right)}{\sqrt{\lambda_n}\sin(\kappa T) - \kappa \sinh\left(\sqrt{\lambda_n}\,T\right)} \right| \\ &= \left| \frac{d}{d\kappa} \left(\frac{\sqrt{\lambda_n}\left(\lambda_n + \kappa^2\right)}{\sqrt{\lambda_n}\sin(\kappa T) - \kappa \sinh\left(\sqrt{\lambda_n}\,T\right)} \right) \right|_{\kappa = \xi} |\kappa_\varepsilon - \kappa| .\end{aligned} \tag{21}$$

Our next goal is to study the term

$$\left| \frac{d}{d\kappa} \left(\frac{\sqrt{\lambda_n}\left(\lambda_n + \kappa^2\right)}{\sqrt{\lambda_n}\sin(\kappa T) - \kappa \sinh\left(\sqrt{\lambda_n}\,T\right)} \right) \right|_{\kappa = \xi} .$$

for $\lambda_n \le M_\varepsilon$. We compute the derivative with respect to κ of the expression

$$\frac{\sqrt{\lambda_n}\left(\lambda_n+\kappa^2\right)}{\sqrt{\lambda_n}\sin(\kappa T)-\kappa\sinh\left(\sqrt{\lambda_n}T\right)}.$$

The result is

$$\begin{aligned}&\frac{d}{d\kappa}\left(\frac{\sqrt{\lambda_n}\left(\lambda_n+\kappa^2\right)}{\sqrt{\lambda_n}\sin(\kappa T)-\kappa\sinh\left(\sqrt{\lambda_n}\,T\right)}\right)\\&=\frac{2\sqrt{\lambda_n}\kappa\left(\sqrt{\lambda_n}\sin(\kappa T)-\kappa\sinh\left(\sqrt{\lambda_n}\,T\right)\right)-\sqrt{\lambda_n}\left(\lambda_n+\kappa^2\right)\left(\sqrt{\lambda_n}\,T\cos(\kappa T)-\sinh\left(\sqrt{\lambda_n}\,T\right)\right)}{\left(\sqrt{\lambda_n}\sin(\kappa T)-\kappa\sinh\left(\sqrt{\lambda_n}\,T\right)\right)^2}.\end{aligned}$$

It is obvious to see that

$$\sqrt{\lambda_n}\,T\cos(\kappa T)-\sinh\left(\sqrt{\lambda_n}\,T\right)=\frac{1}{\kappa}\left(\sqrt{\lambda_n}\sin(\kappa T)-\kappa\sinh\left(\sqrt{\lambda_n}\,T\right)\right)+\sqrt{\lambda_n}\left(T\cos(\kappa T)-\frac{\sin(\kappa T)}{\kappa}\right).$$

For $\kappa>0$, we have

$$\left|T\cos(\kappa T)-\frac{\sin(\kappa T)}{\kappa}\right|\le T+\frac{1}{\kappa}.$$

Thus, we find that

$$\left|\sqrt{\lambda_n}T\cos(\kappa T)-\sinh\left(\sqrt{\lambda_n}T\right)\right|\le\frac{1}{\kappa}\left|\sqrt{\lambda_n}\sin(\kappa T)-\kappa\sinh\left(\sqrt{\lambda_n}T\right)\right|+\sqrt{\lambda_n}\left(T+\frac{1}{\kappa}\right).$$

Since Lemma 3.2 , it follows that

$$\left|\sqrt{\lambda_n}\sin(\kappa T)-\kappa\sinh\left(\sqrt{\lambda_n}\,T\right)\right|\ge\sqrt{\lambda_n}(\kappa T-\sin(\kappa T)).$$

Then,

$$\frac{\left|\sqrt{\lambda_n}\,T\cos(\kappa T)-\sinh\left(\sqrt{\lambda_n}\,T\right)\right|}{\left|\sqrt{\lambda_n}\sin(\kappa T)-\kappa\sinh\left(\sqrt{\lambda_n}\,T\right)\right|}\le\frac{1}{\kappa}+\frac{T+\frac{1}{\kappa}}{\kappa T-\sin(\kappa T)}.$$

Substituting into the derivative expression, we obtain

$$\begin{aligned}\left|\frac{d}{d\kappa}\left(\frac{\sqrt{\lambda_n}\left(\lambda_n+\kappa^2\right)}{\sqrt{\lambda_n}\sin(\kappa T)-\kappa\sinh\left(\sqrt{\lambda_n}\,T\right)}\right)\right|\le&\frac{2\sqrt{\lambda_n}|\kappa|}{\left|\sqrt{\lambda_n}\sin(\kappa T)-\kappa\sinh\left(\sqrt{\lambda_n}\,T\right)\right|}\\&+\frac{\sqrt{\lambda_n}\left(\lambda_n+\kappa^2\right)}{\left|\sqrt{\lambda_n}\sin(\kappa T)-\kappa\sinh\left(\sqrt{\lambda_n}\,T\right)\right|}\left(\frac{1}{\kappa}+\frac{T+\frac{1}{\kappa}}{\kappa T-\sin(\kappa T)}\right).\end{aligned}$$

Applying the lower bound $\left|\sqrt{\lambda_n}\sin(\kappa T)-\kappa\sinh\left(\sqrt{\lambda_n}\,T\right)\right|\ge\sqrt{\lambda_n}(\kappa T-\sin(\kappa T))$, we get

$$\frac{2\sqrt{\lambda_n}|\kappa|}{\left|\sqrt{\lambda_n}\sin(\kappa T)-\kappa\sinh\left(\sqrt{\lambda_n}\,T\right)\right|}\le\frac{2|\kappa|}{\kappa T-\sin(\kappa T)},$$

and

$$\frac{\sqrt{\lambda_n}\left(\lambda_n+\kappa^2\right)}{\left|\sqrt{\lambda_n}\sin(\kappa T)-\kappa\sinh\left(\sqrt{\lambda_n}\,T\right)\right|}\le\frac{\lambda_n+\kappa^2}{\kappa T-\sin(\kappa T)}.$$

Therefore, we find that

$$\left|\frac{d}{d\kappa}\left(\frac{\sqrt{\lambda_n}\left(\lambda_n+\kappa^2\right)}{\sqrt{\lambda_n}\sin(\kappa T)-\kappa\sinh\left(\sqrt{\lambda_n}\,T\right)}\right)\right|\le\frac{2|\kappa|}{\kappa T-\sin(\kappa T)}+\frac{\lambda_n+\kappa^2}{\kappa T-\sin(\kappa T)}\left(\frac{1}{\kappa}+\frac{T+\frac{1}{\kappa}}{\kappa T-\sin(\kappa T)}\right).$$

Since $\lambda_n \le M_\varepsilon$, we obtain the uniform bound

$$\sup_{\lambda_n \le M_\varepsilon} \left| \frac{d}{d\kappa} \left(\frac{\sqrt{\lambda_n}\left(\lambda_n + \kappa^2\right)}{\sqrt{\lambda_n}\sin(\kappa T) - \kappa \sinh\left(\sqrt{\lambda_n}\, T\right)} \right) \right| \le \frac{2|\kappa|}{\kappa T - \sin(\kappa T)} + \frac{M_\varepsilon + \kappa^2}{\kappa T - \sin(\kappa T)} \left(\frac{1}{\kappa} + \frac{T + \frac{1}{\kappa}}{\kappa T - \sin(\kappa T)} \right)$$
$$= A(\kappa, T) + B(\kappa, T) \frac{M_\varepsilon + \kappa^2}{\kappa T - \sin(\kappa T)}. \tag{22}$$

Combining (20), (21), (22), we find that

$$\|h^\varepsilon - \theta^\varepsilon\|_{L^2}^2 \le \left(A(\kappa, T) + B(\kappa, T) \frac{M_\varepsilon + \kappa^2}{\kappa T - \sin(\kappa T)} \right)^2 \varepsilon^2 \|g\|_{L^2}^2.$$

Hence, we obtain

$$\|h^\varepsilon - \theta^\varepsilon\|_{L^2} \le \left(A(\kappa, T) + B(\kappa, T) \frac{M_\varepsilon + \kappa^2}{\kappa T - \sin(\kappa T)} \right) \varepsilon \|g\|_{L^2}.$$

Step 3. Estimate of $\|f - \theta^\varepsilon\|_{L^2}$:

Assume $f \in \mathbb{H}^\sigma(\Omega)$, the truncation error is:

$$\begin{aligned}
\|\theta^\varepsilon - f\|_{L^2}^2 &= \sum_{\lambda_n > M_\varepsilon} \left| \frac{\sqrt{\lambda_n}\left(\lambda_n + \kappa^2\right)}{\sqrt{\lambda_n}\sin(\kappa T) - \kappa \sinh\left(\sqrt{\lambda_n} T\right)} g_n \right|^2 \\
&\le \sum_{\lambda_n > M_\varepsilon} \lambda_n^{-\sigma} \lambda_n^{\sigma} |f_n|^2 \\
&\le M_\varepsilon^{-\sigma} \sum_{\lambda_n > M_\varepsilon} \lambda_n^{\sigma} |g_n|^2 \le M_\varepsilon^{-\sigma} \|f\|_{\mathbb{H}^\sigma}^2.
\end{aligned}$$

Thus

$$\|\theta^\varepsilon - f\|_{L^2} \le M_\varepsilon^{-\sigma/2} \|f\|_{\mathbb{H}^\sigma}.$$

By combining three observations of step $1, 2, 3$, we arrive at

$$\begin{aligned}
\|f^\varepsilon - f\|_{L^2(\Omega)} &\le \|h^\varepsilon - \theta^\varepsilon\|_{L^2(\Omega)} + \|f^\varepsilon - h^\varepsilon\|_{L^2(\Omega)} + \|\theta^\varepsilon - f\|_{L^2(\Omega)} \\
&\le \widetilde{C}(\kappa, T)\left(M_\varepsilon + 1\right)\varepsilon + \left(A(\kappa, T) + B(\kappa, T) \frac{M_\varepsilon + \kappa^2}{\kappa T - \sin(\kappa T)} \right) \varepsilon \|g\|_{L^2(\Omega)} + M_\varepsilon^{-\sigma/2} \|f\|_{\mathbb{H}^\sigma}.
\end{aligned}$$

□

References

[1] Yang, F.; Guariglia, E.; Silvestrov, S., *Fractional–wavelet analysis of positive definite distributions and wavelets on* $\mathcal{D}'(\mathbb{C})$. In: *Engineering Mathematics II: Algebraic, Stochastic and Analysis Structures for Networks, Data Classification and Optimization*, pp. 337–353. Springer, 2016.

[2] Guariglia, E., *Fractional calculus, zeta functions and Shannon entropy. Open Mathematics* **19**(1), 87–100 (2021).

[3] Kozhanov, A. I.; Shipina, T. N., *Loaded differential equations and linear inverse problems for elliptic equations. Complex Variables and Elliptic Equations* **66**(6–7), 910–928 (2021).

[4] Qian, A.; Li, Y., *Optimal error bound and generalized Tikhonov regularization for identifying an unknown source in the heat equation. Journal of Mathematical Chemistry* **49**, 765–775 (2011).

[5] Ahmed, H. M.; Rabie, W. B.; Ragusa, M. A., *Optical solitons and other solutions to Kaup–Newell equation with Jacobi elliptic function expansion method. Analysis and Mathematical Physics* **11**, 1–16 (2021).

[6] Boujemaa, H.; Oulgiht, B.; Ragusa, M. A., *A new class of fractional Orlicz–Sobolev space and singular elliptic problems. Journal of Mathematical Analysis and Applications* **526**(1), 127342 (2023).

[7] Eddine, N. C.; Nguyen, P. D.; Ragusa, M. A., *Existence and multiplicity of solutions for a class of critical anisotropic elliptic equations of Schrödinger–Kirchhoff type. Mathematical Methods in the Applied Sciences* **46**, 16782–16801 (2023).

[8] Inglese, G., *An inverse problem in corrosion detection. Inverse Problems* **13**(4), 977 (1997).
[9] Lesnic, D.; Elliott, L.; Ingham, D. B., *An iterative boundary element method for solving numerically the Cauchy problem for the Laplace equation. Engineering Analysis with Boundary Elements* **20**(2), 123–133 (1997).
[10] Yang, F., *The truncation method for identifying an unknown source in the Poisson equation. Applied Mathematics and Computation* **217**(22), 9334–9339 (2011).
[11] Yang, F.; Fu, C. L., *The modified regularization method for identifying the unknown source on Poisson equation. Applied Mathematical Modelling* **36**(2), 756–763 (2012).
[12] Inglese, I., *An inverse problem in corrosion detection. Inverse Problems* **13** (1997), 977.
[13] Zhdanov, M. S., *Geophysical Inverse Theory and Regularization Problems.* Elsevier Science B.V., (year not supplied).
[14] Yamashita, Y., *Theoretical studies on the inverse problem in electrocardiography and the uniqueness of the solution. IEEE Transactions on Biomedical Engineering* (1982), 719–725.
[15] Wen, J.; Huang, L. M.; Liu, X. Z., *A modified quasi-reversibility method for inverse source problem of Poisson equation. Inverse Problems in Science and Engineering* **29**(12), 2098–2109 (2021).
[16] Zhao, Z.; Meng, Z.; You, L.; Xie, O., *Identifying an unknown source in the Poisson equation by the method of Tikhonov regularization in Hilbert scales. Applied Mathematical Modelling* **38**(19–20), 4686–4693 (2014).
[17] Qian, A.; Li, Y., *Optimal error bound and generalized Tikhonov regularization for identifying an unknown source in the heat equation. Journal of Mathematical Chemistry* **49**(3), 765–775 (2011).
[18] Tuan, N. H.; Tri, V. V.; O'Regan, D., *On a nonlinear parabolic equation with fractional Laplacian and integral conditions. Applied Analysis* **101**(17), 5974–5988 (2022).
[19] Tuan, N. H.; Thang, L. D.; Khoa, V. A., *A modified integral equation method of the nonlinear elliptic equation with globally and locally Lipschitz source. Applied Mathematics and Computation* **265**, 245–265 (2015).
[20] Tuan, N. H.; Thang, L. D.; Trong, D. D.; Khoa, V. A., *Approximation of mild solutions of the linear and nonlinear elliptic equations. Inverse Problems in Science and Engineering* **23**(7), 1237–1266 (2015).
[21] Tuan, N. H.; Thang, L. D.; Trong, D. D.; Khoa, V. A., *On an inverse boundary value problem of a nonlinear elliptic equation in three dimensions. Journal of Mathematical Analysis and Applications* **426**(2), 1232–1261 (2015).
[22] Tuan, N. H.; Caraballo, T., *On initial and terminal value problems for fractional nonclassical diffusion equations. Proceedings of the American Mathematical Society* **149**(1), 143–161 (2021).
[23] Li, L.; Zhou, X.; Gao, H., *The stability and exponential stabilization of the heat equation with memory. Journal of Mathematical Analysis and Applications* **466**(1), 199–214 (2018).
[24] Zhao, J.; Liu, S., *Two regularization methods for inverse source problem on the Poisson equation. Complex Variables and Elliptic Equations* **60**(10), 1374–1391 (2015).
[25] Zhao, J.; Liu, S.; Liu, T., *Two Tikhonov-type regularization methods for inverse source problem on the Poisson equation. Mathematical Methods in the Applied Sciences* **36**(11), 1399–1408 (2013).
[26] Kirsch, A., *An Introduction to the Mathematical Theory of Inverse Problems.* 2nd ed., Applied Mathematical Sciences, vol. 120, Springer, New York, 2011.
[27] Nam, B. Đ.; Thach, T. N.; Tien, N. V., *Inverse Source Problem for the Poisson Equation with Final and Integral Conditions. Bulletin of the Malaysian Mathematical Sciences Society* **48**:67 (2025). DOI: 10.1007/s40840-025-01854-0.